ADVANCED ALGEBRA

Anonymous

Advanced Algebra

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Licensing

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Preface

Intermediate Algebra is the second part of a two-part course in Algebra. Written in a clear and concise manner, it carefully builds on the basics learned in Elementary Algebra and introduces the more advanced topics required for further study of applications found in most disciplines. Used as a standalone textbook, it offers plenty of review as well as something new to engage the student in each chapter. Written as a blend of the traditional and graphical approaches to the subject, this textbook introduces functions early and stresses the geometry behind the algebra. While CAS independent, a standard scientific calculator will be required and further research using technology is encouraged.

Intermediate Algebra clearly lays out the steps required to build the skills needed to solve a variety of equations and interpret the results. With robust and diverse exercise sets, students have the opportunity to solve plenty of practice problems. In addition to embedded video examples and other online learning resources, the importance of practice with pencil and paper is stressed. This text respects the traditional approaches to algebra pedagogy while enhancing it with the technology available today. In addition, Intermediate Algebra was written from the ground up in an open and modular format, allowing the instructor to modify it and leverage their individual expertise as a means to maximize the student experience and success.

The importance of Algebra cannot be overstated; it is the basis for all mathematical modeling used in all disciplines. After completing a course sequence based on Elementary and Intermediate Algebra, students will be on firm footing for success in higher-level studies at the college level.

CHAPTER OVERVIEW

1: Algebra Fundamentals

1.1: Review of Real Numbers and Absolute Value
1.2: Operations with Real Numbers
1.3: Square and Cube Roots of Real Numbers
1.4: Algebraic Expressions and Formulas
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1.1: Review of Real Numbers and Absolute Value

Learning Objectives

- Review the set of real numbers.
- Review the real number line and notation.
- Define the geometric and algebraic definition of absolute value.

Real Numbers

Algebra is often described as the generalization of arithmetic. The systematic use of *variables*[1], letters used to represent numbers, allows us to communicate and solve a wide variety of real-world problems. For this reason, we begin by reviewing real numbers and their operations.

A *set*[2] is a collection of objects, typically grouped within braces {}, where each object is called an *element*[3]. When studying mathematics, we focus on special sets of numbers.

$$\mathbb{N}=\underbrace{\{1,2,3,4,5,\ldots\}}_{\text{Natural Numbers}}$$

$$W=\underbrace{\{0,1,2,3,4,5,\ldots\}}_{\text{Whole Numbers}}$$

$$\mathbb{Z}=\underbrace{\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}}_{\text{Integers}}$$

The three periods (…) are called an ellipsis and indicate that the numbers continue without bound. A **subset**[4], denoted $\subseteq$, is a set consisting of elements that belong to a given set. Notice that the sets of *natural*[5] and *whole numbers*[6] are both subsets of the set of integers and we can write:

$$\mathbb{N}\subseteq\mathbb{Z}\text{ and }W\subseteq\mathbb{Z}$$

A set with no elements is called the **empty set**[7] and has its own special notation:

$$\{\quad\}=\varnothing\text{ Empty Set}$$

Rational numbers[8], denoted $\mathbb{Q}$, are defined as any number of the form $\frac{a}{b}$ where a and b are integers and b is nonzero. We can describe this set using **set notation**[9]:

$$\mathbb{Q}=\left\{\frac{a}{b}|a,b\in\mathbb{Z},b\neq 0\right\}\text{ Rational Numbers}$$

The vertical line | inside the braces reads, "*such that*" and the symbol $\in$ indicates set membership and reads, "*is an element of*." The notation above in its entirety reads, "*the set of all numbers* $\frac{a}{b}$ *such that a and b are elements of the set of integers and b is not equal to zero.*" Decimals that terminate or repeat are rational. For example,

$$0.05=\frac{5}{100}\text{ and }0.\overline{6}=0.6666\ldots=\frac{2}{3}$$

The set of integers is a subset of the set of rational numbers, $\mathbb{Z}\subseteq\mathbb{Q}$, because every integer can be expressed as a ratio of the integer and 1. In other words, any integer can be written over 1 and can be considered a rational number. For example,

$$7=\frac{7}{1}$$

Irrational numbers[10] are defined as any numbers that cannot be written as a ratio of two integers. Nonterminating decimals that do not repeat are irrational. For example,

$$\pi=3.14159\ldots\text{ and }\sqrt{2}=1.41421\ldots$$

Finally, the set of **real numbers**[11], denoted $\mathbb{R}$, is defined as the set of all rational numbers combined with the set of all irrational numbers. Therefore, all the numbers defined so far are subsets of the set of real numbers. In summary,

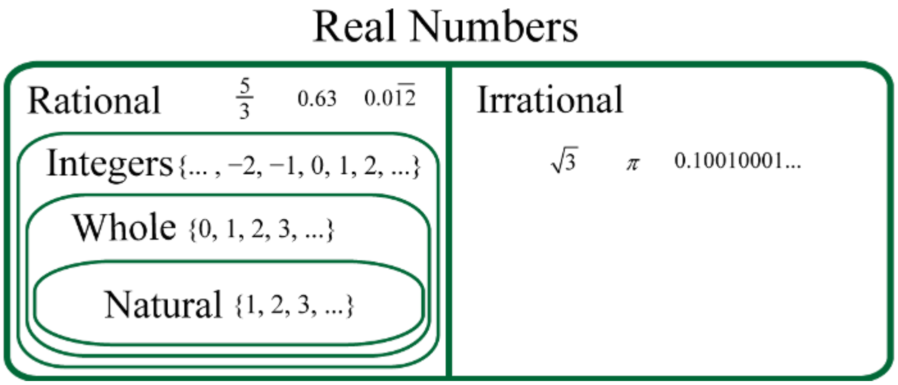

Figure $1.1.1$: Real Numbers

The set of **even integers**[12] is the set of all integers that are evenly divisible by 2. We can obtain the set of even integers by multiplying each integer by 2.

$$\{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\} \textit{ Even Integers}$$

The set of **odd integers**[13] is the set of all nonzero integers that are not evenly divisible by 2.

$$\{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}\textit{Odd Integers}$$

A **prime number**[14] is an integer greater than 1 that is divisible only by 1 and itself. The smallest prime number is 2 and the rest are necessarily odd.

$$\{2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots\}\textit{Prime Numbers}$$

Any integer greater than 1 that is not prime is called a **composite number**[15] and can be uniquely written as a product of primes. When a composite number, such as 42, is written as a product, $42 = 2 \cdot 21$, we say that $2 \cdot 21$ is a **factorization**[16] of 42 and that 2 and 21 are **factors**[17]. Note that factors divide the number evenly. We can continue to write composite factors as products until only a product of primes remains.

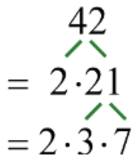

Figure $1.1.2$: Prime Factorization of the number 42

Therefore, the **prime factorization**[18] of 42 is $2 \cdot 3 \cdot 7$.

Example 1.1.1:

Determine the prime factorization of 210.

Solution

Begin by writing 210 as a product with 10 as a factor. Then continue factoring until only a product of primes remains.

$$\begin{aligned} 210 &= 10 \cdot 21 \\ &= 2 \cdot 5 \cdot 3 \cdot 7 \\ &= 2 \cdot 3 \cdot 5 \cdot 7 \end{aligned}$$

Since the prime factorization is unique, it does not matter how we choose to initially factor the number; the end result will be the same.

Answer:

$2 \cdot 3 \cdot 5 \cdot 7$

A **fraction**[19] is a rational number written as a quotient, or ratio, of two integers *a* and *b* where $b \neq 0$.

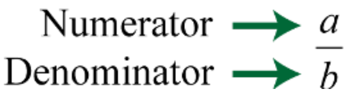

Figure 1.1.3

The integer above the fraction bar is called the **numerator**[20] and the integer below is called the **denominator**[21]. Two equal ratios expressed using different numerators and denominators are called **equivalent fractions**[22]. For example,

$$\frac{50}{100}=\frac{1}{2}$$

Consider the following factorizations of 50 and 100:

$$50=2\cdot 25$$

$$100=4\cdot 25$$

The numbers 50 and 100 share the factor 25. A shared factor is called a **common factor**[23]. Making use of the fact that $\frac{25}{25}=1$, we have

$$\frac{50}{100}=\frac{2\cdot\cancel{25}}{4\cdot\cancel{25}}=\frac{2}{4}\cdot 1=\frac{2}{4}$$

Dividing $\frac{25}{25}$ and replacing this factor with a 1 is called **cancelling**[24]. Together, these basic steps for finding equivalent fractions define the process of **reducing**[25]. Since factors divide their product evenly, we achieve the same result by dividing both the numerator and denominator by 25 as follows:

$$\frac{50\div 25}{100\div 25}=\frac{2}{4}$$

Finding equivalent fractions where the numerator and denominator are **relatively prime**[26], or have no common factor other than 1, is called **reducing to lowest terms**[27]. This can be done by dividing the numerator and denominator by the **greatest common factor (GCF)**.[28] The GCF is the largest number that divides a set of numbers evenly. One way to find the GCF of 50 and 100 is to list all the factors of each and identify the largest number that appears in both lists. Remember, each number is also a factor of itself.

$$\{1,2,5,10,25,50\}\ \textit{Factors of}\ \ 50$$

$$\{1,2,4,5,10,20,25,50,100\}\ \ \textit{Factors of}\ \ 100$$

Common factors are listed in bold, and we see that the greatest common factor is 50. We use the following notation to indicate the GCF of two numbers: $\text{GCF}(50,100)=50$. After determining the GCF, reduce by dividing both the numerator and the denominator as follows:

$$\frac{50\div 50}{100\div 50}=\frac{1}{2}$$

Example 1.1.2:

Reduce to lowest terms: $\frac{108}{72}$.

Solution

A quick way to find the GCF of the numerator and denominator requires us to first write each as a product of primes. The GCF will be the product of all the common prime factors.

$$\left.\begin{array}{l}108=2\cdot 2\cdot 3\cdot 3\cdot 3\\ 72=2\cdot 2\cdot 2\cdot 3\cdot 3\end{array}\right\}\text{GCF}(108,72)=2\cdot 2\cdot 3\cdot 3=36$$

In this case, the product of the common prime factors is 36.

$$\frac{108}{72}=\frac{108\div 36}{72\div 36}=\frac{3}{2}$$

We can convert the improper fraction $\frac{3}{2}$ to a mixed number $1\frac{1}{2}$; however, it is important to note that converting to a mixed number is not part of the reducing process. We consider improper fractions, such as $\frac{3}{2}$, to be reduced to lowest terms. In algebra it is often preferable to work with improper fractions, although in some applications, mixed numbers are more appropriate.

Answer:

$\frac{3}{2}$

Recall the relationship between multiplication and division:

$$\begin{array}{r} \mathit{dividend} \rightarrow \\ \mathit{divisor} \rightarrow \end{array} \frac{12}{6} = 2 \quad \leftarrow \ \mathit{quotient} \text{because } 6 \cdot 2 = 12$$

In this case, the **dividend**[29] 12 is evenly divided by the **divisor**[30] 6 to obtain the **quotient**[31] 2. It is true in general that if we multiply the divisor by the quotient we obtain the dividend. Now consider the case where the dividend is zero and the divisor is nonzero:

$$\frac{0}{6} = 0 \text{ since } 6 \cdot 0 = 0$$

This demonstrates that zero divided by any nonzero real number must be zero. Now consider a nonzero number divided by zero:

$$\frac{12}{0} = ? \text{ or } 0 \cdot ? = 12$$

Zero times anything is zero and we conclude that there is no real number such that $0 \cdot ? = 12$. Thus, the quotient $12 \div 0$ is **undefined**[32]. Try it on a calculator, what does it say? For our purposes, we will simply write "undefined." To summarize, given any real number $a \neq 0$, then

$$0 \div a = \frac{0}{a} = 0 \quad \mathit{zero} \text{ and } a \div 0 = \frac{a}{0} \mathit{undefined}$$

We are left to consider the case where the dividend and divisor are both zero.

$$\frac{0}{0} = ? \text{ or } 0 \cdot ? = 0$$

Here, any real number seems to work. For example, $0 \cdot 5 = 0$ and also, $0 \cdot 3 = 0$. Therefore, the quotient is uncertain or **indeterminate**[33].

$$0 \div 0 = \frac{0}{0} \quad \mathit{indeterminate}$$

In this course, we state that $0 \div 0$ is undefined.

The Number Line and Notation

A **real number line**[34], or simply **number line**, allows us to visually display real numbers by associating them with unique points on a line. The real number associated with a point is called a **coordinate**[35]. A point on the real number line that is associated with a coordinate is called its **graph**[36]. To construct a number line, draw a horizontal line with arrows on both ends to indicate that it continues without bound. Next, choose any point to represent the number zero; this point is called the **origin**[37].

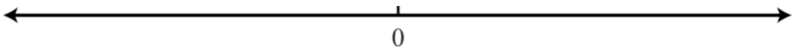

Figure 1.1.4

Positive real numbers lie to the right of the origin and negative real numbers lie to the left. The number zero (0) is neither positive nor negative. Typically, each tick represents one unit.

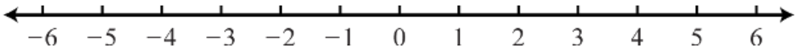

Figure 1.1.5

As illustrated below, the scale need not always be one unit. In the first number line, each tick mark represents two units. In the second, each tick mark represents $\frac{1}{7}$:

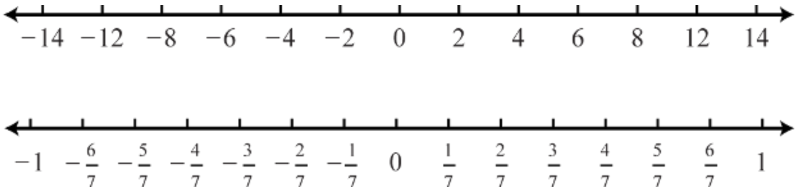

Figure 1.1.6

The graph of each real number is shown as a dot at the appropriate point on the number line. A partial graph of the set of integers $\mathbb{Z}$, follows:

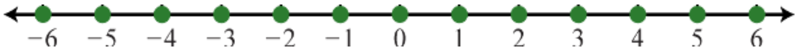

Figure 1.1.7

Example 1.1.3:

Graph the following set of real numbers: $\{-\frac{5}{2},\ 0,\ \frac{3}{2},\ 2\}$.

Solution

Graph the numbers on a number line with a scale where each tick mark represents $\frac{1}{2}$ unit.

Answer:

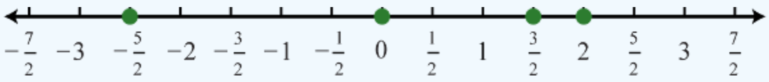

Figure 1.1.8

The **opposite**[38] of any real number a is $-a$. Opposite real numbers are the same distance from the origin on a number line, but their graphs lie on opposite sides of the origin and the numbers have opposite signs.

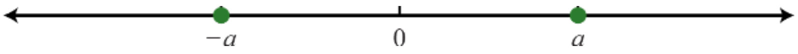

Figure 1.1.9

Given the integer -7, the integer the same distance from the origin and with the opposite sign is $+7$, or just 7.

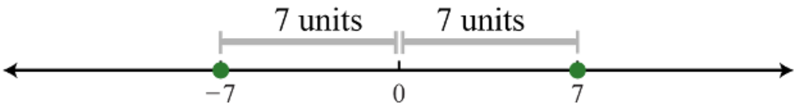

Figure 1.1.10

Therefore, we say that the opposite of -7 is $-(-7) = 7$. This idea leads to what is often referred to as the **double-negative property**[39]. For any real number a,

$$-(-a) = a$$

Example 1.1.4:

Calculate: $-(-(-\frac{3}{8}))$.

Solution

Here we apply the double-negative within the innermost parentheses first.

$$\begin{aligned} -(-(-\tfrac{3}{8})) &= -(\tfrac{3}{8}) \\ &= -\tfrac{3}{8} \end{aligned}$$

Answer:

$-\frac{3}{8}$

In general, an odd number of sequential negative signs results in a negative value and an even number of sequential negative signs results in a positive value.

When comparing real numbers on a number line, the larger number will always lie to the right of the smaller one. It is clear that 15 is greater than 5, but it may not be so clear to see that -1 is greater than -5 until we graph each number on a number line.

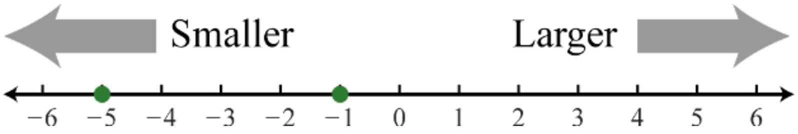

Figure 1.1.11

We use symbols to help us efficiently communicate relationships between numbers on the number line.

$$\begin{array}{cc} \textit{Equality Relationships} & \textit{Order Relationships} \\ = \text{"is equal to"} & < \text{"is less than"} \\ \neq \text{"is not equal to"} & > \text{"is greater than"} \end{array}$$

$$\approx \text{"is approximately equal to"} \quad \leq \quad \text{"is less than or equal to"}$$

$$\geq \quad \text{"is greater than or equal to"}$$

The relationship between the **integers**[40] in the previous illustration can be expressed two ways as follows:

$$-5 < -1 \text{"Negative five is less than negative one."}$$

or

$$-1 > -5 \text{"Negative one is greater than negative five."}$$

The symbols $<$ and $>$ are used to denote **strict inequalities**[41], and the symbols $\leq$ and $\geq$ are used to denote **inclusive inequalities**[42]. In some situations, more than one symbol can be correctly applied. For example, the following two statements are both true:

$$-10 < 0 \text{ and } -10 \leq 0$$

In addition, the "*or equal to*" component of an inclusive inequality allows us to correctly write the following:

$$-10 \leq -10$$

The logical use of the word "*or*" requires that only one of the conditions need be true: the "*less than*" or the "*equal to*."

Example 1.1.5:

Fill in the blank with $<, =,$ or $>$: -2 ____ -12.

Solution

Use $>$ because the graph of -2 is to the right of the graph of -12 on a number line. Therefore, $-2 > -12$, which reads, "*negative two is greater than negative twelve.*"

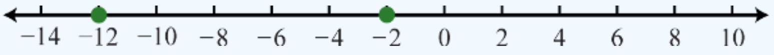

Figure 1.1.12

Answer:

$-2 > -12$

An **algebraic inequality**[43], such as $x \geq 2$, is read, "*x is greater than or equal to* 2." Here the letter *x* is a variable, which can represent any real number. However, the statement $x \geq 2$ imposes a condition on the variable. **Solutions**[44] are the values for *x* that satisfy the condition. This inequality has infinitely many solutions for *x*, some of which are $2, 3, 4.1, 5, 20,$ and 20.001. Since it is impossible to list all of the solutions, a system is needed that allows a clear communication of this infinite set. Common ways of expressing solutions to an inequality are by graphing them on a number line, using interval notation, or using set notation.

To express the solution graphically, draw a number line and shade in all the values that are solutions to the inequality. This is called the **graph of the solution set**[45]. Interval and set notation follow:

$$\text{"x is greater than or equal to 2"} \quad x \geq 2$$

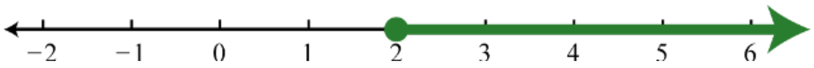

Figure 1.1.13

$$\text{Interval notation}: \ [2, \infty)$$

$$\text{Set notation}: \ \{x \in \mathbb{R} | x \geq 2\}$$

In this example, there is an inclusive inequality, which means that the lower-bound 2 is included in the solution set. Denote this with a closed dot on the number line and a square bracket in interval notation. The symbol ∞ is read as "**infinity**[46]" and indicates that the set is unbounded to the right on a number line. If using a standard keyboard, use (inf) as a shortened form to denote infinity. Now compare the notation in the previous example to that of the strict, or noninclusive, inequality that follows:

$$\text{"x is less than 3"} \quad x < 3$$

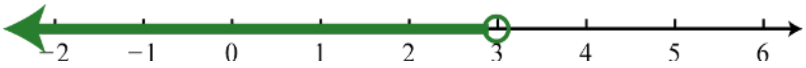

Figure 1.1.14

$$\textit{Interval notation}: (-\infty, 3)$$

$$\textit{Set notation}: \{x \in \mathbb{R} | x < 3\}$$

Strict inequalities imply that solutions may get very close to the boundary point, in this case 3, but not actually include it. Denote this idea with an open dot on the number line and a round parenthesis in interval notation. The symbol $-\infty$ is read as "**negative infinity**[47]" and indicates that the set is unbounded to the left on a number line. Infinity is a bound to the real numbers, but is not itself a real number: it cannot be included in the solution set and thus is always enclosed with a parenthesis.

Interval notation is textual and is determined after graphing the solution set on a number line. The numbers in interval notation should be written in the same order as they appear on the number line, with smaller numbers in the set appearing first. Set notation, sometimes called set-builder notation, allows us to describe the set using familiar mathematical notation. For example,

$$\{x \in \mathbb{R} | x \geq 2\}$$

Here, $x \in \mathbb{R}$ describes the type of number. This implies that the variable x represents a real number. The statement $x \geq 2$ is the condition that describes the set using mathematical notation. At this point in our study of algebra, it is assumed that all variables represent real numbers. For this reason, you can omit the "$\in \mathbb{R}$", and write

$$\{x | x \geq 2\}$$

Example 1.1.6:

Graph the solution set and give the interval and set notation equivalents: $x < -20$.

Solution

Use an open dot at -20, because of the strict inequality $<$, and shade all real numbers to the left.

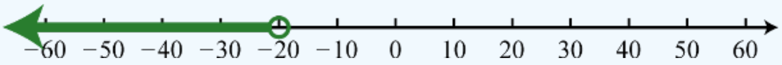

Figure 1.1.15

Answer:

Interval notation: $(-\infty, -20)$; set notation: $\{x | x < -20\}$

A **compound inequality**[48] is actually two or more inequalities in one statement joined by the word "and" or by the word "or". Compound inequalities with the logical "or" require that either condition must be satisfied. Therefore, the solution set of this type of compound inequality consists of all the elements of the solution sets of each inequality. When we join these individual solution sets it is called the **union**[49], denoted $\cup$. For example,

$$x < 3 \text{ or } x \geq 6$$

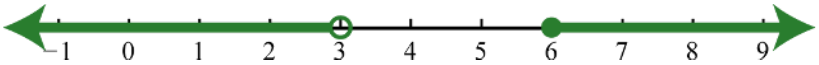

Figure 1.1.16

$$\textit{Interval notation}: (-\infty, 3) \cup [6, \infty)$$

$$\textit{Set notation}: \{x | \quad x < 3 \quad or \quad x \geq 6\}$$

An inequality such as,

$$-1 \leq x < 3$$

reads, "*negative one is less than or equal to x and x is less than three*." This is actually a compound inequality because it can be decomposed as follows:

$$-1 \leq x \text{ and } x < 3$$

The logical "and" requires that both conditions must be true. Both inequalities will be satisfied by all the elements in the **intersection**[50], denoted $\cap$, of the solution sets of each.

Example 1.1.7:

Graph and give the interval notation equivalent: $-1 \leq x < 3$.

Solution

Determine the intersection, or overlap, of the two solution sets to $x < 3$ and $x \geq -1$. The solutions to each inequality are sketched above the number line as a means to determine the intersection, which is graphed on the number line below.

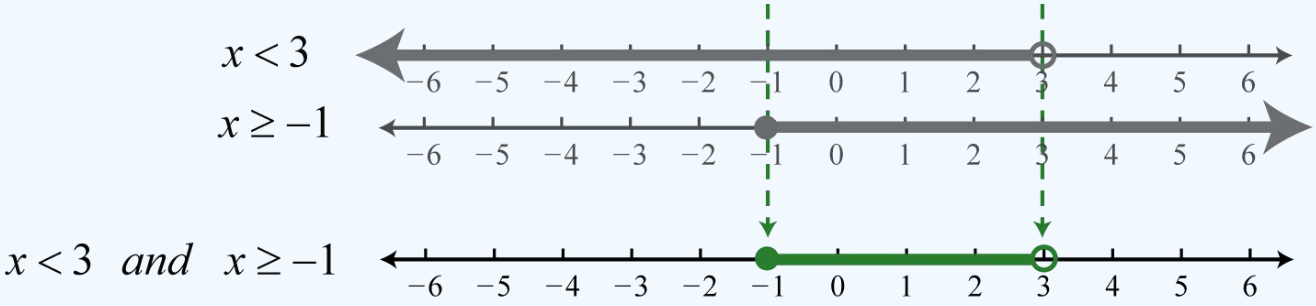

Figure 1.1.17

Here, 3 is not a solution because it solves only one of the inequalities. Alternatively, we may interpret $-1 \leq x < 3$ as all possible values for *x* between, or bounded by, -1 and 3 where -1 is included in the solution set.

Answer:

Interval notation: $[-1,\ 3)$; set notation: $\{x \mid -1 \leq x < 3\}$

In this text, we will often point out the equivalent notation used to express mathematical quantities electronically using the standard symbols available on a keyboard.

$$\times\ {}''*''\quad \geq\ {}''>=''$$

$$\div\ {}''/''\quad \leq\ {}''<=''$$

$$\neq\ {}''!=''$$

Many calculators, computer algebra systems, and programming languages use the notation presented above, in quotes.

Absolute Value

The **absolute value**[51] of a real number *a*, denoted $|a|$, is defined as the distance between zero (the origin) and the graph of that real number on the number line. Since it is a distance, it is always positive. For example,

$$|-4| = 4 \text{ and } |4| = 4$$

Both 4 and -4 are four units from the origin, as illustrated below:

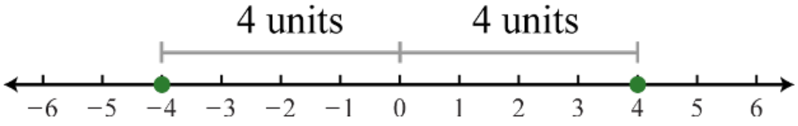

Figure 1.1.18

Also, it is worth noting that,

$$|0| = 0$$

The algebraic definition of the absolute value of a real number *a* follows:

$$|a| = \begin{cases} a \text{ if } a \geq 0 \\ -a \text{ if } a < 0 \end{cases}$$

This is called a **piecewise definition**[52]. The result depends on the quantity *a*. If *a* is nonnegative, as indicated by the inequality $a \geq 0$, then the absolute value will be that number *a*. If *a* is negative, as indicated by the inequality $a < 0$, then the absolute value will be the opposite of that number, −*a*. The results will be the same as the geometric definition. For example, to determine $|-4|$ we make note that the value is negative and use the second part of the definition. The absolute value will be the opposite of -4.

$$\begin{aligned}|-4| &= -(-4)\\ &= 4\end{aligned}$$

At this point, we can determine what real numbers have certain absolute values.

Example 1.1.8:

Determine the values represented by \(*x*: | x |=6\).

Solution

Think of a real number whose distance to the origin is 6 units. There are two solutions: the distance to the right of the origin and the distance to the left of the origin, namely $\{\pm 6\}$. The symbol $\pm$ is read "*plus or minus*" and indicates that there are two answers, one positive and one negative.

$$|-6| = 6 \text{ and } |6| = 6$$

Answer:

x=±6

Example 1.1.9:

Determine the values represented by \(*x*: | x |=−6\).

Solution

Here we wish to find a value where the distance to the origin is negative. Since negative distance is not defined, this equation has no solution. Use the empty set $\varnothing$ to denote this.

Answer:

$\varnothing$

The absolute value can be expressed textually using the notation abs(*a*). We often encounter negative absolute values, such as $-|3|$ or $-abs(3)$. Notice that the negative sign is in front of the absolute value symbol. In this case, work the absolute value first and then find the opposite of the result.

$$\begin{array}{ccc} -|3| & & -|-3| \\ \downarrow & \text{and} & \downarrow \\ =-3 & & =-3 \end{array}$$

Try not to confuse this with the double negative property, which states that $-(-3) = +3$.

Example 1.1.10:

Simplify: $-(-|-50|)$.

Solution

First, find the absolute value of -50 and then apply the double-negative property.

$$\begin{aligned}-(-|-50|) &= -(-50)\\ &= 50\end{aligned}$$

Answer:

50

Key Takeaways

- Algebra is often described as the generalization of arithmetic. The systematic use of variables, used to represent real numbers, allows us to communicate and solve a wide variety of real-world problems. Therefore, it is important to review the subsets of real numbers and their properties.
- The number line allows us to visually display real numbers by associating them with unique points on a line.
- Special notation is used to communicate equality and order relationships between numbers on a number line.

- The absolute value of a real number is defined geometrically as the distance between zero and the graph of that number on a number line. Alternatively, the absolute value of a real number is defined algebraically in a piecewise manner. If a real number a is nonnegative, then the absolute value will be that number a. If a is negative, then the absolute value will be the opposite of that number, $-a$.

Exercise $1.1.1$

Use set notation to list the described elements.

1. Every other positive odd number up to 21.
2. Every other positive even number up to 22.
3. The even prime numbers.
4. Rational numbers that are also irrational.
5. The set of negative integers.
6. The set of negative even integers.
7. Three consecutive odd integers starting with 13.
8. Three consecutive even integers starting with 22.

Answer

1. $\{1,5,9,13,17,21\}$

3. $\{2\}$

5. $\{\ldots,-3,-2,-1\}$

7. $\{13,15,17\}$

Exercise $1.1.2$

Determine the prime factorization of the given composite number.

1. 195
2. 78
3. 330
4. 273
5. 180
6. 350

Answer

1. $3\cdot5\cdot13$

3. $2\cdot3\cdot5\cdot11$

5. $2\cdot2\cdot3\cdot3\cdot5$

Exercise $1.1.3$

Reduce to lowest terms.

1. $\frac{42}{30}$
2. $\frac{105}{70}$
3. $\frac{84}{120}$
4. $\frac{315}{420}$
5. $\frac{60}{45}$
6. $\frac{144}{120}$
7. $\frac{64}{128}$
8. $\frac{72}{216}$
9. $\frac{0}{25}$

10. $\frac{33}{0}$

Answer

1. $\frac{7}{5}$
3. $\frac{7}{10}$
5. $\frac{4}{3}$
7. $\frac{1}{2}$
9. 0

Exercise 1.1.4

Graph the following sets of numbers.

1. $\{-5, 5, 10, 15\}$
2. $\{-4, -2, 0, 2, 4\}$
3. $\{-\frac{3}{2}, -\frac{1}{2},\ 0,\ 1,\ 2\}$
4. $\{-\frac{3}{4}, -\frac{1}{4},\ 0,\ \frac{1}{2},\ \frac{3}{4}\}$
5. $\{-5, -4, -3, -1, 1\}$
6. $\{-40, -30, -20, 10, 30\}$

Answer

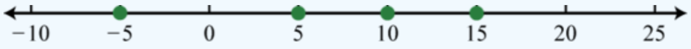
−10
−5
0
5
10
15
20
25

Figure 1.1.19

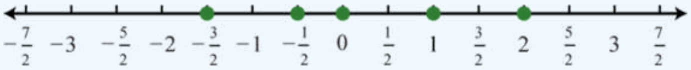
$-\frac{7}{2}$
−3
$-\frac{5}{2}$
−2
$-\frac{3}{2}$
−1
$-\frac{1}{2}$
0
$\frac{1}{2}$
1
$\frac{3}{2}$
2
$\frac{5}{2}$
3
$\frac{7}{2}$

Figure 1.1.20

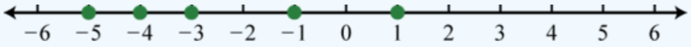
−6
−5
−4
−3
−2
−1
0
1
2
3
4
5
6

Figure 1.1.21

Exercise 1.1.5

Simplify.

1. $-(-10)$
2. $-(-\frac{3}{5})$
3. $-(-(-12))$
4. $-(-(-\frac{5}{3}))$
5. $-(-(-(-\frac{1}{2})))$
6. $-(-(-(-(-\frac{3}{4}))))$

Answer

1. 10
3. -12
5. $\frac{1}{2}$

Exercise 1.1.6

Fill in the blank with $<, =, or >$.

1. -10 ______ -15
2. -101 ______ -100

3. -33 ______ 0
4. 0 ______ -50
5. $-(-(-2))$ ______ $-(-3)$
6. $-(-(-\frac{1}{2}))$ ______ $-\frac{1}{4}$
7. $-(-(-\frac{2}{3}))$ ______ $-(-\frac{1}{2})$
8. $-(-\frac{2}{3})$ ______ $-(-(-(-\frac{2}{3})))$

Answer

1. $>$

3. $<$

5. $<$

7. $<$

Exercise 1.1.7

True or False.

1. $0 = 0$
2. $5 \leq 5$
3. $1.0\overline{32}$ is irrational.
4. 0 is a nonnegative number.
5. Any integer is a rational number.
6. The constant π is rational.

Answer

1. True

3. False

5. True

Exercise 1.1.8

Graph the solution set and give the interval notation equivalent.

1. $x < -1$
2. $x > -3$
3. $x \geq -8$
4. $x \leq 6$
5. $-10 \leq x < 4$
6. $3 < x \leq 7$
7. $-40 < x < 0$
8. $-12 \leq x \leq -4$
9. $x < 5 \text{ and } x \geq 0$
10. $x \leq -10 \text{ and } x \geq -40$
11. $x \leq 7 \text{ and } x < 10$
12. $x < 1 \text{ and } x > 3$
13. $x < -2 \text{ or } x \geq 5$
14. $x \leq 0 \text{ or } x \geq 4$
15. $x < 6 \text{ or } x > 2$
16. $x < 0 \text{ or } x \leq 5$

Answer

1. $(-\infty, -1)$;

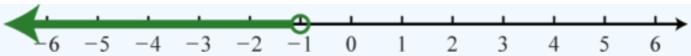
−6
−5
−4
−3
−2
−1
0
1
2
3
4
5
6

Figure 1.1.22

3. $[8, \infty)$;

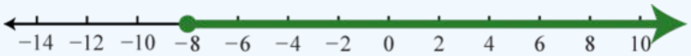
−14
−12
−10
−8
−6
−4
−2
0
2
4
6
8
10

Figure 1.1.23

5. $[-10, 4)$;

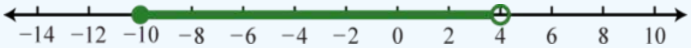
−14
−12
−10
−8
−6
−4
−2
0
2
4
6
8
10

Figure 1.1.24

7. $(-40, 0)$;

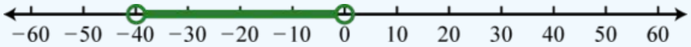
−60
−50
−40
−30
−20
−10
0
10
20
30
40
50
60

Figure 1.1.25

9. $[0, 5)$;

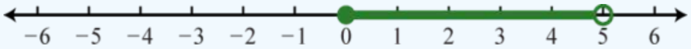
−6
−5
−4
−3
−2
−1
0
1
2
3
4
5
6

Figure 1.1.26

11. $(-\infty, 7)$;

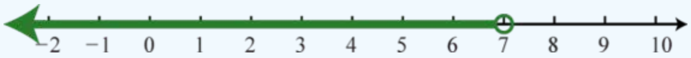
−2
−1
0
1
2
3
4
5
6
7
8
9
10

Figure 1.1.27

13. $(-\infty, -2) \cup [5, \infty)$;

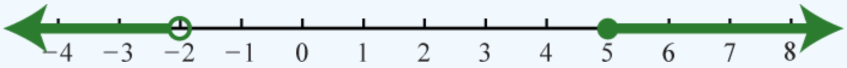
−4
−3
−2
−1
0
1
2
3
4
5
6
7
8

Figure 1.1.28

15. $(-\infty, \infty) = \mathbb{R}$;

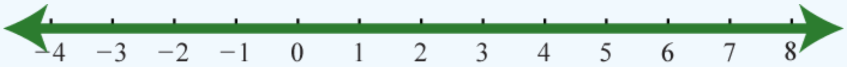
−4
−3
−2
−1
0
1
2
3
4
5
6
7
8

Figure 1.1.29

Exercise 1.1.9

Write an equivalent inequality.

1. All real numbers less than -15.
2. All real numbers greater than or equal to -7.
3. All real numbers less than 6 and greater than zero.
4. All real numbers less than zero and greater than -5.
5. All real numbers less than or equal to 5 or greater than 10.
6. All real numbers between -2 and 2.

Answer

1. $x < -15$

3. $0 < x < 6$

5. $x \leq 5 \text{ or } x > 10$

Exercise 1.1.10

Determine the inequality given the answers expressed in interval notation.

1. $(-\infty, 12)$
2. $[-8, \infty)$
3. $(-\infty, 0]$
4. $(0, \infty)$
5. $(-6, 14)$
6. $(0, 12]$
7. $[5, 25)$
8. $[-30, -10]$
9. $(-\infty, 2) \cup [3, \infty)$
10. $(-\infty, -19] \cup [-12, \infty)$
11. $(-\infty, -2) \cup (0, \infty)$
12. $(-\infty, -15] \cup (-5, \infty)$

Answer

1. $x < 12$

3. $x \leq 0$

5. $-6 < x < 14$

7. $5 \leq x < 25$

9. $x < 2 \text{ or } x \geq 3$

11. $x < -2 \text{ or } x > 0$

Exercise 1.1.11

Simplify.

1. $|-9|$
2. $|14|$
3. $-|-4|$
4. $-|8|$
5. $-|-\frac{5}{8}|$
6. $-(-|\frac{7}{2}|)$
7. $-|-(-7)|$
8. $-|-(-10)|$
9. $-(-|-2|)$
10. $-(-|-10|)$
11. $-(-|-(-5)|)$
12. $-(-(-|-20|))$

Answer

1. 9

3. -4

5. $-\frac{5}{8}$

7. -7

9. 2

11. 5

Exercise 1.1.12

Determine the values represented by *a*.

1. $|a| = 10$
2. $|a| = 7$
3. $|a| = \frac{1}{2}$
4. $|a| = \frac{9}{4}$
5. $|a| = 0$
6. $|a| = -1$

Answer

1. $a = \pm 10$

3. $a = \pm\frac{1}{2}$

5. $a = 0$

Exercise 1.1.13

1. Research and discuss the origins and evolution of algebra.
2. Research and discuss reasons why algebra is a required subject today.
3. Solution sets to inequalities can be expressed using a graph, interval notation, or set notation. Discuss the merits and drawbacks of each method. Which do you prefer?
4. Research and discuss the Fundamental Theorem of Algebra. Illustrate its idea with an example and share your results.

Answer

1. Answer may vary

3. Answer may vary

Footnotes

[1] Letters used to represent numbers.

[2] Any collection of objects.

[3] An object within a set.

[4] A set consisting of elements that belong to a given set.

[5] The set of counting numbers: $\{1, 2, 3, 4, 5, \ldots\}$.

[6] The set of natural numbers combined with zero: $\{0, 1, 2, 3, 4, 5, \ldots\}$.

[7] A subset with no elements, denoted $\varnothing$ or $\{\}$.

[8] Numbers of the form $\frac{a}{b}$, where a and b are integers and b is nonzero.

[9] Notation used to describe a set using mathematical symbols.

[10] Numbers that cannot be written as a ratio of two integers.

[11] The set of all rational and irrational numbers.

[12] Integers that are divisible by 2.

[13] Nonzero integers that are not divisible by 2.

[14] Integer greater than 1 that is divisible only by 1 and itself.

[15] Integers greater than 1 that are not prime.

[16] Any combination of factors, multiplied together, resulting in the product.

[17] Any of the numbers that form a product.

[18] The unique factorization of a natural number written as a product of primes.

[19] A rational number written as a quotient of two integers: ab, where $b \neq 0$.

[20] The number above the fraction bar.

[21] The number below the fraction bar. \

[22] Two equal fractions expressed using different numerators and denominators.

[23] A factor that is shared by more than one real number.

[24] The process of dividing out common factors in the numerator and the denominator.

[25] The process of finding equivalent fractions by dividing the numerator and the denominator by common factors.

[26] Numbers that have no common factor other than 1.

[27] Finding equivalent fractions where the numerator and the denominator share no common integer factor other than 1.

[28] The largest shared factor of any number of integers.

[29] A number to be divided by another number.

[30] The number that is divided into the dividend.

[31] The result of division.

[32] A quotient such as $\frac{5}{0}$ is left without meaning and is not assigned an interpretation.

[33] A quotient such as $\frac{0}{0}$ is a quantity that is uncertain or ambiguous.

[34] A line that allows us to visually represent real numbers by associating them with points on the line.

[35] The real number associated with a point on a number line.

[36] A point on the number line associated with a coordinate.

[37] The point on the number line that represents zero.

[38] Real numbers whose graphs are on opposite sides of the origin with the same distance to the origin.

[39] The opposite of a negative number is positive: $-(-a) = a$.

[40] The set of positive and negative whole numbers combined with zero: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

[41] Express ordering relationships using the symbol $<$ for "less than" and $>$ for "greater than."

[42] Use the symbol $\leq$ to express quantities that are "less than or equal to" and $\geq$ for quantities that are "greater than or equal to" each other.

[43] Algebraic expressions related with the symbols $\leq, <, \geq$ and $>$.

[44] Values that can be used in place of the variable to satisfy the given condition.

[45] Solutions to an algebraic expression expressed on a number line.

[46] The symbol ∞ indicates the interval is unbounded to the right.

[47] The symbol $-\infty$ indicates the interval is unbounded to the left.

[48] Two or more inequalities in one statement joined by the word "and" or by the word "or."

[49] The set formed by joining the individual solution sets indicated by the logical use of the word "or" and denoted with the symbol $\cup$.

[50] The set formed by the shared values of the individual solution sets that is indicated by the logical use of the word "and," denoted with the symbol $\cap$.

[51] The absolute value of a number represents the distance from the graph of the number to zero on a number line.

[52] A definition that changes depending on the value of the variable.

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1.2: Operations with Real Numbers

Learning Objectives

- Review the properties of real numbers.
- Simplify expressions involving grouping symbols and exponents.
- Simplify using the correct order of operations.

Working with Real Numbers

In this section, we continue to review the properties of real numbers and their operations. The result of adding real numbers is called the **sum**[53] and the result of subtracting is called the **difference**[54]. Given any real numbers *a*, *b*, and *c*, we have the following properties of addition:

Table 1.2.1

Additive Identity Property:[55]	a+0=0+a=a
Additive Inverse Property:[56]	a+(−a)=(−a)+a=0
Associative Property:[57]	(a+b)+c=a+(b+c)
Commutative Property:[58]	a+b=b+a

It is important to note that addition is commutative and subtraction is not. In other words, the order in which we add does not matter and will yield the same result. However, this is not true of subtraction.

$$5+10=10+5 \quad 5-10\neq 10-5$$

$$15=15 \quad -5\neq 5$$

We use these properties, along with the double-negative property for real numbers, to perform more involved sequential operations. To simplify things, make it a general rule to first replace all sequential operations with either addition or subtraction and then perform each operation in order from left to right.

Example 1.2.1:

Simplify: $-10-(-10)+(-5)$.

Solution

Replace the sequential operations and then perform them from left to right.

$$-10-(-10)+(-5)=-10+10-5 \quad \textit{Replace } -(-) \textit{ with addition } (+).$$

$$\textit{Replace } +(-) \textit{ with subtraction} (-).$$

$$=0-5$$

$$=-5$$

Answer

-5

Adding or subtracting fractions requires a **common denominator**[59]. Assume the common denominator *c* is a nonzero integer and we have

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c} \text{ and } \frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$$

Example 1.2.2:

Simplify: $\frac{2}{9}-\frac{1}{15}+\frac{8}{45}$.

Solution

First determine the least common multiple (LCM) of $9, 15, and 45$. The least common multiple of all the denominators is called the **least common denominator**[60] (LCD). We begin by listing the multiples of each given denominator:

$$\{9, 18, 27, 36, 45, 54, 63, 72, 81, 90, \ldots\}\ \mathit{Multiples\ of}\ 9$$

$$\{15, 30, 45, 60, 75, 90, \ldots\}\ \mathit{Multiples\ of}\ 15$$

$$\{45, 90, 135 \ldots\}\ \mathit{Multiples\ of}\ 45$$

Here we see that the LCM$(9, 15, 45) = 45$. Multiply the numerator and the denominator of each fraction by values that result in equivalent fractions with the determined common denominator.

$$\begin{aligned}\frac{2}{9} - \frac{1}{15} + \frac{8}{45} &= \frac{2}{9} \cdot \frac{5}{5} - \frac{1}{15} \cdot \frac{3}{3} + \frac{8}{45} \\ &= \frac{10}{45} - \frac{3}{45} + \frac{8}{45}\end{aligned}$$

Once we have equivalent fractions, with a common denominator, we can perform the operations on the numerators and write the result over the common denominator.

$$\begin{aligned}&= \frac{10-3+8}{45} \\ &= \frac{15}{45}\end{aligned}$$

And then reduce if necessary,

$$\begin{aligned}&= \frac{15 \div 15}{45 \div 15} \\ &= \frac{1}{3}\end{aligned}$$

Answer

$\frac{1}{3}$

Finding the LCM using lists of multiples, as described in the previous example, is often very cumbersome. For example, try making a list of multiples for 12 and 81. We can streamline the process of finding the LCM by using prime factors.

$$12 = 2^2 \cdot 3$$

$$81 = 3^4$$

The least common multiple is the product of each prime factor raised to the highest power. In this case,

$$LCM(12, 81) = 2^2 \cdot 3^4 = 324$$

Often we will find the need to translate English sentences involving addition and subtraction to mathematical statements. Below are some common translations.

$$n + 2\ \mathit{The\ sum\ of\ a\ number\ and\ 2.}$$

$$2 - n\ \mathit{The\ difference\ of\ 2\ and\ a\ number.}$$

$$n - 2\ \mathit{Here\ 2\ is\ subtracted\ from\ a\ number.}$$

Example $1.2.3$:

What is 8 subtracted from the sum of 3 and $\frac{1}{2}$?

Solution

We know that subtraction is not commutative; therefore, we must take care to subtract in the correct order. First, add 3 and $\frac{1}{2}$ and then subtract 8 as follows:

$$\overbrace{\left(3 + \frac{1}{2}\right)}^{\text{"the sum of } 3 \text{ and } \frac{1}{2}\text{"}} \overbrace{-\quad 8}^{\text{"subtract 8 from the sum"}}$$

Figure $1.2.1$

Perform the indicated operations.

$$\begin{aligned}(3+\tfrac{1}{2})-8 &= (\tfrac{3}{1}\cdot\tfrac{2}{2}+\tfrac{1}{2})-8\\ &= (\tfrac{6+1}{2})-8\\ &= \tfrac{7}{2}-\tfrac{8}{1}\cdot\tfrac{2}{2}\\ &= \tfrac{7-16}{2}\\ &= -\tfrac{9}{2}\end{aligned}$$

Answer

$-\frac{9}{2}$

The result of multiplying real numbers is called the **product**[61] and the result of dividing is called the **quotient**[62]. Given any real numbers *a*, *b*, and *c*, we have the following properties of multiplication:

Table 1.2.2

Zero Factor Property:[63]	a⋅0=0⋅a=0
Multiplicative Identity Property:[64]	a⋅1=1⋅a=a
Associative Property:[65]	(a⋅b)⋅c=a⋅(b⋅c)
Commutative Property:[66]	a⋅b=b⋅a

It is important to note that multiplication is commutative and division is not. In other words, the order in which we multiply does not matter and will yield the same result. However, this is not true of division.

$$5\cdot 10=10\cdot 5 \quad 5\div 10\neq 10\div 5$$

$$50=50 \quad 0.5\neq 2$$

We will use these properties to perform sequential operations involving multiplication and division. Recall that the product of a positive number and a negative number is negative. Also, the product of two negative numbers is positive.

Example 1.2.4:

Multiply: 5(−3)(−2)(−4).

Solution

Multiply two numbers at a time as follows:

$$\begin{aligned}5(-3)(-2)(-4) &= \underbrace{5(-3)}(-2)(-4)\\ &= \underbrace{-15(-2)}(-4)\\ &= \underbrace{30(-4)}\\ &= -120\end{aligned}$$

Figure 1.2.2

Answer

−120

Because multiplication is commutative, the order in which we multiply does not affect the final answer. However, when sequential operations involve multiplication and division, order does matter; hence we must work the operations from *left to right* to obtain a correct result.

Example 1.2.5:

Simplify: $10÷(−2)(−5)$.

Solution

Perform the division first; otherwise the result will be incorrect.

Correct!

$$10\div(-2)(-5) = \underbrace{10\div(-2)}_{\textit{division first}}(-5)$$

$$= -5(-5)$$

$$= 25$$ ✓

Incorrect!

$$10\div(-2)(-5) = 10\div\underbrace{(-2)(-5)}_{\textit{multiplication first}}$$

$$= 10\div 10$$

$$= 1$$ ✗

Figure 1.2.3

Notice that the order in which we multiply and divide does affect the result. Therefore, it is important to perform the operations of multiplication and division as they appear from left to right.

Answer

25

The product of two fractions is the fraction formed by the product of the numerators and the product of the denominators. In other words, to multiply fractions, multiply the numerators and multiply the denominators:

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$$

Example 1.2.6:

Multiply $-\frac{4}{5}\cdot\frac{25}{12}$.

Solution

Multiply the numerators and multiply the denominators. Reduce by dividing out any common factors.

$$-\frac{4}{5}\cdot\frac{25}{12} = -\frac{4\cdot 25}{5\cdot 12}$$

$$= -\frac{\overset{1}{\cancel{4}}\cdot\overset{5}{\cancel{25}}}{\underset{1}{\cancel{5}}\cdot\underset{3}{\cancel{12}}}$$

$$= -\frac{5}{3}$$

Figure 1.2.4

Answer:

$-\frac{5}{3}$

Two real numbers whose product is 1 are called **reciprocals**[67]. Therefore, $\frac{a}{b}$ and $\frac{b}{a}$ are reciprocals because $\frac{a}{b}\cdot\frac{b}{a}=\frac{ab}{ab}=1$. For example,

$$\frac{2}{3}\cdot\frac{3}{2}=\frac{6}{6}=1$$

Because their product is 1, $\frac{2}{3}$ and $\frac{3}{2}$ are reciprocals. Some other reciprocals are listed below:

$$\frac{5}{8} \text{ and } \frac{8}{5} \qquad 7 \text{ and } \frac{1}{7} \qquad -\frac{4}{5} \text{ and } -\frac{5}{4}$$

This definition is important because dividing fractions requires that you multiply the dividend by the reciprocal of the divisor.

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} \cdot \frac{\frac{d}{c}}{\frac{d}{c}} = \frac{\frac{a}{b} \cdot \frac{d}{c}}{1} = \frac{a}{b} \cdot \frac{d}{c}$$

In general,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

Example 1.2.7:

Simplify: $\frac{5}{4} \div \frac{3}{5} \cdot \frac{1}{2}$.

Solution

Perform the multiplication and division from left to right.

$$\begin{aligned} \frac{5}{4} \div \frac{3}{5} \cdot \frac{1}{2} &= \frac{5}{4} \cdot \frac{5}{3} \cdot \frac{1}{2} \\ &= \frac{5 \cdot 5 \cdot 1}{4 \cdot 3 \cdot 2} \\ &= \frac{25}{24} \end{aligned}$$

In algebra, it is often preferable to work with improper fractions. In this case, we leave the answer expressed as an improper fraction.

Answer

$\frac{25}{24}$

Exercise 1.2.1

Simplify: $\frac{1}{2} \cdot \frac{3}{4} \div \frac{1}{8}$.

Answer

3 www.youtube.com/v/4zV-fyepZkk

Grouping Symbols and Exponents

In a computation where more than one operation is involved, grouping symbols help tell us which operations to perform first. The **grouping symbols**[68] commonly used in algebra are:

$$\begin{aligned} (\ ) &\quad \textit{Parentheses} \\ [\ ] &\quad \textit{Brackets} \\ \{\ \} &\quad \textit{Braces} \\ - &\quad \textit{Fraction bar} \end{aligned}$$

All of the above grouping symbols, as well as absolute value, have the same order of precedence. Perform operations inside the innermost grouping symbol or absolute value first.

Example 1.2.8:

Simplify: $2 - (\frac{4}{5} - \frac{2}{15})$.

Solution

Perform the operations within the parentheses first.

$$\begin{aligned} 2 - \left(\frac{4}{5} - \frac{2}{15}\right) &= 2 - \left(\frac{4}{5} \cdot \frac{3}{3} - \frac{2}{15}\right) \\ &= 2 - \left(\frac{12}{15} - \frac{2}{15}\right) \\ &= 2 - \left(\frac{10}{15}\right) \end{aligned}$$

$$\begin{aligned} &= \frac{2}{1}\cdot\frac{3}{3}-\frac{2}{3} \\ &= \frac{6-2}{3} \\ &= \frac{4}{3} \end{aligned}$$

Answer:

$\frac{4}{3}$

Example 1.2.9:

Simplify: $\frac{5-|4-(-3)|}{|-3|-(5-7)}$.

Solution

The fraction bar groups the numerator and denominator. Hence, they should be simplified separately.

$$\begin{aligned} \frac{5-|4-(-3)|}{|-3|-(5-7)} &= \frac{5-|4+3|}{|-3|-(-2)} \\ &= \frac{5-|7|}{|-3|+2} \\ &= \frac{5-7}{3+2} \\ &= \frac{-2}{5} \\ &= -\frac{2}{5} \end{aligned}$$

Answer:

$-\frac{2}{5}$

If a number is repeated as a factor numerous times, then we can write the product in a more compact form using **exponential notation**[69]. For example,

$$5\cdot 5\cdot 5\cdot 5 = 54$$

The **base**[70] is the factor and the positive integer **exponent**[71] indicates the number of times the base is repeated as a factor. In the above example, the base is 5 and the exponent is 4. Exponents are sometimes indicated with the caret (^) symbol found on the keyboard, $5^4 = 5*5*5*5$. In general, if *a* is the base that is repeated as a factor *n* times, then

$$a^n = \underbrace{a\cdot a\cdot a\cdot\ldots\cdot a}_{n\text{ factors of } a}$$

Figure 1.2.5

When the exponent is 2 we call the result a **square**[72], and when the exponent is 3 we call the result a **cube**[73]. For example,

$$5^2 = 5\cdot 5 = 25 \text{ "5 squared"}$$

$$5^3 = 5\cdot 5\cdot 5 = 125 \text{ "5 cubed"}$$

If the exponent is greater than 3, then the notation a^n is read, "*a raised to the nth power*." The base can be any real number,

$$(2.5)^2 = (2.5)(2.5) = 6.25$$

$$(-\tfrac{2}{3})^3 = (-\tfrac{2}{3})(-\tfrac{2}{3})(-\tfrac{2}{3}) = -\tfrac{8}{27}$$

$$(-2)^4 = (-2)(-2)(-2)(-2) = 16$$

$$-2^4 = -1\cdot 2\cdot 2\cdot 2\cdot 2 = -16$$

Notice that the result of a negative base with an even exponent is positive. The result of a negative base with an odd exponent is negative. These facts are often confused when negative numbers are involved. Study the following four examples carefully:

Table $1.2.3$

The base is (-3)**.**	***The base is*** 3**.**

The base is (-3)**.**	***The base is*** 3**.**
$(-3)^4=(-3)(-3)(-3)(-3)=+81$ $(-3)^3=(-3)(-3)(-3)=-27$	$-3^4=-1\cdot 3\cdot 3\cdot 3\cdot 3=-81$ $-3^3=-1\cdot 3\cdot 3\cdot 3=-27$

The parentheses indicate that the negative number is to be used as the base.

Example 1.2.10:

Calculate:

1. $(-\frac{1}{3})^3$
2. $(-\frac{1}{3})^4$

Solution

Here $-\frac{1}{3}$ is the base for both problems.

1.Use the base as a factor three times.

$$\begin{aligned}(-\tfrac{1}{3})^3 &= (-\tfrac{1}{3})(-\tfrac{1}{3})(-\tfrac{1}{3})\\ &= -\tfrac{1}{27}\end{aligned}$$

2.Use the base as a factor four times.

\((−\frac{1}{3})^{4}=(−\frac{1}{3})(−\frac{1}{3})(−\frac{1}{3})(−\frac{1}{3})

$$= +\frac{1}{81}$$

Answers:

1. $-\frac{12}{7}$
2. $\frac{1}{81}$

Exercise 1.2.2

Simplify:

1. -2^4
2. $(-2)^4$

Answer

1. −16
2. 16

www.youtube.com/v/o3X52psRJtg

Order of Operations

When several operations are to be applied within a calculation, we must follow a specific order to ensure a single correct result.

1. Perform all calculations within the innermost **parentheses** or grouping symbol first.
2. Evaluate all **exponents**.
3. Apply **multiplication and division** from left to right.
4. Perform all remaining **addition and subtraction** operations last from left to right.

Note that multiplication and division *should* be worked from *left to right*. Because of this, it is often reasonable to perform division before multiplication.

Example 1.2.11:

Simplify: $5^3-24\div 6\cdot\frac{1}{2}+2$.

Solution

First, evaluate 5^3 and then perform multiplication and division as they appear from left to right.

$$\begin{aligned}5^3-24\div 6\cdot\frac{1}{2}+2&=5^3-24\div 6\cdot\frac{1}{2}+2\\&=125-24\div 6\cdot\frac{1}{2}+2\\&=125-4\cdot\frac{1}{2}+2\\&=125-2+2\\&=123+2\\&=125\end{aligned}\tag{1.2.1}$$

Multiplying first would have led to an incorrect result.

$$\begin{aligned}5^3-24\div 6\cdot\frac{1}{2}+2&=125-24\div\underbrace{6\cdot\frac{1}{2}}_{\textit{Incorrect}}+2\\&=125-\underbrace{24\div 3}+2\\&=\underbrace{125-8}+2\\&=\underbrace{117+2}\\&=119\quad\times\end{aligned}$$

Figure 1.2.6

Answer:

125

Example 1.2.12:

Simplify: $-10-5^2+(-3)^4$.

Solution

Take care to correctly identify the base when squaring.

$$\begin{aligned}-10-5^2+(-3)^4&=-10-25+81\\&=-35+81\\&=46\end{aligned}$$

Answer:

46

We are less likely to make a mistake if we work one operation at a time. Some problems may involve an absolute value, in which case we assign it the same order of precedence as parentheses.

Example 1.2.13:

Simplify: $7-5\left|-2^2+(-3)^2\right.$.

Solution

Begin by performing the operations within the absolute value first.

$$\begin{aligned} 7-5\left|-2^{2}+(-3)^{2}\right| &= 7-5|-4+9| \\ &= 7-5|5| \\ &= 7-5\cdot 5 \\ &= 7-25 \\ &= -18 \end{aligned}$$

Subtracting $7-5$ first will lead to incorrect results.

$$\begin{aligned} 7-5\left|-2^{2}+(-3)^{2}\right| &= \underbrace{7\ -\ 5}_{\text{Incorrect}}\left|-2^{2}+(-3)^{2}\right| \\ &= 2|-4+9| \\ &= 2|5| \\ &= 10 \quad ✗ \end{aligned}$$

Figure $1.2.7$

Answer:

-18

Exercise $1.2.3$

Simplify: $-6^{2}-\left[-15-(-2)^{3}\right]-(-2)^{4}$.

Answer

-45

www.youtube.com/v/DnaviQzLPA0

Key Takeaways

- Addition is commutative and subtraction is not. Furthermore, multiplication is commutative and division is not.
- Adding or subtracting fractions requires a common denominator; multiplying or dividing fractions does not.
- Grouping symbols indicate which operations to perform first. We usually group mathematical operations with parentheses, brackets, braces, and the fraction bar. We also group operations within absolute values. All groupings have the same order of precedence: the operations within the innermost grouping are performed first.
- When using exponential notation a^{n}, the base *a* is used as a factor *n* times. Parentheses indicate that a negative number is to be used as the base. For example, $(-5)^{2}$ is positive and -5^{2} is negative.
- To ensure a single correct result when applying operations within a calculation, follow the order of operations. First, perform operations in the innermost parentheses or groupings. Next, simplify all exponents. Perform multiplication and division operations from left to right. Finally, perform addition and subtraction operations from left to right.

Exercise $1.2.4$

Perform the operations. Reduce all fractions to lowest terms.

1. $33-(-15)+(-8)$
2. $-10-9+(-6)$
3. $-23+(-7)-(-10)$
4. $-1-(-1)-1$
5. $\frac{1}{2}+\frac{1}{3}-\frac{1}{6}$
6. $-\frac{1}{5}+\frac{1}{2}-\frac{1}{10}$
7. $\frac{2}{3}-\left(-\frac{1}{4}\right)-\frac{1}{6}$
8. $-\frac{3}{2}-\left(-\frac{2}{9}\right)-\frac{5}{6}$
9. $\frac{3}{4}-\left(-\frac{1}{2}\right)-\frac{5}{8}$
10. $-\frac{1}{5}-\frac{3}{2}-\left(-\frac{7}{10}\right)$
11. Subtract 3 from 10.
12. Subtract -2 from 16.

13. Subtract $-\frac{5}{6}$ from 4.
14. Subtract $-\frac{1}{2}$ from $\frac{3}{2}$.
15. Calculate the sum of -10 and 25.
16. Calculate the sum of -30 and -20.
17. Find the difference of 10 and 5.
18. Find the difference of -17 and -3.

Answer

1. 40
3. -20
5. $\frac{2}{3}$
7. $\frac{3}{4}$
9. $\frac{5}{8}$
11. 7
13. $\frac{29}{6}$
15. 15
17. 5

Exercise 1.2.5

The formula $d = |b - a|$ gives the distance between any two points on a number line. Determine the distance between the given numbers on a number line.

1. 10 and 15
2. 6 and 22
3. 0 and 12
4. -8 and 0
5. -5 and -25
6. -12 and -3

Answer

1. 5 units
3. 12 units
5. 20 units

Exercise 1.2.6

Determine the reciprocal of the following.

1. $\frac{1}{3}$
2. $\frac{2}{5}$
3. $-\frac{3}{4}$
4. -12
5. a where $a \neq 0$
6. $\frac{1}{a}$
7. $\frac{a}{b}$ where $a \neq 0$
8. $\frac{1}{ab}$

Answer

1. 3

3. $-\frac{4}{3}$

5. $\frac{1}{a}$

7. $\frac{b}{a}$

Exercise 1.2.7

Perform the operations.

1. $-4(-5) \div 2$
2. $(-15)(-3) \div (-9)$
3. $-22 \div (-11)(-2)$
4. $50 \div (-25)(-4)$
5. $\frac{2}{3}(-\frac{9}{10})$
6. $-\frac{5}{8}(-\frac{16}{25})$
7. $\frac{7}{6}(-\frac{6}{7})$
8. $-\frac{15}{9}(\frac{9}{5})$
9. $\frac{4}{5}(-\frac{2}{5}) \div \frac{16}{25}$
10. $(-\frac{9}{2})(-\frac{3}{2}) \div \frac{27}{16}$
11. $\frac{8}{5} \div \frac{5}{2} \cdot \frac{15}{40}$
12. $\frac{3}{16} \div \frac{5}{8} \cdot \frac{1}{2}$
13. Find the product of 12 and 7.
14. Find the product of $-\frac{2}{3}$ and 12.
15. Find the quotient of -36 and 12.
16. Find the quotient of $-\frac{3}{4}$ and 9.
17. Subtract 10 from the sum of 8 and -5.
18. Subtract -2 from the sum of -5 and -3.
19. Joe earns $\$18.00$ per hour and "time and a half" for every hour he works over 40 hours. What is his pay for 45 hours of work this week?
20. Billy purchased 12 bottles of water at $\$0.75$ per bottle, 5 pounds of assorted candy at $\$4.50$ per pound, and 15 packages of microwave popcorn costing $\$0.50$ each for his party. What was his total bill?
21. James and Mary carpooled home from college for the Thanksgiving holiday. They shared the driving, but Mary drove twice as far as James. If Mary drove for 210 miles, then how many miles was the entire trip?
22. A $6\frac{3}{4}$ foot plank is to be cut into 3 pieces of equal length. What will be the length of each piece?
23. A student earned $72, 78, 84,$ and 90 points on her first four algebra exams. What was her average test score? (Recall that the average is calculated by adding all the values in a set and dividing that result by the number of elements in the set.)
24. The coldest temperature on Earth, -129°F, was recorded in 1983 at Vostok Station, Antarctica. The hottest temperature on Earth, 136°F, was recorded in 1922 at Al' Aziziyah, Libya. Calculate the temperature range on Earth.

Answer

1. 10

3. -4

5. $-\frac{3}{5}$

7. -1

9. $-\frac{1}{2}$

11. $\frac{6}{25}$

13. 84

15. -3

17. -7

19. $\$855$

21. 315 miles

23. 81 points

Exercise $1.2.8$

Perform the operations.

1. $7-\{3-[-6-(10)]\}$
2. $-(9-12)-[6-(-8-3)]$
3. $\frac{1}{2}\{5-(10-3)\}$
4. $\frac{2}{3}\{-6+(6-9)\}$
5. $5\{2[3(4-\frac{3}{2})]\}$
6. $\frac{1}{2}\{-6[-(\frac{1}{2}-\frac{5}{3})]\}$
7. $\frac{5-|5-(-6)|}{|-5|-|-3|}$
8. $\frac{|9-12|-(-3)}{|-16|-3(4)}$
9. $\frac{-|-5-(-7)|-(-2)}{|-2|+|-3|}$
10. $\frac{1-|9-(3-4)|}{-|-2|+(-8-(-10))}$

Answer

1. -12

3. -1

5. 75

7. -3

9. 0

Exercise $1.2.9$

Perform the operations.

1. 12^2
2. $(-12)^2$
3. -12^2
4. $-(-12)^2$
5. -5^4
6. $(-5)^4$
7. $(-\frac{1}{2})^3$
8. $-(-\frac{1}{2})^3$
9. $-(-\frac{3}{4})^2$
10. $-(-\frac{5}{2})^3$
11. $(-1)^{22}$
12. $(-1)^{13}$
13. $-(-1)^{12}$
14. $-(-1)^5$
15. -10^2
16. -10^4

Answer

1. 144

3. -144

5. -625

7. $-\frac{1}{8}$

9. $-\frac{9}{16}$

11. 1

13. -1

15. -100

Exercise 1.2.10

Simplify.

1. $5-3(4-3^2)$
2. $8-5(3-3^2)$
3. $(-5)^2+3(2-4^2)$
4. $6-2(-5^2+4\cdot 7)$
5. $5-3[3(2-3^2)+(-3)^2]$
6. $10-5[(2-5)^2-3]$
7. $[5^2-3^2]-[2-(5+(-4)^2)]$
8. $-7^2-[(2-7)^2-(-8)^2]$
9. $\frac{3}{16}\div(\frac{5}{12}-\frac{1}{2}+\frac{2}{3})\cdot 4$
10. $6\cdot\left[\left(\frac{2}{3}\right)^2-\left(\frac{1}{2}\right)^2\right]\div(-2)^2$
11. $\frac{3-2\cdot 5+4}{2^2-3^2}$
12. $\frac{(3+(-2)^2)\cdot 4-3}{-4^2+1}$
13. $\frac{-5^2+(-3)^2\cdot 2-3}{8^2+6(-10)}$
14. $\frac{(-4)^2+(-3)^3}{-9^2-(-12+2^2)*10}$
15. $-5^2-2|-5|$
16. $-2^4+6|2^4-5^2|$
17. $-(4-|7^2-8^2|)$
18. $-3(5-2|-6|)$
19. $(-3)^2-|-2+(-3)^3|-4^2$
20. $-5^2-2|3^3-2^4|-(-2)^5$
21. $5\cdot|-5|-(2-|-7|)^3$
22. $10^2+2(|-5|^3-6^3)$
23. $\frac{2}{3}-|\frac{1}{2}-(-\frac{4}{3})^2|$
24. $-24|\frac{10}{3}-\frac{1}{2}\div\frac{1}{5}|$
25. Calculate the sum of the squares of the first three consecutive positive odd integers.
26. Calculate the sum of the squares of the first three consecutive positive even integers.
27. What is 6 subtracted from the sum of the squares of 5 and 8?
28. What is 5 subtracted from the sum of the cubes of 2 and 3?

Answer

1. 20

3. -17

5. 41

7. 35

9. $\frac{9}{7}$

11. $\frac{3}{5}$

13. $-\frac{5}{2}$

15. -35

17. 11

19. -36

21. 150

23. $-\frac{11}{18}$

25. 35

27. 83

Exercise 1.2.11

1. What is PEMDAS and what is it missing?
2. Does 0 have a reciprocal? Explain.
3. Explain why we need a common denominator in order to add or subtract fractions.
4. Explain why $(-10)^4$ is positive and -10^4 is negative.

Answer

1. Answer may vary

3. Answer may vary

Footnotes

[53] The result of adding.

[54] The result of subtracting.

[55] Given any real number $a, a+0=0+a=a$.

[56] Given any real number $a, a+(-a)=(-a)+a=0$.

[57] Given real numbers a, b and c, $(a+b)+c=a+(b+c)$.

[58] Given real numbers a and b, $a+b=b+a$.

[59] A denominator that is shared by more than one fraction.

[60] The least common multiple of a set of denominators.

[61] The result of multiplying.

[62] The result of dividing.

[63] Given any real number $a, a \cdot 0=0 \cdot a=0$.

[64] Given any real number $a, a \cdot 1=1 \cdot a=a$.

[65] Given any real numbers a, b and c, $(a \cdot b) \cdot c=a \cdot(b \cdot c)$.

[66] Given any real numbers a and $b, a \cdot b=b \cdot a$.

[67] Two real numbers whose product is 1.

[68] Parentheses, brackets, braces, and the fraction bar are the common symbols used to group expressions and mathematical operations within a computation.

[69] The compact notation a^n used when a factor a is repeated n times.

[70] The factor a in the exponential notation a^n.

[71] The positive integer n in the exponential notation a^n that indicates the number of times the base is used as a factor.

[72] The result when the exponent of any real number is 2.

[73] The result when the exponent of any real number is 3.

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1.3: Square and Cube Roots of Real Numbers

Learning Objectives

- Calculate the exact and approximate value of the square root of a real number.
- Calculate the exact and approximate value of the cube root of a real number.
- Simplify the square and cube root of a real number.
- Apply the Pythagorean theorem.

The Definition of Square and Cube Roots

A *square root*[74] of a number is a number that when multiplied by itself yields the original number. For example, 4 is a square root of 16, because $4^2 = 16$. Since $(-4)^2 = 16$, we can say that -4 is a square root of 16 as well. Every positive real number has two square roots, one positive and one negative. For this reason, we use the *radical sign*[75] $\sqrt{\ }$ to denote the *principal (nonnegative) square root*[76] and a negative sign in front of the radical $-\sqrt{\ }$ to denote the negative square root.

$$\sqrt{16} = 4 \ \textit{Positive Square Root of } 16$$

$$-\sqrt{16} = -4 \textit{Negative Square Root of } 16$$

Zero is the only real number with exactly one square root.

$$\sqrt{0} = 0$$

If the *radicand*[77], the number inside the radical sign, is nonzero and can be factored as the square of another nonzero number, then the square root of the number is apparent. In this case, we have the following property:

$$\sqrt{a^2} = a, \ \text{if } a \geq 0$$

It is important to point out that a is required to be nonnegative. Note that $\sqrt{(-3)^2} \neq -3$ because the radical denotes the principal square root. Instead,

$$\sqrt{(-3)^2} = \sqrt{9} = 3$$

This distinction will be carefully considered later in the course.

Example 1.3.1:

Find the square root:

1. $\sqrt{121}$
2. $\sqrt{0.25}$
3. $\sqrt{\frac{4}{9}}$

Solution

a. $\sqrt{121} = \sqrt{11^2} = 11$
b. $\sqrt{0.25} = \sqrt{0.5^2} = 0.5$
c. $\sqrt{\frac{4}{9}} = \sqrt{\left(\frac{2}{3}\right)^2} = \frac{2}{3}$

Example 1.3.2:

Find the negative square root:

a. $-\sqrt{64}$
b. $-\sqrt{1}$

Solution

a. $-\sqrt{64} = -\sqrt{8^2} = -8$
b. $-\sqrt{1} = -\sqrt{1^2} = -1$

The radicand may not always be a perfect square. If a positive integer is not a perfect square, then its square root will be irrational. Consider $\sqrt{5}$, we can obtain an approximation by bounding it using the perfect squares 4 and 9 as follows:

$$\sqrt{4} < \sqrt{5} < \sqrt{9}$$
$$2 < \sqrt{5} < 3$$

With this we conclude that $\sqrt{5}$ is somewhere between 2 and 3. This number is better approximated on most calculators using the square root button, $\sqrt{}$.

$$\sqrt{5} \approx 2.236 \mathrm{because} 2.236 \wedge 2 \approx 5$$

Next, consider the square root of a negative number. To determine the square root of -9, you must find a number that when squared results in -9,

$$\sqrt{-9} = ? \quad \text{or} \quad (?)^2 = -9$$

However, any real number squared always results in a positive number,

$$(3)^2 = 9 \text{ and } (-3)^2 = 9$$

The square root of a negative number is currently left undefined. Try calculating $\sqrt{-9}$ on your calculator; what does it say? For now, we will state that $\sqrt{-9}$ is not a real number. The square root of a negative number is defined later in the course.

A *cube root*[78] of a number is a number that when multiplied by itself three times yields the original number. Furthermore, we denote a cube root using the symbol $\sqrt[3]{}$, where 3 is called the *index*[79]. For example,

$$\sqrt[3]{8} = 2, \text{ because } 2^3 = 8$$

The product of three equal factors will be positive if the factor is positive, and negative if the factor is negative. For this reason, any real number will have only one real cube root. Hence the technicalities associated with the principal root do not apply. For example,

$$\sqrt[3]{-8} = -2, \text{ because } (-2)^3 = -8$$

In general, given any real number a, we have the following property:

$$\sqrt[3]{a^3} = a$$

When simplifying cube roots, look for factors that are perfect cubes.

Example 1.3.3

Find the cube root:

a. $\sqrt[3]{125}$
b. $\sqrt[3]{0}$
c. $\sqrt[3]{\frac{8}{27}}$

Solution

a. $\sqrt[3]{125} = \sqrt[3]{5^3} = 5$
b. $\sqrt[3]{0} = \sqrt[3]{0^3} = 0$
c. $\sqrt[3]{\frac{8}{27}} = \sqrt[3]{\left(\frac{2}{3}\right)^3} = \frac{2}{3}$

Example 1.3.4

Find the cube root:

a. $\sqrt[3]{-27}$
b. $\sqrt[3]{-1}$

Solution

a. $\sqrt[3]{-27} = \sqrt[3]{(-3)^3} = -3$
b. $\sqrt[3]{-1} = \sqrt[3]{(-1)^3} = -1$

It may be the case that the radicand is not a perfect cube. If this is the case, then its cube root will be irrational. For example, $\sqrt[3]{2}$ is an irrational number, which can be approximated on most calculators using the root button $\sqrt[x]{}$.Depending on the calculator, we typically type in the index prior to pushing the button and then the radicand as follows:

$$3 \quad \sqrt[x]{y} \quad 2 \quad =$$

Therefore, we have

$$\sqrt[3]{2} \approx 1.260, \text{ because } 1.260 \wedge 3 \approx 2$$

We will extend these ideas using any integer as an index later in this course. It is important to point out that a square root has index 2; therefore, the following are equivalent:

$$\sqrt[2]{a} = \sqrt{a}$$

Simplifying Square and Cube Roots

It will not always be the case that the radicand is a perfect square. If not, we use the following two properties to simplify the expression. Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$ where $B \neq 0$,

- **Product Rule for Radicals**:[80]

$$\sqrt[n]{A \cdot B} = \sqrt[n]{A} \cdot \sqrt[n]{B} \tag{1.3.1}$$

- **Quotient Rule for Radicals**:[81]

$$\sqrt[n]{\frac{A}{B}} = \frac{\sqrt[n]{A}}{\sqrt[n]{B}} \tag{1.3.2}$$

A **simplified radical**[82] is one where the radicand does not consist of any factors that can be written as perfect powers of the index. Given a square root, the idea is to identify the largest square factor of the radicand and then apply the property shown above. As an example, to simplify $\sqrt{12}$, notice that 12 is not a perfect square. However, 12 does have a perfect square factor, $12 = 4 \cdot 3$. Apply the property as follows:

$$\begin{aligned} \sqrt{12} &= \sqrt{4 \cdot 3} \quad \textit{Apply the product rule for radicals.} \\ &= \sqrt{4} \cdot \sqrt{3} \quad \textit{Simplify} \\ &= 2 \cdot \sqrt{3} \end{aligned}$$

The number $2\sqrt{3}$ is a simplified irrational number. You are often asked to find an approximate answer rounded off to a certain decimal place. In that case, use a calculator to find the decimal approximation using either the original problem or the simplified equivalent.

$$\sqrt{12} = 2\sqrt{3} \approx 3.46$$

As a check, calculate $\sqrt{12}$ and $2\sqrt{3}$ on a calculator and verify that the results are both approximately 3.46.

Example 1.3.5

Simplify: $\sqrt{135}$.

Solution

Begin by finding the largest perfect square factor of 135.

$$\begin{aligned} 135 &= 3^3 \cdot 5 \\ &= 3^2 \cdot 3 \cdot 5 \\ &= 9 \cdot 15 \end{aligned}$$

Therefore,

$$\begin{aligned} \sqrt{135} &= \sqrt{9 \cdot 15} \quad \textit{Apply the product rule for radicals.} \\ &= \sqrt{9} \cdot \sqrt{15} \quad \textit{Simplify.} \\ &= 3 \cdot \sqrt{15} \end{aligned}$$

Answer

$3\sqrt{15}$

Example 1.3.6

Simplify: $\sqrt{\frac{108}{169}}$.

Solution

We begin by finding the prime factorizations of both 108 and 169. This will enable us to easily determine the largest perfect square factors.

$$\begin{aligned}108 &= 2^2 \cdot 3^3 = 2^2 \cdot 3^2 \cdot 3 \\ 169 &= 13^2\end{aligned}$$

Therefore,

$$\begin{aligned}\sqrt{\frac{108}{169}} &= \sqrt{\frac{2^2 \cdot 3^2 \cdot 3}{13^2}} \textit{Apply the product and quotient rule for radicals.} \\ &= \frac{\sqrt{2^2} \cdot \sqrt{3^2} \cdot \sqrt{3}}{\sqrt{13^2}} \textit{Simplify.} \\ &= \frac{2 \cdot 3 \cdot \sqrt{3}}{13} \\ &= \frac{6\sqrt{3}}{13}\end{aligned}$$

Answer

$\frac{6\sqrt{3}}{13}$

Example 1.3.7

Simplify $-5\sqrt{162}$.

Solution

$$\begin{aligned}-5\sqrt{162} &= -5 \cdot \sqrt{81 \cdot 2} \\ &= -5 \cdot \sqrt{81} \cdot \sqrt{2} \\ &= -5 \cdot 9 \cdot \sqrt{2} \\ &= -45 \cdot \sqrt{2} \\ &= -45\sqrt{2}\end{aligned}$$

Answer

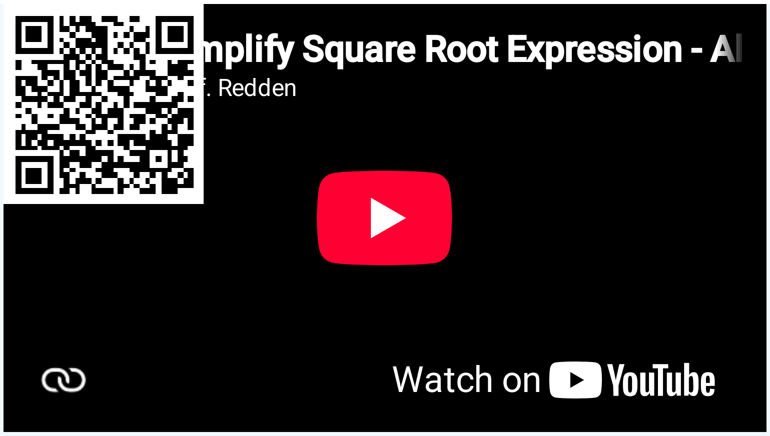
$-45\sqrt{2}$

Exercise 1.3.1

Simplify $4\sqrt{150}$

Answer

$20\sqrt{6}$



A cube root is simplified if it does not contain any factors that can be written as perfect cubes. The idea is to identify the largest cube factor of the radicand and then apply the product or quotient rule for radicals. As an example, to simplify $\sqrt[3]{80}$, notice that 80 is not a perfect cube. However, $80 = 8 \cdot 10$ and we can write,

$$\sqrt[3]{80} = \sqrt[3]{8 \cdot 10} \textit{Apply the product rule for radicals.}$$

$$5pt] = \sqrt[3]{8} \cdot \sqrt[3]{10} \textit{Simplify.}$$

$$= 2 \cdot \sqrt[3]{10}$$

Example $1.3.8$:

Simplify $\sqrt[3]{162}$

Solution

Begin by finding the largest perfect cube factor of 162.

$$\begin{aligned} 162 &= 3^4 \cdot 2 \\ &= 3^3 \cdot 3 \cdot 2 \\ &= 27 \cdot 6 \end{aligned}$$

Therefore,

$$\sqrt[3]{162} = \sqrt[3]{27 \cdot 6} \textit{Apply the product rule for radicals.}$$

$$= \sqrt[3]{27} \cdot \sqrt[3]{6} \textit{Simplify.}$$

$$= 3 \cdot \sqrt[3]{6}$$

Answer

$3\sqrt[3]{6}$

Example $1.3.9$:

Simplify: $\sqrt[3]{-\frac{16}{343}}$.

Solution

$$\begin{aligned} \sqrt[3]{-\frac{16}{343}} &= \frac{\sqrt[3]{-1 \cdot 8 \cdot 2}}{\sqrt[3]{7^3}} \\ &= \frac{\sqrt[3]{-1} \cdot \sqrt[3]{8} \cdot \sqrt[3]{2}}{\sqrt[3]{7^3}} \\ &= \frac{-1 \cdot 2 \cdot \sqrt[3]{2}}{7} \\ &= \frac{-2\sqrt[3]{2}}{7} \end{aligned}$$

Answer

$\frac{-2\sqrt[3]{2}}{7}$

Exercise 1.3.2

Simplify $-2\sqrt[3]{-256}$.

Answer

$8\sqrt[3]{4}$

www.youtube.com/v/oEzfrahfb0A

Consider the following two calculations,

$$\begin{gathered}\sqrt{81}=\sqrt{9^2}=9\\ \sqrt{81}=\sqrt{9^2}=(\sqrt{9})^2=(3)^2=9\end{gathered}$$

Notice that it does not matter if we apply the exponent first or the square root first. This is true for any positive real number. We have the following,

$$\sqrt{a^2}=(\sqrt{a})^2=a,\ \text{ if } a\geq 0$$

Example 1.3.10:

Simplify: $(\sqrt{10})^2$.

Solution

Apply the fact that $(\sqrt{a})^2=a$ if a is nonnegative.

$$(\sqrt{10})^2=10$$

Pythagorean Theorem

A **right triangle**[83] is a triangle where one of the angles measures $90°$. The side opposite the right angle is the longest side, called the **hypotenuse**[84], and the other two sides are called **legs**[85]. Numerous real-world applications involve this geometric figure. The **Pythagorean theorem**[86] states that given any right triangle with legs measuring a and b units, the square of the measure of the hypotenuse c is equal to the sum of the squares of the measures of the legs, $a^2+b^2=c^2$. In other words, the hypotenuse of any right triangle is equal to the square root of the sum of the squares of its legs.

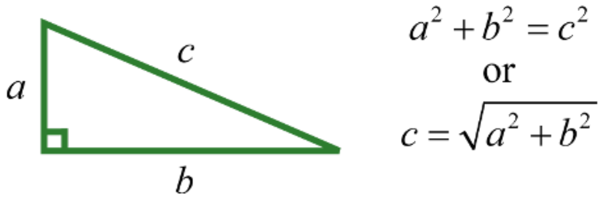


Figure 1.3.1

Example 1.3.11:

Calculate the diagonal of a square with sides measuring 5 units.

Solution

The diagonal of a square will form an isosceles right triangle where the two equal legs measure 5 units each.

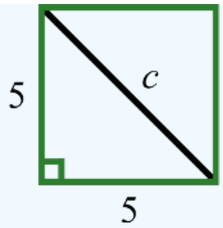


Figure 1.3.2

We can use the Pythagorean theorem to determine the length of the hypotenuse.

$$\begin{aligned} c &= \sqrt{a^2+b^2} \\ &= \sqrt{5^2+5^2} \\ &= \sqrt{25+25} \\ &= \sqrt{50} \\ &= \sqrt{25\cdot 2} \\ &= \sqrt{25}\cdot\sqrt{2} \\ &= 5\cdot\sqrt{2} \end{aligned}$$

Answer: $5\sqrt{2}$ units

The Pythagorean theorem actually states that having side lengths satisfying the property $a^2+b^2=c^2$ is a necessary and sufficient condition of right triangles. In other words, if we can show that the sum of the squares of the lengths of the legs of the triangle is equal to the square of the hypotenuse, then it must be a right triangle.

Example 1.3.12:

Determine whether or not a triangle with legs $a=1$ cm and $b=2$ cm and hypotenuse $b=\sqrt{5}$ cm is a right triangle.

Solution

If the legs satisfy the condition $a^2+b^2=c^2$ then the Pythagorean theorem guarantees that the triangle is a right triangle.

$$\begin{aligned} a^2+b^2 &= c^2 \\ (1)^2+(2)^2 &= (\sqrt{5})^2 \\ 1+4 &= 5 \\ 5 &= 5\checkmark \end{aligned}$$

Answer: Yes, the described triangle is a right triangle.

Key Takeaways

- The square root of a number is a number that when squared results in the original number. The principal square root of a positive real number is the positive square root. The square root of a negative number is currently left undefined.
- When simplifying the square root of a number, look for perfect square factors of the radicand. Apply the product or quotient rule for radicals and then simplify.
- The cube root of a number is a number that when cubed results in the original number. Every real number has only one real cube root.
- When simplifying cube roots, look for perfect cube factors of the radicand. Apply the product or quotient rule for radicals and then simplify.
- The Pythagorean theorem gives us a necessary and sufficient condition of right triangles: $a^2+b^2=c^2$ if and only if a, b and c represent the lengths of the sides of a right triangle.

Exercise 1.3.3

1. $\sqrt{81}$
2. $\sqrt{49}$
3. $-\sqrt{16}$
4. $-\sqrt{100}$
5. $\sqrt{\frac{25}{16}}$

6. $\sqrt{\frac{9}{64}}$
7. $\sqrt{\frac{1}{4}}$
8. $\sqrt{\frac{1}{100}}$
9. $\sqrt{-1}$
10. $\sqrt{-25}$
11. $\sqrt{036}$
12. $\sqrt{1.21}$
13. $\sqrt{(-5)^2}$
14. $\sqrt{(-6)^2}$
15. $2\sqrt{64}$
16. $3\sqrt{36}$
17. $-10\sqrt{4}$
18. $-8\sqrt{25}$
19. $\sqrt[3]{64}$
20. $\sqrt[3]{125}$
21. $\sqrt[3]{-27}$
22. $\sqrt[3]{-1}$
23. $\sqrt[3]{0}$
24. $\sqrt[3]{0.008}$
25. $\sqrt[3]{0.064}$
26. $-\sqrt[3]{-8}$
27. $-\sqrt[3]{1000}$
28. $\sqrt[3]{(-8)^3}$
29. $\sqrt[3]{(-15)^3}$
30. $\sqrt[3]{\frac{1}{216}}$
31. $\sqrt[3]{\frac{27}{64}}$
32. $\sqrt[3]{-\frac{1}{8}}$
33. $\sqrt[3]{-\frac{1}{27}}$
34. $5\sqrt[3]{343}$
35. $4\sqrt[3]{512}$
36. $-10\sqrt[3]{8}$
37. $-6\sqrt[3]{-64}$
38. $8\sqrt[3]{-8}$

Answer

1. 9

3. -4

5. $\frac{5}{4}$

7. $\frac{1}{2}$

9. Not a real number.

11. 0.6

13. 5

15. 16

17. -20

19. 4

21. -3

23. 0

25. 0.4

27. -10

29. -15

31. $\frac{3}{4}$

33. $-\frac{1}{3}$

35. 32

37. 24

Exercise $1.3.4$

Use a calculator to approximate to the nearest hundredth.

1. $\sqrt{3}$
2. $\sqrt{10}$
3. $\sqrt{19}$
4. $\sqrt{7}$
5. $3\sqrt{5}$
6. $-2\sqrt{3}$
7. $\sqrt[3]{3}$
8. $\sqrt[3]{6}$
9. $\sqrt[3]{28}$
10. $\sqrt[3]{9}$
11. $4\sqrt[3]{10}$
12. $-3\sqrt[3]{12}$
13. Determine the set consisting of the squares of the first twelve positive integers.
14. Determine the set consisting of the cubes of the first twelve positive integers.

Answer

1. 1.73

3. 4.36

5. 6.71

7. 1.44

9. 3.04

11. 8.62

13. $\{1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144\}$

Exercise $1.3.5$

Simplify.

1. $\sqrt{18}$
2. $\sqrt{50}$
3. $\sqrt{24}$
4. $\sqrt{40}$
5. $\sqrt{\frac{50}{81}}$
6. $\sqrt{\frac{54}{25}}$
7. $4\sqrt{72}$
8. $3\sqrt{27}$
9. $-5\sqrt{80}$

10. $-6\sqrt{128}$
11. $3\sqrt{-40}$
12. $5\sqrt{-160}$
13. $\sqrt[3]{16}$
14. $\sqrt[3]{54}$
15. $\sqrt[3]{81}$
16. $\sqrt[3]{24}$
17. $\sqrt[3]{\frac{48}{125}}$
18. $\sqrt[3]{\frac{135}{64}}$
19. $7\sqrt[3]{500}$
20. $25\sqrt[3]{686}$
21. $-2\sqrt[3]{-162}$
22. $5\sqrt[3]{-96}$
23. $(\sqrt{64})^2$
24. $(\sqrt{25})^2$
25. $(\sqrt{2})^2$
26. $(\sqrt{6})^2$

Answer

1. $3\sqrt{2}$

3. $2\sqrt{6}$

5. $\frac{5\sqrt{2}}{9}$

7. $24\sqrt{2}$

9. $-20\sqrt{5}$

11. Not a real number.

13. $2\sqrt[3]{2}$

15. $3\sqrt[3]{3}$

17. $\frac{2\sqrt[3]{6}}{5}$

19. $35\sqrt[3]{4}$

21. $6\sqrt[3]{6}$

23. 64

25. 2

Exercise 1.3.6

1. If the two legs of a right triangle measure 3 units and 4 units, then find the length of the hypotenuse.
2. If the two legs of a right triangle measure 6 units and 8 units, then find the length of the hypotenuse.
3. If the two equal legs of an isosceles right triangle measure 7 units, then find the length of the hypotenuse.
4. If the two equal legs of an isosceles right triangle measure 10 units, then find the length of the hypotenuse.
5. Calculate the diagonal of a square with sides measuring 3 centimeters.
6. Calculate the diagonal of a square with sides measuring 10 centimeters.
7. Calculate the diagonal of a square with sides measuring $\sqrt{6}$ centimeters.
8. Calculate the diagonal of a square with sides measuring $\sqrt{10}$ centimeters.
9. Calculate the length of the diagonal of a rectangle with dimensions 4 centimeters by 8 centimeters.
10. Calculate the length of the diagonal of a rectangle with dimensions 8 meters by 10 meters.
11. Calculate the length of the diagonal of a rectangle with dimensions $\sqrt{3}$ meters by 2 meters.
12. Calculate the length of the diagonal of a rectangle with dimensions $\sqrt{6}$ meters by $\sqrt{10}$ meters.

13. To ensure that a newly built gate is square, the measured diagonal must match the distance calculated using the Pythagorean theorem. If the gate measures 4 feet by 4 feet, what must the diagonal measure in inches? (Round off to the nearest tenth of an inch.)
14. If a doorframe measures 3.5 feet by 6.6 feet, what must the diagonal measure to ensure that the frame is a perfect rectangle?

Answer

1. 5 units

3. $7\sqrt{2}$ units

5. $3\sqrt{2}$ centimeters

7. $2\sqrt{3}$ centimeters

9. $4\sqrt{5}$ centimeters

11. $\sqrt{7}$ meters

13. The diagonal must measure approximately 67.9 inches.

Exercise 1.3.7

Determine whether or not the given triangle with legs a and b and hypotenuse c is a right triangle or not.

1. $a = 3, b = 7$, and $c = 10$
2. $a = 5, b = 12$, and $c = 13$
3. $a = 8, b = 15$, and $c = 17$
4. $a = 7, b = 24$, and $c = 30$
5. $a = 3, b = 2$, and $c = \sqrt{13}$
6. $a = \sqrt{7}, b = 4$, and $c = \sqrt{11}$
7. $a = 4, b = \sqrt{3}$, and $c = \sqrt{19}$
8. $a = \sqrt{6}, b = \sqrt{15}, and\backslash(c = 21$

Answer

1. Not a right triangle.

3. Right triangle.

5. Right triangle.

7. Right triangle.

Exercise 1.3.8

1. What does your calculator say after taking the square root of a negative number? Share your results on the discussion board and explain why it says that.
2. Research and discuss the history of the Pythagorean theorem.
3. Research and discuss the history of the square root.
4. Discuss the importance of the principal square root. Why is it that the same issue does not come up with cube roots? Provide some examples with your explanation.

Answer

1. Answer may vary

3. Answer may vary

Footnotes

[74] That number that when multiplied by itself yields the original number.

[75] The symbol $\sqrt{\;}$ used to denote a square root.

[76] The non-negative square root.

[77] The number within a radical.

[78] The number that when multiplied by itself three times yields the original number, denoted by $\sqrt[3]{\;}$.

[79] The positive integer n in the notation $\sqrt[n]{\;}$ that is used to indicate an nth root.

[80] Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$,$\sqrt[n]{A \cdot B} = \sqrt[n]{A} \cdot \sqrt[n]{B}$

[81] Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$, $\sqrt[n]{\frac{A}{B}} = \frac{\sqrt[n]{A}}{\sqrt[n]{B}}$.

[82] A radical where the radicand does not consist of any factors that can be written as perfect powers of the index.

[83] A triangle with an angle that measures $90°$.

[84] The longest side of a right triangle; it will always be the side opposite the right angle.

[85] The sides of a right triangle that are not the hypotenuse.

[86] The hypotenuse of any right triangle is equal to the square root of the sum of the squares of the lengths of the triangle’s legs.

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1.4: Algebraic Expressions and Formulas

Learning Objectives

- Identify the parts of an algebraic expression.
- Apply the distributive property.
- Evaluate algebraic expressions.
- Use formulas that model common applications.

Algebraic Expressions and the Distributive Property

In algebra, letters called variables are used to represent numbers. Combinations of variables and numbers along with mathematical operations form **algebraic expressions**[87], or just **expressions**. The following are some examples of expressions with one variable, x:

Table 1.4.1

$2x+3$	x^2-9	$\frac{1}{x}+\frac{x}{x+2}$	$3\sqrt{x}+x$

Terms[88] in an algebraic expression are separated by addition operators and **factors**[89] are separated by multiplication operators. The numerical factor of a term is called the **coefficient**[90]. For example, the algebraic expression $x^2y^2+6xy-3$ can be thought of as x^2y^2 $+6xy+(-3)$ and has three terms. The first term, x^2y^2, represents the quantity $1x^2y^2=1\cdot x\cdot x\cdot y\cdot y$ where 1 is the coefficient and x and y are the variables. All of the variable factors with their exponents form the **variable part of a term**[91]. If a term is written without a variable factor, then it is called a **constant term**[92]. Consider the components of $x^2y^2+6xy-3$,

Table 1.4.2

Terms	*Coefficient*	*Variable Part*
x^2y^2	1	x^2y^2
$6xy$	6	xy
-3	-3	

The third term in this expression, -3, is called a constant term because it is written without a variable factor. While a variable represents an unknown quantity and may change, the constant term does not change.

Example 1.4.1:

List all coefficients and variable parts of each term: $10a^2-5ab-b^2$.

Solution

We want to think of the third term in this example $-b^2$ as $-1b^2$.

Table 1.4.3

Terms	*Coefficient*	*Variable Part*
$10a^2$	10	a^2
$-5ab$	-5	ab
$-b^2$	-1	b^2

Answer: Coefficients: $\{-5,\ -1,\ 10\}$; Variable parts: $\{a^2,\ ab,\ b^2\}$

In our study of algebra, we will encounter a wide variety of algebraic expressions. Typically, expressions use the two most common variables, x and y. However, expressions may use any letter (or symbol) for a variable, even Greek letters, such as alpha (α) and beta (β). Some letters and symbols are reserved for constants, such as $\pi\approx 3.14159$ and $e\approx 2.71828$. Since there is only a limited number of letters, you will also use subscripts, $x_1, x_2, x_3, x_4, \ldots$, to indicate different variables.

The properties of real numbers are important in our study of algebra because a variable is simply a letter that represents a real number. In particular, the **distributive property**[93] states that if given any real numbers a, b and c, then,

$$a\ (b+c)=ab+ac$$

This property is one that we apply often when simplifying algebraic expressions. To demonstrate how it will be used, we simplify $2(5-3)$ in two ways, and observe the same correct result.

Table 1.4.4

Working parenthesis first.	***Using the distributive property.***
$2(5-3)=2(2)$ $=4$	$2(5-3)=2\cdot5-2\cdot3$ $=10-6$ $=4$

Certainly, if the contents of the parentheses can be simplified we should do that first. On the other hand, when the contents of parentheses cannot be simplified any further, we multiply every term within it by the factor outside of it using the distributive property. Applying the distributive property allows us to multiply and remove the parentheses.

Example 1.4.2:

Simplify: $5(-2a+5b)-2c$.

Solution

Multiply only the terms grouped within the parentheses for which we are applying the distributive property.

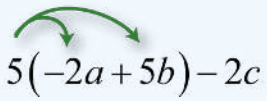

Figure 1.4.1

$$=5\cdot(-2a)+5\cdot5b-2c$$

$$=-10a+25b-2c$$

Answer: $-10a+25b-2c$

Recall that multiplication is commutative and therefore we can write the distributive property in the following manner, $(b+c)a=ba+ca$.

Example 1.4.3:

Simplify: $(3x-4y+1)\cdot3$.

Solution

Multiply all terms within the parenthesis by 3.

$$(3x-4y+1)\cdot3=3x\cdot3-4y\cdot3+1\cdot3$$

$$=9x-12y+3$$

Answer: $9x-12y+3$

Terms whose variable parts have the same variables with the same exponents are called **like terms**[94], or **similar terms**[95]. Furthermore, constant terms are considered to be like terms. If an algebraic expression contains like terms, apply the distributive property as follows:

$$5x+7x=(5+7)x=12x$$

$$4x^2+5x^2-7x^2=(4+5-7)x^2=2x^2$$

In other words, if the variable parts of terms are exactly the same, then we can add or subtract the coefficients to obtain the coefficient of a single term with the same variable part. This process is called **combining like terms**[96]. For example,

$$12x^2y^3 + 3x^2y^3 = 15x^2y^3$$

Notice that the variable factors and their exponents do not change. Combining like terms in this manner, so that the expression contains no other similar terms, is called **simplifying the expression**[97]. Use this idea to simplify algebraic expressions with multiple like terms.

Example 1.4.4:

Simplify:

$x^2 - 10x + 8 + 5x^2 - 6x - 1$.

Solution

Identify the like terms and add the corresponding coefficients.

$$\begin{aligned} &\underline{1x^2} - \underline{\underline{10x}} + \underline{\underline{\underline{8}}} + \underline{5x^2} - \underline{\underline{6x}} - \underline{\underline{\underline{1}}} \quad \textit{Combine like terms.} \\ &= 6x^2 - 16x + 7 \end{aligned}$$

Answer: $6x^2 - 16x + 7$

Example 1.4.5:

Simplify: $a^2b^2 - ab - 2\left(2a^2b^2 - 5ab + 1\right)$.

Solution

Distribute -2 and then combine like terms.

$$\begin{aligned} a^2b^2 - ab - 2\left(2a^2b^2 - 5ab + 1\right) &= a^2b^2 - ab - 4a^2b^2 + 10ab - 2 \\ &= -3a^2b^2 + 9ab - 2 \end{aligned}$$

Answer: $-3a^2b^2 + 9ab - 2$

Evaluating Algebraic Expressions

An algebraic expression can be thought of as a generalization of particular arithmetic operations. Performing these operations after substituting given values for variables is called **evaluating**[98]. In algebra, a variable represents an unknown value. However, if the problem specifically assigns a value to a variable, then you can replace that letter with the given number and evaluate using the order of operations.

Example 1.4.6:

Evaluate:

a. $5x - 2$ where $x = \frac{2}{3}$
b. $y^2 - y - 6$ where $y = -4$

Solution

To avoid common errors, it is a best practice to first replace all variables with parentheses, and then replace, or **substitute**[99], the appropriate given value.

a.

$$\begin{aligned} 5x - 2 &= 5(\ \ ) - 2 \\ &= 5\left(\frac{2}{3}\right) - 2 \\ &= \frac{10}{3} - \frac{2}{1} \cdot \frac{3}{3} \\ &= \frac{10 - 6}{3} \\ &= \frac{4}{3} \end{aligned}$$

b.

$$\begin{aligned} y^2-y-6 &= (\ \ )^2-(\ \ )-6 \\ &= (-4)^2-(-4)-6 \\ &= 16+4-6 \\ &= 14 \end{aligned}$$

Answer:

a. $\frac{4}{3}$

b. 14

Often algebraic expressions will involve more than one variable.

Example 1.4.7:

Evaluate a^3-8b^3 where $a=-1$ and $b=\frac{1}{2}$.

Solution

After substituting in the appropriate values, we must take care to simplify using the correct order of operations.

$$\begin{aligned} a^3-8b^3 &= (\ \ )^3-8(\ \ )^3 \textit{Replace variables with parentheses.} \\ &= (-1\ )^3-8(\tfrac{1}{2})^3 \textit{Substitute in the appropriate values.} \\ &= -1-8\left(\tfrac{1}{8}\right) \textit{Simplify.} \\ &= -1-1 \\ &= -2 \end{aligned}$$

Answer: -2

Example 1.4.8:

Evaluate $\frac{x^2-y^2}{2x-1}$ where $x=-\frac{3}{2}$ and $y=-3$.

Solution

$$\begin{aligned} \tfrac{x^2-y^2}{2x-1} &= \tfrac{(\ \ )^2-(\ \ )^2}{2(\ \ )-1} \\ &= \tfrac{\left(-\frac{3}{2}\right)^2-(-3)^2}{2\left(-\frac{3}{2}\right)-1} \\ &= \tfrac{\frac{9}{4}-9}{-3-1} \end{aligned}$$

At this point we have a complex fraction. Simplify the numerator and then multiply by the reciprocal of the denominator.

$$\begin{aligned} &= \frac{\frac{9}{4}-\frac{9}{1}\cdot\frac{4}{4}}{-4} \\ &= \frac{\frac{-27}{4}}{\frac{-4}{1}} \\ &= \frac{-27}{4}\left(-\frac{1}{4}\right) \\ &= \frac{27}{16} \end{aligned}$$

Answer: $\frac{27}{16}$

The answer to the previous example can be written as a mixed number, $\frac{27}{16}=1\frac{11}{16}$. Unless the original problem has mixed numbers in it, or it is an answer to a realworld application, solutions will be expressed as reduced improper fractions.

Example 1.4.9:

Evaluate $\sqrt{b^2-4ac}$ where $a=-1, b=-7$, and $c=\frac{1}{4}$.

Solution

Substitute in the appropriate values and then simplify.

$$\begin{aligned}\sqrt{b^2-4ac} &= \sqrt{(\ )^2-4(\ )\ (\ )}\\ &= \sqrt{(-7)^2-4(-1)(\tfrac{1}{4})}\\ &= \sqrt{49+4(\frac{1}{4})}\\ &= \sqrt{49+1}\\ &= \sqrt{50}\\ &= \sqrt{25\cdot 2}\\ &= 5\sqrt{2}\end{aligned}$$

Aligned: $5\sqrt{2}$

Exercise 1.4.1

Evaluate $\frac{\sqrt{3\pi Vh}}{\pi h}$ where $V=25\pi$ and $h=3$.

Answer

5

www.youtube.com/v/Y4RCMceThu4

Using Formulas

The main difference between algebra and arithmetic is the organized use of variables. This idea leads to reusable **formulas**[100], which are mathematical models using algebraic expressions to describe common applications. For example, the volume of a right circular cone depends on its radius r and height h and is modeled by the formula:

$$V=\frac{1}{3}\pi r^2h$$

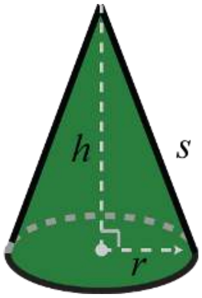

Figure 1.4.1

In this equation, variables and constants are used to describe the relationship between volume and the length of the base and height. If the radius of the base measures 3 meters and the height measures 5 meters, then the volume can be calculated using the formula as follows:

$$\begin{aligned}V &= \frac{1}{3}\pi r^2h\\ &= \frac{1}{3}\pi(3m)^2(5m)\\ &= \frac{1}{\cancel{3}}\pi\cdot \overset{3}{\cancel{9}}\cdot 5m^3\\ &= 15\pi\text{m}^3\end{aligned}$$

Using $\pi\approx 3.14$, we can approximate the volume: $V\approx 15(3.14)=47.1$ cubic meters.

A list of formulas that describe the area and perimeter of common plane figures follows. The letter P represents perimeter and is measured in linear units. The letter A represents area and is measured in square units.

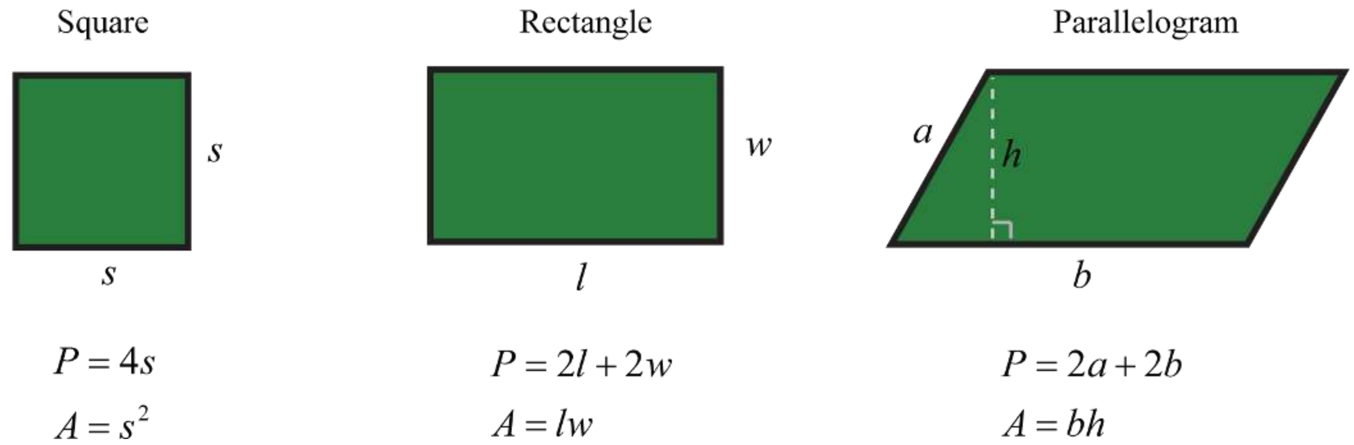
Square
s
s
$P=4s$
$A=s^2$
Rectangle
w
l
$P=2l+2w$
$A=lw$
Parallelogram
a
h
b
$P=2a+2b$
$A=bh$

Figure 1.4.2

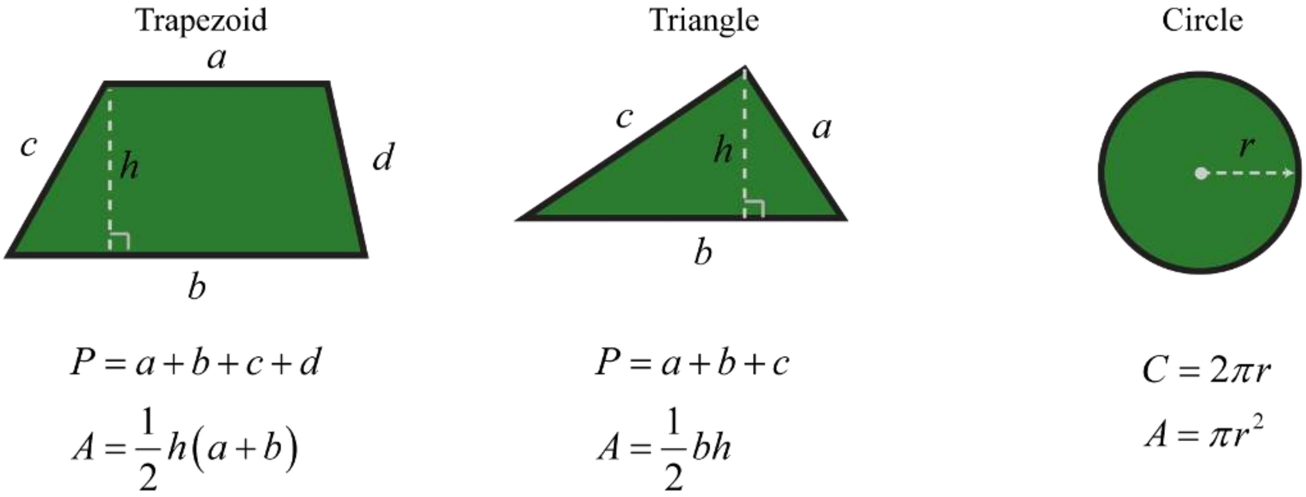
Trapezoid
a
c
h
d
b
$P=a+b+c+d$
$A=\frac{1}{2}h(a+b)$
Triangle
c
h
a
b
$P=a+b+c$
$A=\frac{1}{2}bh$
Circle
r
$C=2\pi r$
$A=\pi r^2$

Figure 1.4.3

A list of formulas that describe the surface area and volume of common figures follows. Here *SA* represents surface area and is measured in square units. The letter *V* represents volume and is measured in cubic units.

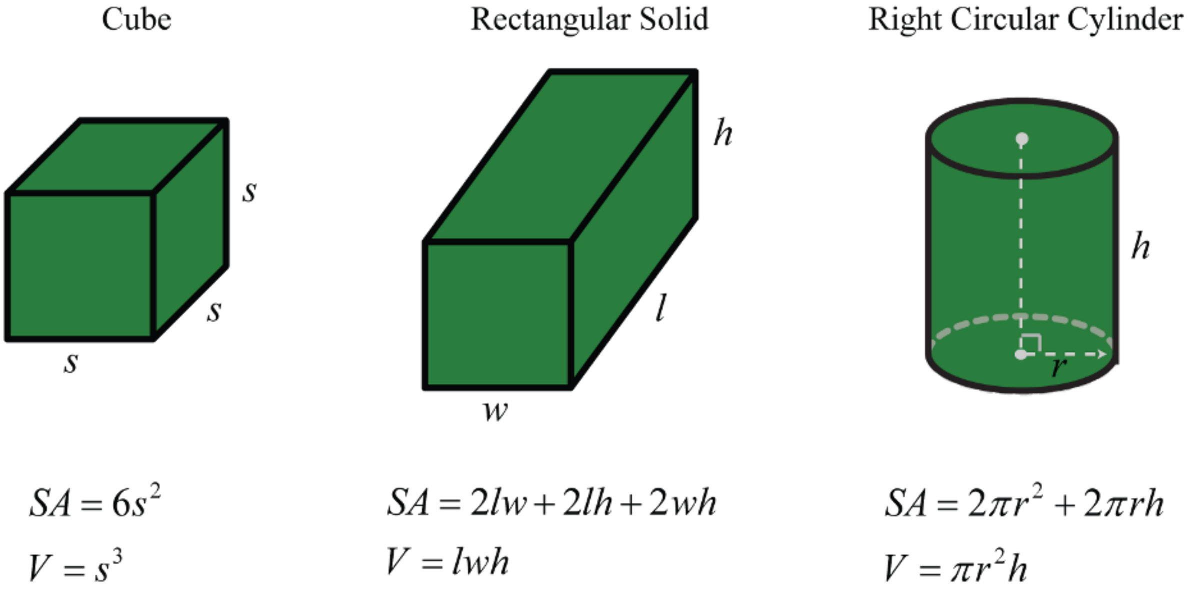
Cube
s
s
s
$SA=6s^2$
$V=s^3$
Rectangular Solid
h
l
w
$SA=2lw+2lh+2wh$
$V=lwh$
Right Circular Cylinder
h
r
$SA=2\pi r^2+2\pi rh$
$V=\pi r^2 h$

Figure 1.4.4

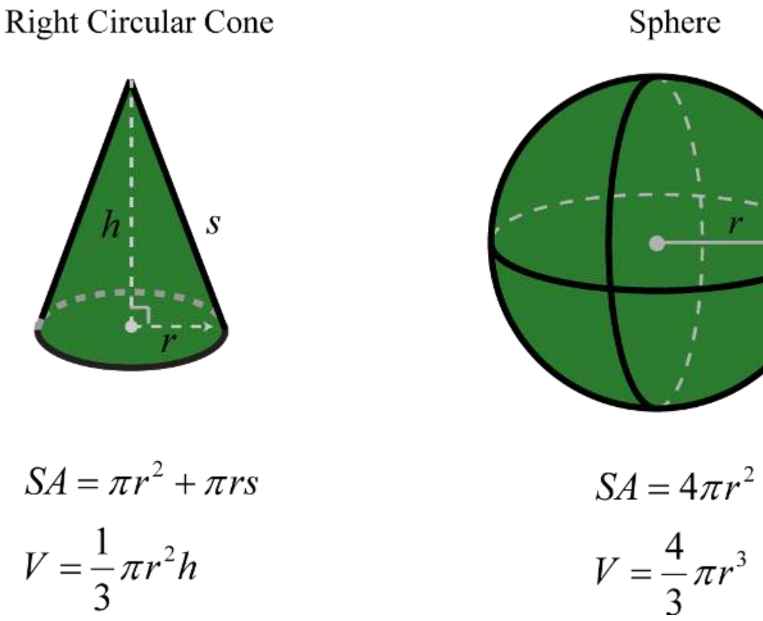

Figure $1.4.5$

Example $1.4.10$:

The diameter of a spherical balloon is 10 inches. Determine the volume rounded off to the nearest hundredth.

Solution

The formula for the volume of a sphere is

$$V = \frac{4}{3}\pi r^3$$

This formula gives the volume in terms of the radius, r. Therefore, divide the diameter by 2 and then substitute into the formula. Here, $r = \frac{10}{2} = 5$ inches and we have

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi(5\text{in})^3 \\ &= \frac{4}{3}\pi \cdot 125\text{in}^3 \\ &= \frac{500\pi}{3}\text{in}^3 \approx 523.60\text{in}^3 \end{aligned}$$

Answer: The volume of the balloon is approximately 523.60 cubic inches.

Formulas can be found in a multitude of subjects. For example, **uniform motion**[101] is modeled by the formula $D = rt$, which expresses distance D, in terms of the average rate, or speed, r and the time traveled at that rate, t. This formula, $D = rt$, is used often and is read, "*distance equals rate times time*."

Example $1.4.11$:

Jim's road trip took $2\,\frac{1}{2}$ hours at an average speed of 66 miles per hour. How far did he travel?

Solution

Substitute the appropriate values into the formula and then simplify.

$$\begin{aligned} D &= r \cdot t \\ &= \left(66\frac{\text{mi}}{\text{hr}}\right) \cdot \left(2\frac{1}{2}\text{hr}\right) \\ &= \frac{66}{1} \cdot \frac{5}{2}\text{mi} \\ &= 33 \cdot 5\text{mi} \\ &= 165\text{mi} \end{aligned}$$

Answer: Jim traveled 165 miles.

Simple interest[102] I is given by the formula $I = prt$, where p represents the principal amount invested at an annual interest rate r for t years.

Example 1.4.12:

Calculate the simple interest earned on a 2-year investment of $\$1,250$ at an annual interest rate of $3\,\frac{3}{4}$.

Solution

Convert $3\,\frac{3}{4}$ to a decimal number before using it in the formula.

$$r = 3\tfrac{3}{4}\% = 3.75\% = 0.0375$$

Use this and the fact that $p = \$1,250$ and $t = 2$ years to calculate the simple interest.

$$\begin{aligned} I &= prt \\ &= (1,250)(0.0375)(2) \\ &= 93.75 \end{aligned}$$

Answer: The simple interest earned is $\$93.75$.

Key Takeaways

- Think of algebraic expressions as generalizations of common arithmetic operations that are formed by combining numbers, variables, and mathematical operations.
- The distributive property $a(b+c) = ab + ac$, is used when multiplying grouped algebraic expressions. Applying the distributive property allows us to remove parentheses.
- Combine like terms, or terms whose variable parts have the same variables with the same exponents, by adding or subtracting the coefficients to obtain the coefficient of a single term with the same variable part. Remember that the variable factors and their exponents do not change.
- To avoid common errors when evaluating, it is a best practice to replace all variables with parentheses and then substitute the appropriate values.
- The use of algebraic expressions allows us to create useful and reusable formulas that model common applications.

Exercise 1.4.2

List all of the coefficients and variable parts of each term.

1. $-5x^2 + x - 1$
2. $y^2 - 9y + 3$
3. $5x^2 - 3xy + y^2$
4. $a^2b^2 + 2ab - 4$
5. $x^2y + xy^2 - 3xy + 9$
6. $x^4 - x^3 + x^2 - x + 2$

Answer

1. Coefficients: $\{-5, 1, -1\}$; variable parts: $\{x^2, x\}$

3. Coefficients: $\{5, -3, 1\}$; variable parts: $\{x^2, xy, y^2\}$

5. Coefficients: $\{1, -3, 9\}$; variable parts: $\{x^2y, xy^2, xy\}$

Exercise 1.4.3

Multiply.

1. $5(3x - 5)$
2. $3(4x - 1)$
3. $-2(2x^2 - 5x + 1)$
4. $-5(6x^2 - 3x - 1)$

5. $\frac{2}{3}(9y^2+12y-3)$
6. $-\frac{3}{4}(8y^2+20y+4)$
7. $12(\frac{1}{3}a^2-\frac{5}{6}a+\frac{7}{12})$
8. $-9(\frac{1}{9}a^2-\frac{5}{3}a+1)$
9. $9(a^2-2b^2)$
10. $-5(3x^2-y^2)$
11. $(5a^2-3ab+b^2)\cdot 6$
12. $(a^2b^2-9ab-3)\cdot 7$
13. $-(5x^2-xy+y^2)$
14. $-(x^2y^2-6xy-1)$

Answer

1. $15x-25$

3. $-4x^2+10x-2$

5. $6y^2+8y-2$

7. $4a^2-10a+7$

9. $9a^2-18b^2$

11. $30a^2-18ab+6b^2$

13. $-5x^2+xy-y^2$

Exercise 1.4.4

Combine like terms.

1. $18x-5x+3x$
2. $30x-50x+10x$
3. $3y-4+2y-12$
4. $12y+7-15y-6$
5. $2x^2-3x+2+5x^2-6x+1$
6. $9x^2+7x-5-10x^2-8x+6$
7. $\frac{3}{5}a^2-\frac{1}{2}+\frac{1}{3}a^2+\frac{4}{5}$
8. $\frac{1}{6}a^2+\frac{2}{3}-\frac{4}{3}a^2-\frac{1}{9}$
9. $\frac{1}{2}y^2+\frac{2}{3}y-3+\frac{3}{5}y^2+\frac{1}{3}y-\frac{7}{3}$
10. $\frac{5}{6}x^2+\frac{1}{8}x-1-\frac{1}{2}x^2+\frac{3}{4}x-\frac{4}{5}$
11. $a^2b^2+5ab-2+7a^2b^2-6ab+12$
12. $a^2-12ab+4b^2-6a^2+10ab-5b^2$
13. $3x^2y+12xy-5xy^2+5xy-8x^2y+2xy^2$
14. $10x^2y+2xy-4xy^2+2x^2y-8xy+5xy^2$
15. $7m^2n-9mn+mn^2-6m^2n+mn-2mn^2$
16. $m^2n-5mn+5mn^2-3m^2n+5mn+2mn^2$
17. $x^{2n}-3x^n+5+2x^{2n}-4x^n-3$
18. $5y^{2n}-3y^n+1-3y^{2n}-2y^n-1$

Answer

1. $16x$

3. $5y-16$

5. $7x^2-9x+3$

7. $\frac{14}{15}a^2+\frac{3}{10}$

9. $\frac{11}{10}y^2+y-\frac{16}{3}$

11. $8a^2b^2 - ab + 10$

13. $-5x^2y + 17xy - 3xy^2$

15. $m^2n - 8mn - mn^2$

17. $3x^{2n} - 7x^n + 2$

Exercise 1.4.5

Simplify.

1. $5 - 2(4x + 8)$
2. $8 - 6(2x - 1)$
3. $2(x^2 - 7x + 1) + 3x - 7$
4. $-5(x^2 + 4x - 1) + 8x^2 - 5$
5. $5ab - 4(ab + 5)$
6. $5(7 - ab) + 2ab$
7. $2 - a^2 + 3(a^2 + 4)$
8. $7 - 3y + 2(y^2 - 3y - 2)$
9. $8x^2 - 3x - 5(x^2 + 4x - 1)$
10. $2 - 5y - 6(y^2 - y + 2)$
11. $a^2b^2 - 5 + 3(a^2b^2 - 3ab + 2)$
12. $a^2 - 3ab - 2(a^2 - ab + 1)$
13. $10y^2 + 6 - (3y^2 + 2y + 4)$
14. $4m^2 - 3mn - (m^2 - 3mn + n^2)$
15. $x^{2n} - 3x^n + 5(x^{2n} - x^n + 1)$
16. $-3(y^{2n} - 2y^n + 1) + 4y^{2n} - 5$

Answer

1. $-8x - 11$

3. $2x^2 - 11x - 5$

5. $ab - 20$

7. $2a^2 + 14$

9. $3x^2 - 23x + 5$

11. $4a^2b^2 - 9ab + 1$

13. $7y^2 - 2y + 2$

15. $6x^{2n} - 8x^n + 5$

Exercise 1.4.6

Evaluate.

1. $-2x + 3$ where $x = -2$
2. $8x - 5$ where $x = -1$
3. $x^2 - x + 5$ where $x = -5$
4. $2x^2 - 8x + 1$ where $x = 3$
5. $\frac{x^2 - x + 2}{2x - 1}$ where $x = -\frac{1}{2}$
6. $\frac{9x^2 + x - 2}{3x - 4}$ where $x = -\frac{2}{3}$
7. $(3y - 2)(y + 5)$ where $y = \frac{2}{3}$
8. $(3x + 2)(5x + 1)$ where $x = -\frac{1}{5}$
9. $(3x - 1)(x - 8)$ where $x = -1$
10. $(7y + 5)(y + 1)$ where $y = -2$

11. $y^6 - y^3 + 2$ where $y = -1$
12. $y^5 + y^3 - 3$ where $y = -2$
13. $a^2 - 5b^2$ where $a = -2$ and $b = -1$
14. $a^3 - 2b^3$ where $a = -3$ and $b = 2$
15. $(x - 2y)(x + 2y)$ where $x = 2$ and $y = -5$
16. $(4x - 3y)(x - y)$ where $x = -4$ and $y = -3$
17. $a^2 - ab + b^2$ where $a = -1$ and $b = -2$
18. $x^2y^2 - xy + 2$ where $x = -3$ and $y = -2$
19. $a^4 - b^4$ where $a = -2$ and $b = -3$
20. $a^6 - 2a^3b^3 - b^6$ where $a = 2$ and $b = -1$

Answer

1. 7

3. 35

5. $-\frac{11}{8}$

7. 0

9. 36

11. 4

13. -1

15. -96

17. 3

19. -65

Exercise 1.4.7

Evaluate $\sqrt{b^2 - 4ac}$ given the following values.

1. $a = 6, b = 1$ and $c = -1$
2. $a = 15, b = 4$ and $c = -4$
3. $a = \frac{3}{4}, b = -2$ and $c = -4$
4. $a = \frac{1}{2}, b = -2$ and $c = -30$
5. $a = 1, b = 2$ and $c = -1$
6. $a = 1, b = -4$ and $c = -50$
7. $a = 1, b = -1$ and $c = -\frac{1}{16}$
8. $a = -2, b = -\frac{1}{3}$ and $c = 1$

Answer

1. 5

3. 4

5. $2\sqrt{2}$

7. $\frac{\sqrt{5}}{2}$

Exercise 1.4.8

Convert the following temperatures to degrees Celsius given $C = \frac{5}{9}(F - 32)$, where F represents degrees Fahrenheit.

1. 95°F
2. 86°F
3. 32°F

4. $-40°F$

Answer

1. $35°C$
3. 0°C

Exercise $1.4.9$

1. Calculate the perimeter and area of a rectangle with dimensions 12 feet by 5 feet.
2. Calculate the perimeter and area of a rectangle with dimensions 5 meters by 1 meter.
3. Calculate the surface area and volume of a sphere with radius 6 centimeters.
4. The radius of the base of a right circular cylinder measures 4 inches and the height measures 10 inches. Calculate the surface area and volume.
5. Calculate the volume of a sphere with a diameter of 18 centimeters.
6. The diameter of the base of a right circular cone measures 6 inches. If the height is $1\frac{1}{2}$ feet, then calculate its volume.
7. Given that the height of a right circular cylinder is equal to the radius of the base, derive a formula for the surface area in terms of the radius of the base.
8. Given that the area of the base of a right circular cylinder is 25π square inches, find the volume if the height is 1 foot.
9. Jose was able to drive from Tucson to Phoenix in 2 hours at an average speed of 58 mph. How far is Phoenix from Tucson?
10. If a bullet train can average 152 mph, then how far can it travel in $\frac{3}{4}$ of an hour?
11. Margaret traveled for $1\frac{3}{4}$ hour at an average speed of 68 miles per hour. How far did she travel?
12. The trip from Flagstaff, AZ to the Grand Canyon national park took $1\frac{1}{2}$ hours at an average speed of 54 mph. How far is the Grand Canyon national park from Flagstaff?
13. Calculate the simple interest earned on a 3-year investment of $\$2,500$ at an annual interest rate of $5\frac{1}{4}\%$.
14. Calculate the simple interest earned on a 1-year investment of $\$5,750$ at an annual interest rate of $2\frac{5}{8}\%$.
15. What is the simple interest earned on a 5-year investment of $\$20,000$ at an annual interest rate of 6%?
16. What is the simple interest earned on a 1-year investment of $\$50,000$ at an annual interest rate of 4.5%?
17. The time t in seconds an object is in free fall is given by the formula $t=\frac{\sqrt{s}}{4}$, where s represents the distance in feet the object has fallen. How long does it take an object to fall 32 feet? (Give the exact answer and the approximate answer to the nearest hundredth.)
18. The current I measured in amperes, is given by the formula $I=\sqrt{\frac{P}{R}}$, where P is the power usage measured in watts, and R is the resistance measured in ohms. If a light bulb uses 60 watts of power and has 240 ohms of resistance, then how many amperes of current are required?

Answer

1. $P=34$ feet; $A=60$ square feet
3. $SA=144\pi$ square centimeters; $V=288\pi$ cubic centimeters
5. 972π cubic centimeters
7. $SA=4\pi r^2$
9. 116 miles
11. 119 miles
13. $\$393.75$
15. $\$6,000$
17. $\sqrt{2}\approx 1.41$ seconds

Exercise 1.4.10

1. Find and post a useful mathematical model. Demonstrate its use with some values.
2. Research and discuss the history of the variable. What can we use if we run out of letters?
3. Find and post a link to a useful resource describing the Greek alphabet.
4. Given the algebraic expression $5-3(9x-1)$, explain why we do not subtract 5 and 3 first.
5. Do we need a separate distributive property for more than two terms? For example, $a(b+c+d)=ab+ac+ad$. Explain.
6. How can we check to see if we have simplified an expression correctly?

Answer

1. Answer may vary
3. Answer may vary
5. Answer may vary

Footnotes

[87] Combinations of variables and numbers along with mathematical operations used to generalize specific arithmetic operations.

[88] Components of an algebraic expression separated by addition operators.

[89] Components of a term separated by multiplication operators.

[90] The numerical factor of a term.

[91] All the variable factors with their exponents.

[92] A term written without a variable factor.

[93] Given any real numbers a, b, and $c, a(b+c)=ab+ac$ or $(b+c)a=ba+ca$.

[94] Constant terms or terms whose variable parts have the same variables with the same exponents.

[95] Used when referring to like terms.

[96] Adding or subtracting like terms within an algebraic expression to obtain a single term with the same variable part.

[97] The process of combining like terms until the expression contains no more similar terms.

[98] The process of performing the operations of an algebraic expression for given values of the variables.

[99] The act of replacing a variable with an equivalent quantity.

[100] A reusable mathematical model using algebraic expressions to describe a common application.

[101] The distance D after traveling at an average rate r for some time t can be calculated using the formula $D=rt$.

[102] Modeled by the formula $I=prt$, where p represents the principal amount invested at an annual interest rate r for t years.

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1.5: Rules of Exponents and Scientific Notation

Learning Objectives

- Review the rules of exponents.
- Review the definition of negative exponents and zero as an exponent.
- Work with numbers using scientific notation.

Review of the Rules of Exponents

In this section, we review the rules of exponents. Recall that if a factor is repeated multiple times, then the product can be written in exponential form x^n. The positive integer exponent n indicates the number of times the base x is repeated as a factor.

$$x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n\,times}$$

Figure 1.5.1

Consider the product of x^4 and x^6,

$$x^4 \cdot x^6 = \underbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}_{10\ factors\ of\ x} = x^{10}$$

Figure 1.5.2

Expanding the expression using the definition produces multiple factors of the base which is quite cumbersome, particularly when n is large. For this reason, we have useful rules to help us simplify expressions with exponents. In this example, notice that we could obtain the same result by adding the exponents.

$$x^4 \cdot x^6 = x^{4+6} = x^{10} \textit{Product rule for exponents}$$

In general, this describes the **product rule for exponents**[103]. In other words, when multiplying two expressions with the same base we add the exponents. Compare this to raising a factor involving an exponent to a power, such as $(x^6)^4$.

$$\begin{aligned} \left(x^6\right)^4 &= \underbrace{x^6 \cdot x^6 \cdot x^6 \cdot x^6}_{4\ factors\ of\ x^6} \\ &= x^{6+6+6+6} \\ &= x^{24} \end{aligned}$$

Figure 1.5.3

Here we have 4 factors of x^6, which is equivalent to multiplying the exponents.

$$(x^6)^4 = x^{6\cdot 4} = x^{24} \textit{Power rule for exponents}$$

This describes the **power rule for exponents**[104]. Now we consider raising grouped products to a power. For example,

$$\begin{aligned} (x^2y^3)^4 &= x^2y^3 \cdot x^2y^3 \cdot x^2y^3 \cdot x^2y^3 \\ &= x^2 \cdot x^2 \cdot x^2 \cdot x^2 \cdot y^3 \cdot y^3 \cdot y^3 \cdot y^3 \quad \textit{Commutative property} \\ &= x^{2+2+2+2} \cdot y^{3+3+3+3} \\ &= x^8y^{12} \end{aligned}$$

After expanding, we are left with four factors of the product x^2y^3. This is equivalent to raising each of the original grouped factors to the fourth power and applying the power rule.

$$(x^2y^3)^4 = (x^2)^4 (y^3)^4 = x^8y^{12}$$

In general, this describes the use of the power rule for a product as well as the power rule for exponents. In summary, the rules of exponents streamline the process of working with algebraic expressions and will be used extensively as we move through our study of algebra. Given any positive integers m and n where $x, y \neq 0$ we have

Table 1.5.1

Product rule for exponents:	$x^m \cdot x^n = x^{m+n}$

Quotient rule for exponents:	$\frac{x^m}{x^n} = x^{m-n}$
Power rule for exponents:	$(x^m)^n = x^{m \cdot n}$
Power rule for a product:[105]	$(xy)^n = x^n y^n$
Power rule for a quotient:[106]	$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

These rules allow us to efficiently perform operations with exponents.

Example $1.5.1$:

Simplify: $\frac{10^4 \cdot 10^{12}}{10^3}$.

Solution

$$\begin{aligned} \frac{10^4 \cdot 10^{12}}{10^3} &= \frac{10^{16}}{10^3} \quad \textit{Product rule} \\ &= 10^{16-3} \quad \textit{Quotient rule} \\ &= 10^{13} \end{aligned}$$

Answer:

10^{13}

In the previous example, notice that we did not multiply the base 10 times itself. When applying the product rule, add the exponents and leave the base unchanged.

Example $1.5.2$:

Simplify: $(x^5 \cdot x^4 \cdot x)^2$.

Solution: Recall that the variable x is assumed to have an exponent of one, $x = x^1$.

$$\begin{aligned} (x^5 \cdot x^4 \cdot x)^2 &= (x^{5+4+1})^2 \\ &= (x^{10})^2 \\ &= x^{10 \cdot 2} \\ &= x^{20} \end{aligned}$$

Answer:

x^{20}

The base could in fact be any algebraic expression.

Example $1.5.3$:

Simplify: $(x+y)^9(x+y)^{13}$.

Solution: Treat the expression $(x+y)$ as the base.

$$\begin{aligned} (x+y)^9(x+y)^{13} &= (x+y)^{9+13} \\ &= (x+y)^{22} \end{aligned}$$

Answer:

$(x+y)^{22}$

The commutative property of multiplication allows us to use the product rule for exponents to simplify factors of an algebraic expression.

Example 1.5.4:

Simplify: $-8x^5y \cdot 3x^7y^3$.

Solution: Multiply the coefficients and add the exponents of variable factors with the same base.

$$\begin{aligned} -8x^5y \cdot 3x^7y^3 &= -8 \cdot 3 \cdot x^5 \cdot x^7 \cdot y^1 \cdot y^3 \quad \textit{Commutative property} \\ &= -24 \cdot x^{5+7} \cdot y^{1+3} \quad \textit{Power rule for exponents} \\ &= -24x^{12}y^4 \end{aligned}$$

Answer:

$-24x^{12}y^4$

Division involves the quotient rule for exponents.

Example 1.5.5:

Simplify: $\frac{33x^7y^5(x-y)^{10}}{11x^6y(x-y)^3}$.

Solution

$$\begin{aligned} \frac{33x^7y^5(x-y)^{10}}{11x^6y(x-y)^3} &= \frac{33}{11} \quad x^{7-6} \cdot y^{5-1} \cdot (x-y)^{10-3} \\ &= 3x^1y^4(x-y)^7 \end{aligned}$$

Answer:

$3xy^4(x-y)^7$

The power rule for a quotient allows us to apply that exponent to the numerator and denominator. This rule requires that the denominator is nonzero and so we will make this assumption for the remainder of the section.

Example 1.5.6:

Simplify: $\left(\frac{-4a^2b}{c^4}\right)^3$.

Solution: First apply the power rule for a quotient and then the power rule for a product.

$$\begin{aligned} \left(\frac{-4a^2b}{c^4}\right)^3 &= \frac{\left(-4a^2b\right)^3}{\left(c^4\right)^3} \quad \textit{Power rule for a quotient} \\ &= \frac{(-4)^3\left(a^2\right)^3(b)^3}{\left(c^4\right)^3} \textit{Power rule for a product} \\ &= \frac{-64a^6b^3}{c^{12}} \end{aligned}$$

Answer:

$-\frac{64a^6b^3}{c^{12}}$

Using the quotient rule for exponents, we can define what it means to have zero as an exponent. Consider the following calculation:

$$1 = \frac{25}{25} = \frac{5^2}{5^2} = 5^{2-2} = 5^0$$

Twenty-five divided by twenty-five is clearly equal to one, and when the quotient rule for exponents is applied, we see that a zero exponent results. In general, given any nonzero real number x and integer n,

$$1 = \frac{x^n}{x^n} = x^{n-n} = x^0$$

This leads us to the definition of **zero as an exponent**[107],

$$x^0 = 1 \, x \neq 0$$

It is important to note that 0^0 is indeterminate. If the base is negative, then the result is still positive one. In other words, any nonzero base raised to the zero power is defined to be equal to one. In the following examples assume all variables are nonzero.

Example 1.5.7:

Simplify:

a. $(-2x)^0$
b. $-2x^0$

Solution

a. Any nonzero quantity raised to the zero power is equal to 1.

$$(-2x)^0 = 1$$

b. In the example, $-2x^0$, the base is x, not $-2x$.

$$\begin{aligned} -2x^0 &= -2 \cdot x^0 \\ &= -2 \cdot 1 \\ &= -2 \end{aligned}$$

Noting that $2^0 = 1$ we can write,

$$\frac{1}{2^3} = \frac{2^0}{2^3} = 2^{0-3} = 2^{-3}$$

In general, given any nonzero real number x and integer n,

$$\frac{1}{x^n} = \frac{x^0}{x^n} = x^{0-n} = x^{-n} x \neq 0$$

This leads us to the definition of **negative exponents**[108]:

$$x^{-n} = \frac{1}{x^n} x \neq 0$$

An expression is completely simplified if it does not contain any negative exponents.

Example 1.5.8:

Simplify: $(-4x^2y)^{-2}$.

Solution

Rewrite the entire quantity in the denominator with an exponent of 2 and then simplify further.

$$\begin{aligned} (-4x^2y)^{-2} &= \frac{1}{(-4x^2y)^2} \\ &= \frac{1}{(-4)^2 (x^2)^2 (y)^2} \\ &= \frac{1}{16x^4y^2} \end{aligned}$$

Answer:

$\frac{1}{16x^4y^2}$

Sometimes negative exponents appear in the denominator.

Example 1.5.9:

Simplify: $\frac{x^{-3}}{y^{-4}}$.

Solution

$$\frac{x^{-3}}{y^{-4}} = \frac{\frac{1}{x^3}}{\frac{1}{y^4}} = \frac{1}{x^3} \cdot \frac{y^4}{1} = \frac{y^4}{x^3}$$

Answer:

$\frac{y^4}{x^3}$

The previous example suggests a property of **quotients with negative exponents**[109]. Given any integers m and n where $x \neq 0$ and $y \neq 0$, then

$$\frac{x^{-n}}{y^{-m}}=\frac{\frac{1}{x^n}}{\frac{1}{y^m}}=\frac{1}{x^n}\cdot\frac{y^m}{1}=\frac{y^m}{x^n}$$

This leads us to the property

$$\frac{x^{-n}}{y^{-m}}=\frac{y^m}{x^n}$$

In other words, negative exponents in the numerator can be written as positive exponents in the denominator and negative exponents in the denominator can be written as positive exponents in the numerator.

Example 1.5.10:

Simplify: $\frac{-5x^{-3}y^3}{z^{-4}}$.

Solution

Take care with the coefficient -5, recognize that this is the base and that the exponent is actually positive one: $-5=(-5)^1$. Hence, the rules of negative exponents do not apply to this coefficient; leave it in the numerator.

$$\begin{aligned}\frac{-5x^{-3}y^3}{z^{-4}}&=\frac{-5x^{-3}y^3}{z^{-4}}\\&=\frac{-5y^3z^4}{x^3}\end{aligned}$$

Answer:

$\frac{-5y^3z^4}{x^3}$

In summary, given integers m and n where $x,y\neq 0$ we have

Table 1.5.3

Zero exponent	$x^0=1$
Negative exponent	$x^{-n}=\frac{1}{x^n}$
Quotients with negative exponents	$\frac{x^{-n}}{y^{-m}}=\frac{y^m}{x^n}$

Furthermore, all of the rules of exponents defined so far extend to any integer exponents. We will expand the scope of these properties to include any real number exponents later in the course.

Exercise 1.5.1

Simplify: $\left(\frac{2x^{-2}y^3}{z}\right)^{-4}$.

Answer

$\frac{x^8z^4}{16y^{12}}$

www.youtube.com/v/EDlugO2Ooxs

Scientific Notation

Real numbers expressed using **scientific notation**[110] have the form,

$$a\times 10^n$$

where n is an integer and $1\leq a<10$.This form is particularly useful when the numbers are very large or very small. For example,

$$\begin{aligned}9,460,000,000,000,000m&=9.46\times 10^{15}\text{m} \quad \textit{One light year}\\0.000000000025\text{m}&=2.5\times 10^{-11}\text{m} \quad \textit{Raduis of a light year}\end{aligned}$$

It is cumbersome to write all the zeros in both of these cases. Scientific notation is an alternative, compact representation of these numbers. The factor 10^n indicates the power of ten to multiply the coefficient by to convert back to decimal form:

$$9.46\times10^{15}=9.46\times\overbrace{1{,}000{,}000{,}000{,}000{,}000}^{15\ zeros}=9{,}460{,}000{,}000{,}000{,}000$$

Figure $1.5.4$

This is equivalent to moving the decimal in the coefficient fifteen places to the right.

A negative exponent indicates that the number is very small:

$$2.5\times10^{-11}=2.5\times\frac{1}{10^{11}}=\frac{2.5}{\underbrace{100{,}000{,}000{,}000}_{11\ zeros}}=0.000000000025$$

Figure $1.5.5$

This is equivalent to moving the decimal in the coefficient eleven places to the left.

Converting a decimal number to scientific notation involves moving the decimal as well. Consider all of the equivalent forms of 0.00563 with factors of 10 that follow:

$$\begin{aligned} 0.00563 &= 0.0563\times10^{-1}\\ &= 0.563\times10^{-2}\\ &= 5.63\times10^{-3}\\ &= 56.3\times10^{-4}\\ &= 563\times10^{-5}\end{aligned}$$

While all of these are equal, 5.63×10^{-3} is the only form expressed in correct scientific notation. This is because the coefficient 5.63 is between 1 and 10 as required by the definition. Notice that we can convert 5.63×10^{-3} back to decimal form, as a check, by moving the decimal three places to the left.

Example $1.5.11$:

Write $1,075,000,000,000$ using scientific notation.

Solution

Here we count twelve decimal places to the left of the decimal point to obtain the number 1.075.

$$1,075,000,000,000=1.075\times10^{12}$$

Answer:

1.075×10^{12}

Example $1.5.12$:

Write 0.000003045 using scientific notation.

Solution

Here we count six decimal places to the right to obtain 3.045.

$$0.000003045=3.045\times10^{-6}$$

Answer:

3.045×10^{-6}

Often we will need to perform operations when using numbers in scientific notation. All the rules of exponents developed so far also apply to numbers in scientific notation.

Example $1.5.13$:

Multiply: $\left(4.36\times10^{-5}\right)\left(5.3\times10^{12}\right)$.

Solution

Use the fact that multiplication is commutative, and apply the product rule for exponents.

$$\begin{aligned}\left(4.36\times10^{-5}\right)\left(5.30\times10^{12}\right) &= (4.36\cdot5.30)\times\left(10^{-5}\cdot10^{12}\right)\\ &= 23.108\times10^{-5+12}\\ &= 2.3108\times10^{1}\times10^{7}\\ &= 2.3108\times10^{1+7}\\ &= 2.3108\times10^{8}\end{aligned}$$

Answer:

2.3108×10^{8}

Example 1.5.14:

Divide: $\left(3.24\times10^{8}\right)\div\left(9.0\times10^{-3}\right)$.

Solution

$$\begin{aligned}\frac{\left(3.24\times10^{8}\right)}{\left(9.0\times10^{-3}\right)} &= \left(\frac{3.24}{9.0}\right)\times\left(\frac{10^{8}}{10^{-3}}\right)\\ &= 0.36\times10^{8-(-3)}\\ &= 0.36\times10^{8+3}\\ &= 3.6\times10^{-1}\times10^{11}\\ &= 3.6\times10^{-1+11}\\ &= 3.6\times10^{10}\end{aligned}$$

Answer:

3.6×10^{10}

Example 1.5.15:

The speed of light is approximately 6.7×10^{8} miles per hour. Express this speed in miles per second.

Solution

A unit analysis indicates that we must divide the number by $3,600$.

$$\begin{aligned}6.7\times10^{8}\text{ miles per hour} &= \frac{6.7\times10^{8}\text{ miles}}{1\ \cancel{\text{hour}}}\cdot\left(\frac{1\ \cancel{\text{hour}}}{60\ \cancel{\text{minutes}}}\right)\cdot\left(\frac{1\ \cancel{\text{minutes}}}{60\text{ seconds}}\right)\\ &= \frac{6.7\times10^{8}\text{ miles}}{3600\text{ seconds}}\\ &= \left(\frac{6.7}{3600}\right)\times10^{8}\\ &\approx 0.0019\times10^{8}\quad \textit{rounded to two significant digits}\\ &= 1.9\times10^{-3}\times10^{8}\\ &= 1.9\times10^{-3+8}\\ &= 1.9\times10^{5}\end{aligned}$$

Answer:

The speed of light is approximately 1.9×10^{5} miles per second.

Example 1.5.16:

The Sun moves around the center of the galaxy in a nearly circular orbit. The distance from the center of our galaxy to the Sun is approximately $26,000$ light years. What is the circumference of the orbit of the Sun around the galaxy in meters?

Solution

One light-year measures 9.46×10^{15} meters. Therefore, multiply this by $26,000$ or 2.60×10^{4} to find the length of $26,000$ light years in meters.

$$\begin{aligned}\left(9.46\times10^{15}\right)\left(2.60\times10^{4}\right)&=9.46\cdot2.60\times10^{15}\cdot10^{4}\\&\approx24.6\times10^{19}\\&=2.46\times10^{1}\cdot10^{19}\\&=2.46\times10^{20}\end{aligned}$$

The radius r of this very large circle is approximately 2.46×10^{20} meters. Use the formula $C=2\pi r$ to calculate the circumference of the orbit.

$$\begin{aligned}C&=2\pi r\\&\approx2(3.14)\left(2.46\times10^{20}\right)\\&=15.4\times10^{20}\\&=1.54\times10^{1}\cdot10^{20}\\&=1.54\times10^{21}\end{aligned}$$

Answer:

The circumference of the Sun’s orbit is approximately 1.54×10^{21} meters.

Exercise 1.5.2

Divide: $\left(3.15\times10^{-5}\right)\div\left(12\times10^{-13}\right)$.

Answer

2.625×10^{7}

www.youtube.com/v/jOiRSs7hyW4

Key Takeaways

- When multiplying two quantities with the same base, add exponents: $x^m\cdot x^n=x^{m+n}$.
- When dividing two quantities with the same base, subtract exponents: $\frac{x^m}{x^n}=x^{m-n}$.
- When raising powers to powers, multiply exponents: $(x^m)^n=x^{m\cdot n}$.
- When a grouped quantity involving multiplication and division is raised to a power, apply that power to all of the factors in the numerator and the denominator: $(xy)^n=x^ny^n$ and $\left(\frac{x}{y}\right)^n=\frac{x^n}{y^n}$.
- Any nonzero quantity raised to the 0 power is defined to be equal to $1:x^0=1$.
- Expressions with negative exponents in the numerator can be rewritten as expressions with positive exponents in the denominator: $x^{-n}=\frac{1}{x^n}$.
- Expressions with negative exponents in the denominator can be rewritten as expressions with positive exponents in the numerator: $\frac{1}{x^{-m}}=x^m$.
- Take care to distinguish negative coefficients from negative exponents.
- Scientific notation is particularly useful when working with numbers that are very large or very small.

Exercise 1.5.3

Simplify. (Assume all variables represent nonzero numbers.)

1. $10^4\cdot10^7$
2. $7^3\cdot7^2$
3. $\dfrac{10^2\cdot10^4}{10^5}$
4. $\dfrac{7^5\cdot7^9}{7^2}$
5. $x^3\cdot x^2$
6. $y^5\cdot y^3$
7. $\dfrac{a^8\cdot a^6}{a^5}$
8. $\dfrac{b^4\cdot b^{10}}{b^8}$
9. $\dfrac{x^{2n}\cdot x^{3n}}{x^n}$

10. $\frac{x^n \cdot x^{8n}}{x^{3n}}$
11. $(x^5)^3$
12. $(y^4)^3$
13. $(x^4y^5)^3$
14. $(x^7y)^5$
15. $(x^2y^3z^4)^4$
16. $(xy^2z^3)^2$
17. $(-5x^2yz^3)^2$
18. $(-2xy^3z^4)^5$
19. $(x^2yz^5)^n$
20. $(xy^2z^3)^{2n}$
21. $(x \cdot x^3 \cdot x^2)^3$
22. $(y^2 \cdot y^5 \cdot y)^2$
23. $\frac{a^2 \cdot (a^4)^2}{a^3}$
24. $\frac{a \cdot a^3 \cdot a^2}{(a^2)^3}$
25. $(2x+3)^4(2x+3)^9$
26. $(3y-1)^7(3y-1)^2$
27. $(a+b)^3(a+b)^5$
28. $(x-2y)^7(x-2y)^3$
29. $5x^2y \cdot 3xy^2$
30. $-10x^3y^2 \cdot 2xy$
31. $-6x^2yz^3 \cdot 3xyz^4$
32. $2xyz^2\left(-4x^2y^2z\right)$
33. $3x^ny^{2n} \cdot 5x^2y$
34. $8x^{5n}y^n \cdot 2x^{2n}y$
35. $\frac{40x^5y^3z}{4x^2y^2z}$
36. $\frac{8x^2y^5z^3}{16x^2yz}$
37. $\frac{24a^8b^3(a-5b)^{10}}{8a^5b^3(a-5b)^2}$
38. $\frac{175m^9n^5(m+n)^7}{25m^8n(m+n)^3}$
39. $(-2x^4y^2z)^6$
40. $(-3xy^4z^7)^5$
41. $\left(\frac{-3ab^2}{2c^3}\right)^3$
42. $\left(\frac{-10a^3b}{3c^2}\right)^2$
43. $\left(\frac{-2xy^4}{z^3}\right)^4$
44. $\left(\frac{-7x^9y}{z^4}\right)^3$
45. $\left(\frac{xy^2}{z^3}\right)^n$
46. $\left(\frac{2x^2y^3}{z}\right)^n$
47. $(-5x)^0$
48. $(3x^2y)^0$
49. $-5x^0$
50. $3x^2y^0$
51. $(-2a^2b^0c^3)^5$
52. $(-3a^4b^2c^0)^4$

53. $\frac{(9x^3y^2z^0)^2}{3xy^2}$
54. $\frac{(-5x^0y^5z)^3}{25y^2z^0}$
55. $-2x^{-3}$
56. $(-2x)^{-2}$
57. $a^4 \cdot a^{-5} \cdot a^2$
58. $b^{-8} \cdot b^3 \cdot b^4$
59. $\dfrac{a^8 \cdot a^{-3}}{a^{-6}}$
60. $\dfrac{b^{-10} \cdot b^4}{b^{-2}}$
61. $10x^{-3}y^2$
62. $-3x^{-5}y^{-2}$
63. $3x^{-2}y^2z^{-1}$
64. $-5x^{-4}y^{-2}z^2$
65. $\dfrac{25x^{-3}y^2}{5x^{-1}y^{-3}}$
66. $\dfrac{-9x^{-1}y^3z^{-5}}{3x^{-2}y^2z^{-1}}$
67. $(-5x^{-3}y^2z)^{-3}$
68. $(-7x^2y^{-5}z^{-2})^{-2}$
69. $\left(\dfrac{2x^{-3}z}{y^2}\right)^{-5}$
70. $\left(\dfrac{5x^5z^{-2}}{2y^{-3}}\right)^{-3}$
71. $\left(\dfrac{12x^3y^2z}{2x^7yz^8}\right)^3$
72. $\left(\dfrac{150xy^8z^2}{90x^7y^2z}\right)^2$
73. $\left(\dfrac{-9a^{-3}b^4c^{-2}}{3a^3b^5c^{-7}}\right)^{-4}$
74. $\left(\dfrac{-15a^7b^5c^{-8}}{3a^{-6}b^2c^3}\right)^{-3}$

Answer

1. 10^{11}

3. 10

5. x^5

7. a^9

9. x^{4n}

11. x^{15}

13. $x^{12}y^{15}$

15. $x^8y^{12}z^{16}$

17. $25x^4y^2z^6$

19. $x^{2n}y^nz^{5n}$

21. x^{18}

23. a^7

25. $(2x+3)^{13}$

27. $(a+b)^8$

29. $15x^3y^3$

31. $-18x^3y^2z^7$

33. $15x^{n+2}y^{2n+1}$

35. $10x^3y$

37. $3a^3(a-5b)^8$

39. $64x^{24}y^{12}z^6$

41. $-\frac{27a^3b^6}{8c^9}$

43. $\frac{16x^4y^{16}}{z^{12}}$

45. $\frac{x^ny^{2n}}{z^{3n}}$

47. 1

49. -5

51. $-32a^{10}c^{15}$

53. $27x^5y^2$

55. $-\frac{2}{x^3}$

57. a

59. a^{11}

61. $\frac{10y^2}{x^3}$

63. $\frac{3y^2}{x^2z}$

65. $\frac{5y^5}{x^2}$

67. $-\frac{x^9}{125y^6z^3}$

69. $\frac{x^{15}y^{10}}{32z^5}$

71. $\frac{216y^3}{x^{12}z^{21}}$

73. $\frac{a^{24}b^4}{81c^{20}}$

Exercise $1.5.4$

The value in dollars of a new mobile phone can be estimated by using the formula $V=210(2t+1)^{-1}$, where t is the number of years after purchase.

1. How much was the phone worth new?
2. How much will the phone be worth in 1 year?
3. How much will the phone be worth in 3 years?
4. How much will the phone be worth in 10 years?
5. How much will the phone be worth in 100 years?
6. According to the formula, will the phone ever be worthless? Explain.
7. The height of a particular right circular cone is equal to the square of the radius of the base, $h=r^2$. Find a formula for the volume in terms of r.
8. A sphere has a radius $r=3x^2$.Find the volume in terms of x.

Answer

1. $\$210$

3. $\$30$
5. $\$1.04$
7. $V = \frac{1}{3}\pi r^4$

Exercise 1.5.5

Convert to a decimal number.

1. 5.2×10^8
2. 6.02×10^9
3. 1.02×10^{-6}
4. 7.44×10^{-5}

Answer

1. $520,000,000$
3. 0.00000102

Exercise 1.5.6

Rewrite using scientific notation.

1. $7,050,000$
2. $430,000,000,000$
3. 0.00005001
4. 0.000000231

Answer

1. 7.05×10^6
3. 5.001×10^{-5}

Exercise 1.5.7

Perform the operations.

1. $(1.2 \times 10^9)(3 \times 10^5)$
2. $(4.8 \times 10^{-5})(1.6 \times 10^{20})$
3. $(9.1 \times 10^{23})(3 \times 10^{10})$
4. $(5.5 \times 10^{12})(7 \times 10^{-25})$
5. $\frac{9.6\times10^{16}}{1.2\times10^{-4}}$
6. $\frac{4.8\times10^{-14}}{2.4\times10^{-6}}$
7. $\frac{4\times10^{-8}}{8\times10^{10}}$
8. $\frac{2.3\times10^{23}}{9.2\times10^{-3}}$
9. $987,000,000,000,000 \times 23,000,000$
10. $0.00000000024 \times 0.00000004$
11. $0.000000000522 \div 0.0000009$
12. $81,000,000,000 \div 0.0000648$
13. The population density of Earth refers to the number of people per square mile of land area. If the total land area on Earth is 5.751×10^7 square miles and the population in 2007 was estimated to be 6.67×10^9 people, then calculate the population density of Earth at that time.
14. In 2008 the population of New York City was estimated to be 8.364 million people. The total land area is 305 square miles. Calculate the population density of New York City.
15. The mass of Earth is 5.97×10^{24} kilograms and the mass of the Moon is 7.35×10^{22} kilograms. By what factor is the mass of Earth greater than the mass of the Moon?

16. The mass of the Sun is 1.99×10^{30} kilograms and the mass of Earth is 5.97×10^{24} kilograms. By what factor is the mass of the Sun greater than the mass of Earth? Express your answer in scientific notation.
17. The radius of the Sun is 4.322×10^5 miles and the average distance from Earth to the Moon is 2.392×10^5 miles. By what factor is the radius of the Sun larger than the average distance from Earth to the Moon?
18. One light year, 9.461×10^{15} meters, is the distance that light travels in a vacuum in one year. If the distance from our Sun to the nearest star, Proxima Centauri, is estimated to be 3.991×10^{16} meters, then calculate the number of years it would take light to travel that distance.
19. It is estimated that there are about 1 million ants per person on the planet. If the world population was estimated to be 6.67 billion people in 2007, then estimate the world ant population at that time.
20. The radius of the earth is 6.3×10^6 meters and the radius of the sun is 7.0×10^8 meters. By what factor is the radius of the Sun larger than the radius of the Earth?
21. A gigabyte is 1×10^9 bytes and a megabyte is 1×10^6 bytes. If the average song in the MP3 format consumes about 4.5 megabytes of storage, then how many songs will fit on a 4-gigabyte memory card?
22. Water weighs approximately 18 grams per mole. If one mole is about 6×10^{23} molecules, then approximate the weight of each molecule of water.

Answer

1. 3.6×10^{14}

3. 2.73×10^{34}

5. 8×10^{20}

7. 5×10^{-19}

9. 2.2701×10^{22}

11. 5.8×10^{-4}

13. About 116 people per square mile

15. 81.2

17. 1.807

19. $6.67 \times 10^{15} ants$

21. Approximately 889 songs

Exercise 1.5.8

1. Use numbers to show that $(x+y)^n \neq x^n + y^n$.
2. Why is 0^0 indeterminate?
3. Explain to a beginning algebra student why $2^2 \cdot 2^3 \neq 4^5$.
4. René Descartes (1637) established the usage of exponential form: a^2, a^3, and so on. Before this, how were exponents denoted?

Answer

1. Answer may vary

3. Answer may vary

Footnotes

[103] $x^m \cdot x^n = x^{m+n}$; the product of two expressions with the same base can be simplified by adding the exponents.

[104] $(x^m)^n = x^{mn}$; a power raised to a power can be simplified by multiplying the exponents.

[105] $(xy)^n = x^n y^n$; if a product is raised to a power, then apply that power to each factor in the product.

[106] $(xy)^n = x^n y^n$; if a quotient is raised to a power, then apply that power to the numerator and the denominator.

[107] $x^0 = 1$; any nonzero base raised to the 0 power is defined to be 1.

[108] $x^{-n} = \frac{1}{x^n}$, given any integer n, where x is nonzero.

[109] $\frac{x^{-n}}{y^{-m}} = \frac{y^m}{x^n}$, given any integers m and n, where $x \neq 0$ and $y \neq 0$.

[110] Real numbers expressed the form $a \times 10^n$, where n is an integer and $1 \leq a < 10$.

[111] $\frac{x^m}{x^n} = x^{m-n}$; the quotient of two expressions with the same base can be simplified by subtracting the exponents.

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1.6: Polynomials and Their Operations

Learning Objectives

- Identify a polynomial and determine its degree.
- Add and subtract polynomials.
- Multiply and divide polynomials.

Definitions

A **polynomial**[112] is a special algebraic expression with terms that consist of real number coefficients and variable factors with whole number exponents. Some examples of polynomials follow:

Table 1.6.1

$3x^2$	$7xy+5$	$\frac{3}{2}x^3+3x^2-\frac{1}{2}x+1$	$6x^2y-4xy^3+7$

The **degree of a term**[113] in a polynomial is defined to be the exponent of the variable, or if there is more than one variable in the term, the degree is the sum of their exponents. Recall that $x^0=1$; any constant term can be written as a product of x^0 and itself. Hence the degree of a constant term is 0.

Table 1.6.2

Term	Degree
$3x^2$	2
$6x^2y$	$2+1=3$
$7a^2b^3$	$2+3=5$
8	0, since $8=8x^0$
$2x$	1, since $2x=2x^1$

The **degree of a polynomial**[114] is the largest degree of all of its terms.

Table 1.6.3

Polynomial	Degree
$4x^5-3x^3+2x-1$	5
$6x^2y-5xy^3+7$	4, because $5xy^3$ has degree 4.
$\frac{1}{2}x+\frac{5}{4}$	1, because $\frac{1}{2}x=\frac{1}{2}x^1$

Of particular interest are **polynomials with one variable**[115], where each term is of the form a_nx^n. Here a_n is any real number and n is any whole number. Such polynomials have the standard form:

$$a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$$

Typically, we arrange terms of polynomials in descending order based on the degree of each term. The **leading coefficient**[116] is the coefficient of the variable with the highest power, in this case, a_n.

Example 1.6.1:

Write in standard form: $3x-4x^2+5x^3+7-2x^4$.

Solution

Since terms are defined to be separated by addition, we write the following:

$$\begin{aligned}&3x-4x^2+5x^3+7-2x^4\\&=3x+(-4)x^2+5x^3+7+(-2)x^4\end{aligned}$$

In this form, we can see that the subtraction in the original corresponds to negative coefficients. Because addition is commutative, we can write the terms in descending order based on the degree as follows:

$$\begin{aligned} &= (-2)x^4 + 5x^3 + (-4)x^2 + 3x + 7 \\ &= -2x^4 + 5x^3 - 4x^2 + 3x + 7 \end{aligned}$$

Answer:

$-2x^4 + 5x^3 - 4x^2 + 3x + 7$

We classify polynomials by the number of terms and the degree:

Table $1.6.4$

Expression	Classification	Degree
$5x^7$	**Monomial**[117] (one term)	7
$8x^6 - 1$	**Binomial**[118] (two terms)	6
$-3x^2 + x - 1$	**Trinomial**[119] (three terms)	2
$5x^3 - 2x^2 + 3x - 6$	**Polynomial** (many terms)	3

We can further classify polynomials with one variable by their degree:

Table $1.6.5$

Polynomial	Name
5	**Constant**[120] (degree 0)
$2x + 1$	**Linear**[121] (degree 1)
$3x^2 + 5x - 3$	**Quadratic**[122] (degree 2)
$x^3 + x^2 + x + 1$	**Cubic**[123] (degree 3)
$7x^4 + 3x^3 - 7x + 8$	Fourth-degree polynomial

In this text, we call any polynomial of degree $n \geq 4$ an nth-degree polynomial. In other words, if the degree is 4, we call the polynomial a fourth-degree polynomial. If the degree is 5, we call it a fifth-degree polynomial, and so on.

Example $1.6.2$

State whether the following polynomial is linear or quadratic and give the leading coefficient: $25 + 4x - x^2$.

Solution

The highest power is 2; therefore, it is a quadratic polynomial. Rewriting in standard form we have

$$-x^2 + 4x + 25$$

Here $-x^2 = -1x^2$ and thus the leading coefficient is -1.

Answer:

Quadratic; leading coefficient: -1

Adding and Subtracting Polynomials

We begin by simplifying algebraic expressions that look like $+(a+b)$ or $-(a+b)$. Here, the coefficients are actually implied to be $+1$ and -1 respectively and therefore the distributive property applies. Multiply each term within the parentheses by these factors as follows:

$$\begin{aligned} +(a+b) &= +1(a+b) = (+1)a + (+1)b = a + b \\ -(a+b) &= -1(a+b) = (-1)a + (-1)b = -a - b \end{aligned}$$

Use this idea as a means to eliminate parentheses when adding and subtracting polynomials.

Example 1.6.3:

Add: $9x^2 + (x^2 - 5)$.

Solution

The property $+(a+b) = a+b$ allows us to eliminate the parentheses, after which we can then combine like terms.

$$\begin{aligned} 9x^2 + (x^2 - 5) &= 9x^2 + x^2 - 5 \\ &= 10x^2 - 5 \end{aligned}$$

Answer:

$10x^2 - 5$

Example 1.6.4:

Add: $(3x^2y^2 - 4xy + 9) + (2x^2y^2 - 6xy - 7)$.

Solution

Remember that the variable parts have to be exactly the same before we can add the coefficients.

$$\begin{aligned} &(3x^2y^2 - 4xy + 9) + (2x^2y^2 - 6xy - 7) \\ &= \underline{3x^2y^2} - \underline{\underline{4xy}} + \underline{\underline{\underline{9}}} + \underline{2x^2y^2} - \underline{\underline{6xy}} - \underline{\underline{\underline{7}}} \\ &= 5x^2y^2 - 10xy + 2 \end{aligned}$$

Answer:

$5x^2y^2 - 10xy + 2$

When subtracting polynomials, the parentheses become very important.

Example 1.6.5:

Subtract: $4x^2 - (3x^2 + 5x)$.

Solution

The property $-(a+b) = -a - b$ allows us to remove the parentheses after subtracting each term.

$$\begin{aligned} 4x^2 - (3x^2 + 5x) &= 4x^2 - 3x^2 - 5x \\ &= x^2 - 5x \end{aligned}$$

Answer:

$x^2 - 5x$

Subtracting a quantity is equivalent to multiplying it by -1.

Example 1.6.6:

Subtract: $(3x^2 - 2xy + y^2) - (2x^2 - xy + 3y^2)$.

Solution

Distribute the -1, remove the parentheses, and then combine like terms. Multiplying the terms of a polynomial by -1 changes all the signs.

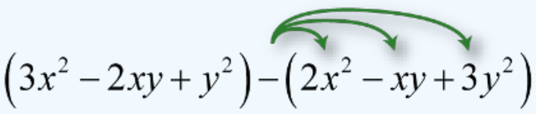

Figure 1.6.1

$$\begin{aligned}&= 3x^2-2xy+y^2-2x^2+xy-3y^2\\ &= x^2-xy-2y^2\end{aligned}$$

Answer:

$x^2-xy-2y^2$

Exercise 1.6.1

Subtract: $(7a^2-2ab+b^2)-(a^2-2ab+5b^2)$.

Answer

$6a^2-4b^2$

www.youtube.com/v/IDtREB_PQ3A

Multiplying Polynomials

Use the product rule for exponents, $x^m \cdot x^n = x^{m+n}$, to multiply a monomial times a polynomial. In other words, when multiplying two expressions with the same base, add the exponents. To find the product of monomials, multiply the coefficients and add the exponents of variable factors with the same base. For example,

$$\begin{aligned}7x^4\cdot 8x^3 &= 7\cdot 8\cdot x^4\cdot x^3 && \textit{Commutative property}\\ &= 56x^{4+3} && \textit{Product rule for exponents}\\ &= 56x^7\end{aligned}$$

To multiply a polynomial by a monomial, apply the distributive property, and then simplify each term.

Example 1.6.7:

Multiply: $5xy^2\,(2x^2y^2-xy+1)$.

Solution

Apply the distributive property and then simplify.

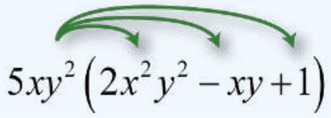

Figure 1.6.2

$$\begin{aligned}&= 5xy^2\cdot 2x^2y^2-5xy^2\cdot xy+5xy^2\cdot 1\\ &= 10x^3y^4-5x^2y^3+5xy^2\end{aligned}$$

Answer:

$10x^3y^4-5x^2y^3+5xy^2$

To summarize, multiplying a polynomial by a monomial involves the distributive property and the product rule for exponents. Multiply all of the terms of the polynomial by the monomial. For each term, multiply the coefficients and add exponents of variables where the bases are the same.

In the same manner that we used the distributive property to distribute a monomial, we use it to distribute a binomial.

$$\begin{aligned}(a+b)(c+d) &= (a+b)\cdot c+(a+b)\cdot d\\ &= ac+bc+ad+bd\\ &= ac+ad+bc+bd\end{aligned}$$

Here we apply the distributive property multiple times to produce the final result. This same result is obtained in one step if we apply the distributive property to a and b separately as follows:

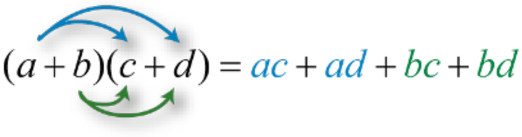

Figure 1.6.3

This is often called the FOIL method. Multiply the first, outer, inner, and then last terms.

Example 1.6.8:

Multiply: $(6x-1)(3x-5)$.

Solution

Distribute $6x$ and -1 and then combine like terms.

$$\begin{aligned}(6x-1)(3x-5) &= 6x\cdot 3x - 6x\cdot 5 + (-1)\cdot 3x - (-1)\cdot 5\\ &= 18x^2 - 30x - 3x + 5\\ &= 18x^2 - 33x + 5\end{aligned}$$

Answer:

$18x^2 - 33x + 5$

Consider the following two calculations:

Table 1.6.6

$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a^2+ab+ba+b^2\\ &= a^2+ab+ab+b^2\\ &= a^2+2ab+b^2\end{aligned}$	$\begin{aligned}(a-b)^2 &= (a-b)(a-b)\\ &= a^2-ab-ba+b^2\\ &= a^2-ab-ab+b^2\\ &= a^2-2ab+b^2\end{aligned}$

This leads us to two formulas that describe **perfect square trinomials**[124]:

$$\begin{aligned}(a+b)^2 &= a^2+2ab+b^2\\ (a-b)^2 &= a^2-2ab+b^2\end{aligned}$$

We can use these formulas to quickly square a binomial.

Example 1.6.9:

Multiply: $(3x+5)^2$

Solution

Here $a = 3x$ and $b = 5$. Apply the formula:

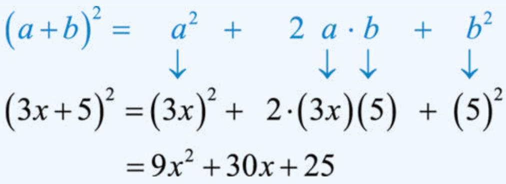
$(a+b)^2 = a^2 + 2\,a\cdot b + b^2$
$(3x+5)^2 = (3x)^2 + 2\cdot(3x)(5) + (5)^2$
$= 9x^2 + 30x + 25$

Figure 1.6.4

Answer:

$9x^2 + 30x + 25$

This process should become routine enough to be performed mentally. Our third special product follows:

$$\begin{aligned}(a+b)(a-b) &= a^2 - ab + ba - b^2\\ &= a^2 - ab + ab - b^2\\ &= a^2 - b^2\end{aligned}$$

This product is called **difference of squares**[125]:

$$(a+b)(a-b)=a^2-b^2$$

The binomials $(a+b)$ and $(a-b)$ are called **conjugate binomials**[126]. When multiplying conjugate binomials the middle terms are opposites and their sum is zero; the product is itself a binomial.

Example 1.6.10:

Multiply: $(3xy+1)(3xy-1)$.

Solution

$$\begin{aligned}(3xy+1)(3xy-1) &= (3xy)^2-3xy+3xy-1^2\\ &= 9x^2y^2-1\end{aligned}$$

Answer:

$9x^2y^2-1$

Exercise 1.6.2

Multiply: $(x^2+5y^2)(x^2-5y^2)$.

Answer

(x^4-25y^4)

www.youtube.com/v/p7R3FdPp6_s

Example 1.6.11:

Multiply: $(5x-2)^3$.

Solution

Here we perform one product at a time.

$$\begin{aligned}(5x-2)^3 &= (5x-2)\underbrace{(5x-2)(5x-2)}_{\text{Multiply first.}}\\ &= (5x-2)(25x^2-10x-10x+4)\\ &= (5x-2)(25x^2-20x+4)\\ &= 125x^3-100x^2+20x-50x^2+40x-8\\ &= 125x^3-150x^2+60x-8\end{aligned}$$

Figure 1.6.5

Answer:

$125x^2-150x^2+60x-8$

Dividing Polynomials

Use the quotient rule for exponents, $\frac{x^m}{x^n}=x^{m-n}$, to divide a polynomial by a monomial. In other words, when dividing two expressions with the same base, subtract the exponents. In this section, we will assume that all variables in the denominator are nonzero.

Example 1.6.12:

Divide: $\frac{24x^7y^5}{8x^3y^2}$.

Solution

Divide the coefficients and apply the quotient rule by subtracting the exponents of the like bases.

$$\begin{aligned}\frac{24x^7y^5}{8x^3y^2} &= \frac{24}{8}x^{7-3}y^{5-2}\\ &= 3x^4y^3\end{aligned}$$

Answer:

$3x^4y^3$

When dividing a polynomial by a monomial, we may treat the monomial as a common denominator and break up the fraction using the following property:

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

Applying this property will result in terms that can be treated as quotients of monomials.

Example 1.6.13:

Divide: $\frac{-5x^4+25x^3-15x^2}{5x^2}$.

Solution

Break up the fraction by dividing each term in the numerator by the monomial in the denominator, and then simplify each term.

$$\begin{aligned}\frac{-5x^4+25x^3-15x^2}{5x^2} &= -\frac{5x^4}{5x^2} + \frac{25x^3}{5x^2} - \frac{15x^2}{5x^2}\\ &= -\frac{5}{5}x^{4-2} + \frac{25}{5}x^{3-2} - \frac{15}{5}x^{2-2}\\ &= -1x^2 + 5x^1 - 3x^0\\ &= -x^2 + 5x - 3 \cdot 1\end{aligned}$$

Answer:

$-x^2+5x-3$

We can check our division by multiplying our answer, the quotient, by the monomial in the denominator, the divisor, to see if we obtain the original numerator, the dividend.

Table 1.6.7

$\frac{Dividend}{Divisor} = Quotient$	$\frac{-5x^4+25x^3-15x^2}{5x^2} = -x^2+5x-3$
or	or
$Dividend = Divisor \cdot Quotient$	$-5x^4+25x^3-15x^2 = 5x^2(-x^2+5x-3)$

The same technique outlined for dividing by a monomial *does not* work for polynomials with two or more terms in the denominator. In this section, we will outline a process called **polynomial long division**[127], which is based on the division algorithm for real numbers. For the sake of clarity, we will assume that all expressions in the denominator are nonzero.

Example 1.6.14:

Divide $\frac{x^3+3x^2-8x-4}{x-2}$:

Solution

Here $x-2$ is the divisor and x^3+3x^2-8x-4 is the dividend. To determine the first term of the quotient, divide the leading term of the dividend by the leading term of the divisor.

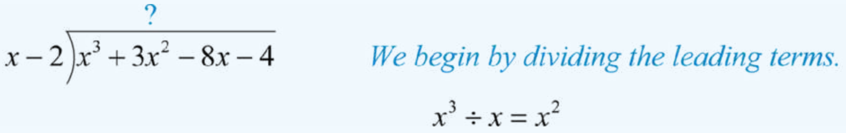
?
$x-2\overline{)x^3+3x^2-8x-4}$
We begin by dividing the leading terms.
$x^3 \div x = x^2$

Figure 1.6.6

Multiply the first term of the quotient by the divisor, remembering to distribute, and line up like terms with the dividend.

$$\begin{array}{r}
x^2\\
x-2\overline{)x^3+3x^2-8x-4}\\
\underline{x^3-2x^2}
\end{array}$$

Distribute and line up like terms.

$$x^2\cdot(x-2)=x^3-2x^2$$

Figure 1.6.7

Subtract the resulting quantity from the dividend. Take care to subtract both terms.

$$\begin{array}{r}
x^2\\
x-2\overline{)x^3+3x^2-8x-4}\\
\underline{-(x^3-2x^2)}\\
5x^2
\end{array}$$

Subtract.

$$-(x^3-2x^2)=-x^3+2x^2$$

Figure 1.6.8

Bring down the remaining terms and repeat the process.

$$\begin{array}{r}
x^2\quad ?\\
x-2\overline{)x^3+3x^2-8x-4}\\
\underline{-(x^3-2x^2)}\\
5x^2-8x-4
\end{array}$$

Bring down the remaining terms.

Figure 1.6.9

Notice that the leading term is eliminated and that the result has a degree that is one less. The complete process is illustrated below:

$$\begin{array}{r}
x^2+5x+2\\
x-2\overline{)x^3+3x^2-8x-4}\\
\underline{-(x^3-2x^2)}\\
5x^2-8x-4\\
\underline{-(5x^2-10x)}\\
2x-4\\
\underline{-(2x-4)}\\
0
\end{array}$$

Figure 1.6.10

Polynomial long division ends when the degree of the remainder is less than the degree of the divisor. Here, the remainder is 0. Therefore, the binomial divides the polynomial evenly and the answer is the quotient shown above the division bar.

$$\frac{x^3+3x^2-8x-4}{x-2}=x^2+5x+2$$

To check the answer, multiply the divisor by the quotient to see if you obtain the dividend as illustrated below:

$$x^3+3x^2-8x-4=(x-2)\left(x^2+5x+2\right)$$

This is left to the reader as an exercise.

Answer:

x^2+5x+2

Next, we demonstrate the case where there is a nonzero remainder.

$$\begin{array}{r} \textit{Quotient} \\ \textit{Divisor}\overline{)\textit{Dividend}} \\ \vdots \\ \hline \textit{Remainder} \end{array}$$

Figure 1.6.11

Just as with real numbers, the final answer adds to the quotient the fraction where the remainder is the numerator and the divisor is the denominator. In general, when dividing we have:

$$\frac{\textit{Dividend}}{\textit{Divisor}}=\textit{Quotient}+\frac{\textit{Remainder}}{\textit{Divisor}}$$

If we multiply both sides by the divisor we obtain,

$$\textit{Dividend}=\textit{Quotient}\times\textit{Divisor}+\textit{Remainder}$$

Example 1.6.15:

Divide: $\frac{6x^2-5x+3}{2x-1}$.

Solution

Since the denominator is a binomial, begin by setting up polynomial long division.

$$\begin{array}{r} ? \\ 2x-1\overline{)\,6x^2-5x+3} \end{array}$$

Figure 1.6.12

To start, determine what monomial times $2x-1$ results in a leading term $6x^2$.This is the quotient of the given leading terms: $(6x^2)\div(2x)=3x$. Multiply $3x$ times the divisor $2x-1$, and line up the result with like terms of the dividend.

$$\begin{array}{r} 3x \\ 2x-1\overline{)6x^2-5x+3} \\ 6x^2-3x \end{array}$$

Figure 1.6.13

Subtract the result from the dividend and bring down the constant term $+3$.

$$\begin{array}{r} 3x\quad ? \\ 2x-1\overline{)6x^2-5x+3} \\ \underline{-(6x^2-3x)} \\ -2x+3 \end{array}$$

Figure 1.6.14

Subtracting eliminates the leading term. Multiply $2x-1$ by -1 and line up the result.

$$\begin{array}{r} 3x-1 \\ 2x-1\overline{)6x^2-5x+3} \\ \underline{-(6x^2-3x)} \\ -2x+3 \\ -2x+1 \end{array}$$

Figure 1.6.15

Subtract again and notice that we are left with a remainder.

$$\begin{array}{rl}
 & 3x-1 \\
2x-1 & \overline{)\,6x^2-5x+3} \\
 & \underline{-(6x^2-3x)} \\
 & -2x+3 \\
 & \underline{-(-2x+1)} \\
 & 2
\end{array}$$

Figure 1.6.16

The constant term 2 has degree 0 and thus the division ends. Therefore,

$$\frac{6x^2-5x+3}{2x-1}=3x-1+\frac{2}{2x-1}$$

To check that this result is correct, we multiply as follows:

$$\begin{aligned} quotient\times divisor+remainder &=(3x-1)(2x-1)+2 \\ &=6x^2-3x-2x+1+2 \\ &=6x^2-5x+2=dividend\ ✓ \end{aligned}$$

Answer:

$3x-1+\frac{2}{2x-1}$

Occasionally, some of the powers of the variables appear to be missing within a polynomial. This can lead to errors when lining up like terms. Therefore, when first learning how to divide polynomials using long division, fill in the missing terms with zero coefficients, called **placeholders**[128].

Example $1.6.16$:

Divide: $\frac{27x^3+64}{3x+4}$.

Solution

Notice that the binomial in the numerator does not have terms with degree 2 or 1. The division is simplified if we rewrite the expression with placeholders:

$$27x^3+64=27x^3+0x^2+0x+64$$

Set up polynomial long division:

$$\begin{array}{rl}
 & ?\\
3x+4 & \overline{)\,27x^3+0x^2+0x+64}
\end{array}$$

Figure 1.6.17

We begin with 27x3÷3x=9x2 and work the rest of the division algorithm.

$$\begin{array}{rl}
 & 9x^2-12x+16 \\
3x+4 & \overline{)\,27x^3+0x^2+0x+64} \\
 & \underline{-(27x^3+36x^2)} \\
 & -36x^2+0x+64 \\
 & \underline{-(-36x^2-48x)} \\
 & 48x+64 \\
 & \underline{-(48x+64)} \\
 & 0
\end{array}$$

Figure 1.6.18

Answer:

$9x^2-12x+16$

Example 1.6.17:

Divide: $\frac{3x^4-2x^3+6x^2+23x-7}{x^2-2x+5}$.

Solution

$$\begin{array}{r} ? \\ x^2-2x+5\overline{\big)3x^4-2x^3+6x^2+23x-7} \end{array}$$

Figure 1.6.19

Begin the process by dividing the leading terms to determine the leading term of the quotient $3x^4 \div x^2 = 3x^2$. Take care to distribute and line up the like terms. Continue the process until the remainder has a degree less than 2.

$$\begin{array}{r} 3x^2+4x-1 \\ x^2-2x+5\overline{\big)3x^4-2x^3+6x^2+23x-7} \\ \underline{-\left(3x^4-6x^3+15x^2\right)} \qquad\qquad \\ 4x^3-9x^2+23x-7 \\ \underline{-\left(4x^3-8x^2+20x\right)} \quad \\ -x^2+3x-7 \\ \underline{-\left(-x^2+2x-5\right)} \\ x-2 \end{array}$$

Figure 1.6.20

The remainder is $x-2$. Write the answer with the remainder:

$\frac{3x^4-2x^3+6x^2+23x-7}{x^2-2x+5}=3x^2+4x-1+\frac{x-2}{x^2-2x+5}$

Answer:

$3x^2+4x-1+\frac{x-2}{x^2-2x+5}$

Polynomial long division takes time and practice to master. Work lots of problems and remember that you may check your answers by multiplying the quotient by the divisor (and adding the remainder if present) to obtain the dividend.

Exercise 1.6.3

Divide: $\frac{6x^4-13x^3+9x^2-14x+6}{3x-2}$.

Answer

$2x^3-3x^2+x-4-\frac{2}{3x-2}$

www.youtube.com/v/K9NRVMreKAQ

Key Takeaways

- Polynomials are special algebraic expressions where the terms are the products of real numbers and variables with whole number exponents.
- The degree of a polynomial with one variable is the largest exponent of the variable found in any term. In addition, the terms of a polynomial are typically arranged in descending order based on the degree of each term.

- When adding polynomials, remove the associated parentheses and then combine like terms. When subtracting polynomials, distribute the -1, remove the parentheses, and then combine like terms.
- To multiply polynomials apply the distributive property; multiply each term in the first polynomial with each term in the second polynomial. Then combine like terms.
- When dividing by a monomial, divide all terms in the numerator by the monomial and then simplify each term.
- When dividing a polynomial by another polynomial, apply the division algorithm.

Exercise 1.6.4

Write the given polynomials in standard form.

1. $1-x-x^2$
2. $y-5+y^2$
3. $y-3y^2+5-y^3$
4. $8-12a^2+a^3-a$
5. $2-x^2+6x-5x^3+x^4$
6. $a^3-5+a^2+2a^4-a^5+6a$

Answer

1. $-x^2-x+1$

3. $-y^3-3y^2+y+5$

5. $x^4-5x^3-x^2+6x+2$

Exercise 1.6.5

Classify the given polynomial as a monomial, binomial, or trinomial and state the degree.

1. x^2-x+2
2. $5-10x^3$
3. $x^2y^2+5xy-6$
4. $-2x^3y^2$
5. x^4-1
6. 5

Answer

1. 7. Trinomial; degree 2

3. Trinomial; degree 4

5. Binomial; degree 4

Exercise 1.6.6

State whether the polynomial is linear or quadratic and give the leading coefficient.

1. $1-9x^2$
2. $10x^2$
3. $2x-3$
4. $100x$
5. $5x^2+3x-1$
6. $x-1$
7. $x-6-2x^2$
8. $1-5x$

Answer

1. Quadratic, -9

3. Linear, 2

5. Quadratic, 5

7. Quadratic, -2

Exercise 1.6.7

1. $(5x^2-3x-2)+(2x^2-6x+7)$
2. $(x^2+7x-12)+(2x^2-x+3)$
3. $(x^2+5x+10)+(x^2-10)$
4. $(x^2-1)+(4x+2)$
5. $(10x^2+3x-2)-(x^2-6x+1)$
6. $(x^2-3x-8)-(2x^2-3x-8)$
7. $(\frac{2}{3}x^2+\frac{3}{4}x-1)-(\frac{1}{6}x^2+\frac{5}{2}x-\frac{1}{2})$
8. $(\frac{4}{5}x^2-\frac{5}{8}x+\frac{10}{6})-(\frac{3}{10}x^2-\frac{2}{3}x+\frac{3}{5})$
9. $(x^2y^2+7xy-5)-(2x^2y^2+5xy-4)$
10. $(x^2-y^2)-(x^2+6xy+y^2)$
11. $(a^2b^2+5ab-2)+(7ab-2)-(4-a^2b^2)$
12. $(a^2+9ab-6b^2)-(a^2-b^2)+7ab$
13. $(10x^2y-8xy+5xy^2)-(x^2y-4xy)+(xy^2+4xy)$
14. $(2m^2n-6mn+9mn^2)-(m^2n+10mn)-m^2n$
15. $(8x^2y^2-5xy+2)-(x^2y^2+5)+(2xy-3)$
16. $(x^2-y^2)-(5x^2-2xy-y^2)-(x^2-7xy)$
17. $(\frac{1}{6}a^2-2ab+\frac{3}{4}b^2)-(\frac{5}{3}a^2+\frac{4}{5}b^2)+\frac{11}{8}ab$
18. $(\frac{5}{2}x^2-2y^2)-(\frac{7}{5}x^2-\frac{1}{2}xy+\frac{7}{3}y^2)-\frac{1}{2}xy$
19. $(x^{2n}+5x^n-2)+(2x^{2n}-3x^n-1)$
20. $(7x^{2n}-x^n+5)-(6x^{2n}-x^n-8)$
21. Subtract $4y-3$ from $y^2+7y-10$.
22. Subtract x^2+3x-2 from $2x^2+4x-1$.
23. A right circular cylinder has a height that is equal to the radius of the base, $h=r$. Find a formula for the surface area in terms of h.
24. A rectangular solid has a width that is twice the height and a length that is 3 times that of the height. Find a formula for the surface area in terms of the height.

Answer

1. $7x^2-9x+5$

3. $2x^2+5x$

5. $9x^2+9x-3$

7. $\frac{1}{2}x^2-\frac{7}{4}x-\frac{1}{2}$

9. $-x^2y^2+2xy-1$

11. $2a^2b^2+12ab-8$

13. $9x^2y+6xy^2$

15. $7x^2y^2-3xy-6$

17. $-\frac{3}{2}a^2-\frac{5}{8}ab-\frac{1}{20}b^2$

19. $-\frac{3}{2}a^2-\frac{5}{8}ab-\frac{1}{20}b^2$

21. y^2+3y-7

23. $SA=4\pi h^2$

Exercise 1.6.8

Multiply

1. $-8x^2 \cdot 2x$
2. $-10x^2y \cdot 5x^3y^2$
3. $2x(5x-1)$
4. $-4x(3x-5)$
5. $7x^2(2x-6)$
6. $-3x^2\left(x^2-x+3\right)$
7. $-5y^4\left(y^2-2y+3\right)$
8. $\frac{5}{2}a^3\left(24a^2-6a+4\right)$
9. $2xy\left(x^2-7xy+y^2\right)$
10. $-2a^2b\left(a^2-3ab+5b^2\right)$
11. $x^n\left(x^2+x+1\right)$
12. $x^n\left(x^{2n}-x^n-1\right)$
13. $(x+4)(x-5)$
14. $(x-7)(x-6)$
15. $(2x-3)(3x-1)$
16. $(9x+1)(3x+2)$
17. $\left(3x^2-y^2\right)\left(x^2-5y^2\right)$
18. $\left(5y^2-x^2\right)\left(2y^2-3x^2\right)$
19. $(3x+5)(3x-5)$
20. $(x+6)(x-6)$
21. $\left(a^2-b^2\right)\left(a^2+b^2\right)$
22. $(ab+7)(ab-7)$
23. $\left(4x-5y^2\right)\left(3x^2-y\right)$
24. $(xy+5)(x-y)$
25. $(x-5)\left(x^2-3x+8\right)$
26. $(2x-7)\left(3x^2-x+1\right)$
27. $\left(x^2+7x-1\right)\left(2x^2-3x-1\right)$
28. $\left(4x^2-x+6\right)\left(5x^2-4x-3\right)$
29. $(x+8)^2$
30. $(x-3)^2$
31. $(2x-5)^2$
32. $(3x+1)^2$
33. $(a-3b)^2$
34. $(7a-b)^2$
35. $\left(x^2+2y^2\right)^2$
36. $\left(x^2-6y\right)^2$
37. $\left(a^2-a+5\right)^2$
38. $\left(x^2-3x-1\right)^2$
39. $(x-3)^3$
40. $(x+2)^3$
41. $(3x+1)^3$
42. $(2x-3)^3$
43. $(x+2)^4$
44. $(x-3)^4$
45. $(2x-1)^4$
46. $(3x-1)^4$
47. $\left(x^{2n}+5\right)\left(x^{2n}-5\right)$
48. $\left(x^n-1\right)\left(x^{2n}+4x^n-3\right)$
49. $\left(x^{2n}-1\right)^2$

50. $(x^{3n}+1)^2$
51. Find the product of $3x-2$ and x^2-5x-2.
52. Find the product of x^2+4 and x^3-1.
53. Each side of a square measures $3x^3$ units. Determine the area in terms of x.
54. Each edge of a cube measures $2x^2$ units. Determine the volume in terms of x.

Answer

1. $-16x^3$
3. $10x^2-2x$
5. $14x^3-42x^2$
7. $-5y^6+10y^5-15y^4$
9. $2x^3y-14x^2y^2+2xy^3$
11. $x^{n+2}+x^{n+1}+x^n$
13. x^2-x-20
15. $6x^2-11x+3$
17. $3x^4-16x^2y^2+5y^4$
19. $9x^2-25$
21. a^4-b^4
23. $12x^3-15x^2y^2-4xy+5y^3$
25. $x^3-8x^2+23x-40$
27. $2x^4+11x^3-24x^2-4x+1$
29. $x^2+16x+64$
31. $4x^2-20x+25$
33. $a^2-6ab+9b^2$
35. $x^4+4x^2y^2+4y^4$
37. $a^4-2a^3+11a-10a+25$
39. $x^3-9x^2+27x-27$
41. $27x^3+27x^2+9x+1$
43. $x^4+8x^3+24x^2+32x+16$
45. $16x^4-32x^3+24x^2-8x+1$
47. $x^{4n}-25$
49. $x^{4n}-2x^{2n}+1$
51. $3x^3-17x^2+4x+4$
53. $9x^6$ square units

Exercise 1.6.9

Divide.

1. $\frac{125x^5y^2}{25x^4y^2}$
2. $\frac{256x^2y^3z^5}{64x^2yz^2}$
3. $\frac{20x^3-12x^2+4x}{4x}$
4. $\frac{15x^4-75x^3+18x^2}{3x^2}$

5. $\frac{12a^2b+28ab^2-4ab}{4ab}$
6. $\frac{-2a^4b^3+16a^2b^2+8ab^3}{2ab^2}$
7. $\frac{x^3+x^2-3x+9}{x+3}$
8. $\frac{x^3-4x^2-9x+20}{x-5}$
9. $\frac{6x^3-11x^2+7x-6}{2x-3}$
10. $\frac{9x^3-9x^2-x+1}{3x-1}$
11. $\frac{16x^3+8x^2-39x+17}{4x-3}$
12. $\frac{12x^3-56x^2+55x+30}{2x-5}$
13. $\frac{6x^4+13x^3-9x^2-x+6}{3x+2}$
14. $\frac{25x^4-10x^3+11x^2-7x+1}{5x-1}$
15. $\frac{20x^4+12x^3+9x^2+10x+5}{2x+1}$
16. $\frac{25x^4-45x^3-26x^2+36x-11}{5x-2}$
17. $\frac{3x^4+x^2-1}{x-2}$
18. $\frac{x^4+x-3}{x+3}$
19. $\frac{x^3-10}{x-2}$
20. $\frac{x^3+15}{x+3}$
21. $\frac{y^5+1}{y+1}$
22. $\frac{y^6+1}{y+1}$
23. $\frac{x^4-4x^3+6x^2-7x-1}{x^2-x+2}$
24. $\frac{6x^4+x^3-2x^2+2x+4}{3x^2-x+1}$
25. $\frac{2x^3-7x^2+8x-3}{x^2-2x+1}$
26. $\frac{2x^4+3x^3-6x^2-4x+3}{x^2+x-3}$
27. $\frac{x^4+4x^3-2x^2-4x+1}{x^2-1}$
28. $\frac{x^4+x-1}{x^2+1}$
29. $\frac{x^3+6x^2y+4xy^2-y^3}{x+y}$
30. $\frac{2x^3-3x^2y+4xy^2-3y^3}{x-y}$
31. $\frac{8a^3-b^3}{2a-b}$
32. $\frac{a^3+27b^3}{a+3b}$
33. Find the quotient of $10x^2-11x+3$ and $2x-1$.
34. Find the quotient of $12x^2+x-11$ and $3x-2$.

Answer

1. $5x$

3. $5x^2-3x+1$

5. $3a+7b-1$

7. x^2-2x+3

9. $3x^2-x+2$

11. $4x^2+5x-6-\frac{1}{4x-3}$

13. $2x^3+3x^2-5x+3$

15. $10x^3+x^2+4x+3+\frac{2}{2x+1}$

17. $3x^3+6x^2+13x+26+\frac{51}{x-2}$

19. $x^2+2x+4-\frac{2}{x-2}$

21. $y^4-y^3+y^2-y+1$

23. $x^2-3x+1-\frac{3}{x^2-x+2}$

25. $2x-3$

27. x^2+4x-1

29. $x^2+5xy-y^2$

31. $4a^2 + 2ab + b^2$

33. $5x - 3$

Footnotes

[112] An algebraic expression consisting of terms with real number coefficients and variables with whole number exponents.

[113] The exponent of the variable. If there is more than one variable in the term, the degree of the term is the sum their exponents.

[114] The largest degree of all of its terms.

[115] A polynomial where each term has the form $a_n x^n$, where a_n is any real number and n is any whole number.

[116] The coefficient of the term with the largest degree.

[117] Polynomial with one term.

[118] Polynomial with two terms.

[119] Polynomial with three terms.

[120] A polynomial with degree 0.

[121] A polynomial with degree 1.

[122] A polynomial with degree 2.

[123] A polynomial with degree 3.

[124] The trinomials obtained by squaring the binomials $(a+b)^2 = a^2 + 2ab + b^2$ and $(a-b)^2 = a^2 - 2ab + b^2$.

[125] The special product obtained by multiplying conjugate binomials $(a+b)(a-b) = a^2 - b^2$.

[126] The binomials $(a+b)$ and $(a-b)$.

[127] The process of dividing two polynomials using the division algorithm.

[128] Terms with zero coefficients used to fill in all missing exponents within a polynomial.

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1.7: Solving Linear Equations

Learning Objectives

- Use the properties of equality to solve basic linear equations.
- Identify and solve conditional linear equations, identities, and contradictions.
- Clear fractions from equations.
- Set up and solve linear equations.

Solving Basic Linear Equations

An **equation**[129] is a statement indicating that two algebraic expressions are equal. A **linear equation with one variable**[130], x, is an equation that can be written in the standard form $ax+b=0$ where a and b are real numbers and $a \neq 0$. For example

$$3x-12=0$$

A **solution**[131] to a linear equation is any value that can replace the variable to produce a true statement. The variable in the linear equation $3x-12=0$ is x and the solution is $x=4$. To verify this, substitute the value 4 in for x and check that you obtain a true statement.

$$\begin{aligned} 3x-12&=0 \\ 3(4)-12&=0 \\ 12-12&=0 \\ 0&=0 \quad ✓ \end{aligned}$$

Alternatively, when an equation is equal to a constant, we may verify a solution by substituting the value in for the variable and showing that the result is equal to that constant. In this sense, we say that solutions "satisfy the equation."

Example 1.7.1:

Is $a=-\frac{1}{2}$ a solution to $-10a+5=25$?

Solution

Recall that when evaluating expressions, it is a good practice to first replace all variables with parentheses, and then substitute the appropriate values. By making use of parentheses, we avoid some common errors when working the order of operations.

$$-10a+5=-10(-\tfrac{1}{2})+5=5+5=10\neq 25 \quad ✗$$

Answer:

No, $a=-\frac{1}{2}$ does not satisfy the equation.

Developing techniques for solving various algebraic equations is one of our main goals in algebra. This section reviews the basic techniques used for solving linear equations with one variable. We begin by defining **equivalent equations**[132] as equations with the same solution set.

$$\left.\begin{aligned} 3x-5&=16 \\ 3x&=21 \\ x&=7 \end{aligned}\right\} \quad \textit{Equivalent equations}$$

Here we can see that the three linear equations are equivalent because they share the same solution set, namely, $\{7\}$. To obtain equivalent equations, use the following **properties of equality**[133]. Given algebraic expressions A and B, where c is a nonzero number:

Table 1.7.1

Addition property of equality:	If $A=B$, then $A+c=B+c$
Subtraction property of equality:	If $A=B$, then $A-c-B-c$
Multiplication property of equality:	If $A=B$, then $cA=cB$
Division property of equality:	If $A=B$, then $\frac{A}{c}=\frac{B}{c}$

Note

Multiplying or dividing both sides of an equation by 0 is carefully avoided. Dividing by 0 is undefined and multiplying both sides by 0 results in the equation $0 = 0$.

We solve algebraic equations by isolating the variable with a coefficient of 1. If given a linear equation of the form $ax + b = c$, then we can solve it in two steps. First, use the appropriate equality property of addition or subtraction to isolate the variable term. Next, isolate the variable using the equality property of multiplication or division. Checking the solution in the following examples is left to the reader.

Example 1.7.2:

Solve: $7x - 2 = 19$.

Solution

$$\begin{aligned} 7x - 2 &= 19 \\ 7x - 2 {\color{#1a9bd7}+2} &= 19 {\color{#1a9bd7}+2} && {\color{#1a9bd7}\textit{Add 2 to both sides.}} \\ 7x &= 21 \\ \frac{7x}{\color{#1a9bd7}7} &= \frac{21}{\color{#1a9bd7}7} && {\color{#1a9bd7}\textit{Divide both sides by 7.}} \\ x &= 3 \end{aligned}$$

Answer:

The solution is 3.

Example 1.7.3:

Solve: $56 = 8 + 12y$.

Solution

When no sign precedes the term, it is understood to be positive. In other words, think of this as $56 = +8 + 12y$. Therefore, we begin by subtracting 8 on both sides of the equal sign.

$$\begin{aligned} 56 {\color{#1a9bd7}-8} &= 8 + 12y {\color{#1a9bd7}-8} \\ 48 &= 12y \\ \frac{48}{\color{#1a9bd7}12} &= \frac{12y}{\color{#1a9bd7}12} \\ 4 &= y \end{aligned}$$

It does not matter on which side we choose to isolate the variable because the **symmetric property**[134] states that $4 = y$ is equivalent to $y = 4$.

Answer:

The solution is 4.

Example 1.7.4:

Solve: $\frac{5}{3}x + 2 = -8$.

Solution

Isolate the variable term using the addition property of equality, and then multiply both sides of the equation by the reciprocal of the coefficient $\frac{5}{3}$.

$$\begin{aligned} \frac{5}{3}x + 2 &= -8 \\ \frac{5}{3}x + 2 {\color{#1a9bd7}-2} &= -8 {\color{#1a9bd7}-2} && {\color{#1a9bd7}\textit{Subtract 2 on both sides.}} \\ \frac{5}{3}x &= -10 \\ {\color{#1a9bd7}\frac{3}{5}} \cdot \frac{5}{3}x &= {\color{#1a9bd7}\frac{3}{\cancel{5}}} \cdot (\overset{-2}{\cancel{-10}}) && {\color{#1a9bd7}\textit{Multiply both sides by } \frac{3}{5}.} \\ 1x &= 3 \cdot (-2) \end{aligned} \tag{1.7.1}$$

$$x = -6$$

Answer:

The solution is -6.

In summary, to retain equivalent equations, we must perform the same operation on both sides of the equation.

Exercise 1.7.1

Solve: $\frac{2}{3}x + \frac{1}{2} = -\frac{5}{6}$.

Answer

$x = -2$

www.youtube.com/v/cQwqXs9AD6M

General Guidelines for Solving Linear Equations

Typically linear equations are not given in standard form, and so solving them requires additional steps. When solving linear equations, the goal is to determine what value, if any, will produce a true statement when substituted in the original equation. Do this by isolating the variable using the following steps:

- Step 1: Simplify both sides of the equation using the order of operations and combine all like terms on the same side of the equal sign.
- Step 2: Use the appropriate properties of equality to combine like terms on opposite sides of the equal sign. The goal is to obtain the variable term on one side of the equation and the constant term on the other.
- Step 3: Divide or multiply as needed to isolate the variable.
- Step 4: Check to see if the answer solves the original equation.

We will often encounter linear equations where the expressions on each side of the equal sign can be simplified. If this is the case, then it is best to simplify each side first before solving. Normally this involves combining same-side like terms.

Note

At this point in our study of algebra the use of the properties of equality should seem routine. Therefore, displaying these steps in this text, usually in blue, becomes optional.

Example 1.7.5:

Solve: $-4a + 2 - a = 1$.

Solution

First combine the like terms on the left side of the equal sign.

$$\begin{aligned} -4a + 2 - a &= 1 && \textit{Combine same} - \textit{side like terms.} \\ -5a + 2 &= 1 && \textit{Subtract 2 on both sides.} \\ -5a &= -1 && \textit{Divide both sides by } -5. \\ a &= \frac{-1}{-5} = \frac{1}{5} \end{aligned}$$

Always use the original equation to check to see if the solution is correct.

$$\begin{aligned} -4a + 2 - a &= -4\left(\frac{1}{5}\right) + 2 - \frac{1}{5} \\ &= -\frac{4}{5} + \frac{2}{1} \cdot \frac{5}{5} - \frac{1}{5} \\ &= \frac{-4 + 10 + 1}{5} \\ &= \frac{5}{5} = 1 \ ✓ \end{aligned}$$

Answer:

The solution is $\frac{1}{5}$.

Given a linear equation in the form $ax+b=cx+d$, we begin the solving process by combining like terms on opposite sides of the equal sign. To do this, use the addition or subtraction property of equality to place like terms on the same side so that they can be combined. In the examples that remain, the check is left to the reader.

Example 1.7.6:

Solve: $-2y-3=5y+11$.

Solution

Subtract $5y$ on both sides so that we can combine the terms involving y on the left side.

$$\begin{aligned} -2y-3-5y &= 5y+11-5y \\ -7y-3 &= 11 \end{aligned}$$

From here, solve using the techniques developed previously.

$$\begin{aligned} -7y-3 &= 11 && \textit{Add 3 to both sides.} \\ -7y &= 14 \\ y &= \frac{14}{-7} && \textit{Divide both sides by } -7. \\ y &= -2 \end{aligned}$$

Answer:

The solution is -2.

Solving will often require the application of the distributive property.

Example 1.7.7:

Solve: $-\frac{1}{2}(10x-2)+3=7(1-2x)$.

Solution

Simplify the linear expressions on either side of the equal sign first.

$$\begin{aligned} -\frac{1}{2}(10x-2)+3 &= 7(1-2x) && \textit{Distribute} \\ -5x+1+3 &= 7-14x && \textit{Combine same}-\textit{side like terms.} \\ -5x+4 &= 7-14x && \textit{Combine opposite}-\textit{side like terms.} \\ 9x &= 3 && \textit{Solve.} \\ x &= \frac{3}{9}=\frac{1}{3} \end{aligned}$$

Answer:

The solution is $\frac{1}{3}$.

Example 1.7.8:

Solve: $5(3-a)-2(5-2a)=3$.

Solution

Begin by applying the distributive property.

$$\begin{aligned} 5(3-a)-2(5-2a) &= 3 \\ 15-5a-10+4a &= 3 \\ 5-a &= 3 \\ -a &= -2 \end{aligned}$$

Here we point out that $-a$ is equivalent to $-1a$; therefore, we choose to divide both sides of the equation by -1.

$$\begin{aligned} -a &= -2 \\ \frac{-1a}{-1} &= \frac{-2}{-1} \\ a &= 2 \end{aligned}$$

Alternatively, we can multiply both sides of $-a=-2$ by negative one and achieve the same result.

$$\begin{aligned} -a&=-2\\ (-1)(-a)&=(-1)(-2)\\ a&=2 \end{aligned}$$

Answer:

The solution is 2.

Exercise $1.7.2$

Solve: $6-3(4x-1)=4x-7$.

Answer

$x=1$

www.youtube.com/v/NAIAZrFjU-o

There are three different types of equations. Up to this point, we have been solving **conditional equations**[135]. These are equations that are true for particular values. An **identity**[136] is an equation that is true for all possible values of the variable. For example,

$$x=x \quad \textit{Identity}$$

has a solution set consisting of all real numbers, $\mathbb{R}$. A **contradiction**[137] is an equation that is never true and thus has no solutions. For example,

$$x+1=x \quad \textit{Contradiction}$$

has no solution. We use the empty set, $\varnothing$, to indicate that there are no solutions.

If the end result of solving an equation is a true statement, like $0=0$, then the equation is an identity and any real number is a solution. If solving results in a false statement, like $0=1$, then the equation is a contradiction and there is no solution.

Example $1.7.9$:

Solve: $4(x+5)+6=2(2x+3)$.

Solution

$$\begin{aligned} 4(x+5)+6&=2(2x+3)\\ 4x+20+6&=4x+6\\ 4x+26&=4x+6\\ 26&=6 \ \text{✗} \end{aligned}$$

Solving leads to a false statement; therefore, the equation is a contradiction and there is no solution.

Answer:

$\varnothing$

Example $1.7.10$:

Solve: $3(3y+5)+5=10(y+2)-y$.

Solution

$$\begin{aligned} 3(3y+5)+5&=10(y+2)-y\\ 9y+15+5&=10y+20-y\\ 9y+20&=9y+20\\ 9y&=9y\\ 0&=0 \ \checkmark \end{aligned}$$

Solving leads to a true statement; therefore, the equation is an identity and any real number is a solution.

Answer:

$\mathbb{R}$

The coefficients of linear equations may be any real number, even decimals and fractions. When this is the case it is possible to use the multiplication property of equality to clear the fractional coefficients and obtain integer coefficients in a single step. If given fractional coefficients, then multiply both sides of the equation by the least common multiple of the denominators (LCD).

Example 1.7.11:

Solve: $\frac{1}{3}x+\frac{1}{5}=\frac{1}{5}x-1$.

Solution

Clear the fractions by multiplying both sides by the least common multiple of the given denominators. In this case, it is the $LCD(3,5)=15$.

$$\begin{aligned} 15\cdot\left(\frac{1}{3}x+\frac{1}{5}\right) &= 15\cdot\left(\frac{1}{5}x-1\right) && \textit{Multiply both sides by } 15. \\ 15\cdot\frac{1}{3}x+15\cdot\frac{1}{5} &= 15\cdot\frac{1}{5}x-15\cdot 1 && \textit{Simplify.} \\ 5x+3 &= 3x-15 && \textit{Solve.} \\ 2x &= -18 \\ x &= \frac{-18}{2}=-9 \end{aligned}$$

Answer:

The solution is -9.

It is important to know that this technique only works for equations. *Do not try to clear fractions when simplifying expressions.* As a reminder:

Table $1.7.2$

Expression	Equation
$\frac{1}{2}x+\frac{5}{3}$	$\frac{1}{2}x+\frac{5}{3}=0$

We simplify expressions and solve equations. If you multiply an expression by 6, you will change the problem. However, if you multiply both sides of an equation by 6, you obtain an equivalent equation.

Table $1.7.3$

Incorrect	Correct
$\frac{1}{2}x+\frac{5}{3}$ $\neq 6\cdot\left(\frac{1}{2}x+\frac{5}{3}\right)$ $=3x+10$ ✗	$\frac{1}{2}x+\frac{5}{3}=0$ $6\cdot\left(\frac{1}{2}x+\frac{5}{3}\right)=6\cdot 0$ $3x+10=0$ ✓

Applications Involving Linear Equations

Algebra simplifies the process of solving real-world problems. This is done by using letters to represent unknowns, restating problems in the form of equations, and by offering systematic techniques for solving those equations. To solve problems using algebra, first translate the wording of the problem into mathematical statements that describe the relationships between the given information and the unknowns. Usually, this translation to mathematical statements is the difficult step in the process. The key to the translation is to carefully read the problem and identify certain key words and phrases.

Table $1.7.4$

Key Words	Translation
Sum, increased by, more than, plus, added to, total	$+$
Difference, decreased by, subtracted from, less, minus	$-$
Product, multiplied by, of, times, twice	$\cdot$

Key Words	Translation
Quotient, divided by, ratio, per	$\div$
Is, total, result	$=$

When translating sentences into mathematical statements, be sure to read the sentence several times and parse out the key words and phrases. It is important to first identify the variable, "*let x represent…*" and state in words what the unknown quantity is. This step not only makes our work more readable, but also forces us to think about what we are looking for.

Example 1.7.12:

When 6 is subtracted from twice the sum of a number and 8 the result is 5. Find the number.

Solution

Let *n* represent the unknown number.

$$\overbrace{2\,\cdot}^{\text{"twice"}}\quad\overbrace{(n+8)}^{\text{"the sum of a number and 8"}}\quad\overbrace{-\ \ 6}^{\text{"6 is subtracted from"}}\quad\overbrace{=}^{\text{"result is"}}\quad 5$$

Figure $1.7.1$

To understand why we included the parentheses in the set up, you must study the structure of the following two sentences and their translations:

Table $1.7.5$

"twice the sum of a number and 8"	2(n+8)
"the sum of ***twice a number*** *and 8"*	2n+8

The key was to focus on the phrase "*twice the sum*," this prompted us to group the sum within parentheses and then multiply by 2. After translating the sentence into a mathematical statement we then solve.

$$\begin{aligned} 2(n+8)-6&=5\\ 2n+16-6&=5\\ 2n+10&=5\\ 2n&=-5\\ n&=\frac{-5}{2}\end{aligned}$$

Check.

$$\begin{aligned} 2(n+8)-6&=2\left(-\frac{5}{2}+8\right)-6\\ &=2\left(\frac{11}{2}\right)-6\\ &=11-6\\ &=5\quad\checkmark\end{aligned}$$

Answer:

The number is $-\frac{5}{2}$.

General guidelines for setting up and solving word problems follow.

- Step 1: Read the problem several times, identify the key words and phrases, and organize the given information.
- Step 2: Identify the variables by assigning a letter or expression to the unknown quantities.
- Step 3: Translate and set up an algebraic equation that models the problem.
- Step 4: Solve the resulting algebraic equation.
- Step 5: Finally, answer the question in sentence form and make sure it makes sense (check it).

For now, set up all of your equations using only one variable. Avoid two variables by looking for a relationship between the unknowns.

Example 1.7.13:

A rectangle has a perimeter measuring 92 meters. The length is 2 meters less than 3 times the width. Find the dimensions of the rectangle.

Solution

The sentence "*The length is 2 meters less than 3 times the width,*" gives us the relationship between the two variables.

Let w represent the width of the rectangle.

Let $3w-2$ represent the length.

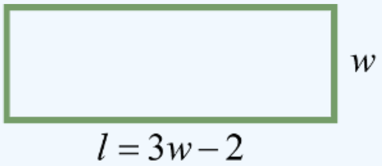

Figure $1.7.2$

The sentence "*A rectangle has a perimeter measuring 92 meters*" suggests an algebraic set up. Substitute 92 for the perimeter and the expression $3w-2$ for the length into the appropriate formula as follows:

$$\begin{array}{cccc} P= & 2l & + & 2w \\ \downarrow & \downarrow & & \\ 92= & 2(3w-2) & + & 2w \end{array}$$

Once you have set up an algebraic equation with one variable, solve for the width, w.

$$\begin{aligned} 92&=2(3w-2)+2w && \textit{Distribute.} \\ 92&=6w-4+2w && \textit{Combine like terms.} \\ 92&=8w-4 && \textit{Solve for } w. \\ 96&=8w && \\ 12&=w && \end{aligned}$$

Use $3w-2$ to find the length.

$$l=3w-2=3(12)-2=36-2=34$$

To check, make sure the perimeter is 92 meters.

$$\begin{aligned} P&=2l+2w \\ &=2(34)+2(12) \\ &=68+24 \\ &=92 \end{aligned}$$

Answer:

The rectangle measures 12 meters by 34 meters.

Example 1.7.14:

Given a $4\frac{3}{8}\%$ annual interest rate, how long will it take $\$2,500$ to yield $\$437.50$ in simple interest?

Solution

Let *t* represent the time needed to earn $\$437.50$ at $4\frac{3}{8}\%$. Organize the information needed to use the formula for simple interest, $I=prt$.

Table $1.7.6$

Given interest for the time period:	$I=\$437.50$
Given principal:	$p=\$2,500$

Given rate:	$r = 4\frac{3}{8}\% = 4.375\ \% = 0.04375$

Next, substitute all of the known quantities into the formula and then solve for the only unknown, *t*.

$$\begin{aligned} I &= prt \\ 437.50 &= 2500(0.04375)t \\ 437.50 &= 109.375t \\ \frac{437.50}{109.375} &= \frac{109.375t}{109.375} \\ 4 &= t \end{aligned}$$

Answer:

It takes 4 years for $\$2,500$ invested at $4\frac{3}{8}\%$ to earn $\$437.50$ in simple interest.

Example $1.7.15$:

Susan invested her total savings of $\$12,500$ in two accounts earning simple interest. Her mutual fund account earned 7% last year and her CD earned 4.5%. If her total interest for the year was $\$670$, how much was in each account?

Solution

The relationship between the two unknowns is that they total $\$12,500$. When a total is involved, a common technique used to avoid two variables is to represent the second unknown as the difference of the total and the first unknown.

Let x represent the amount invested in the mutual fund.

Let $12,500-x$ represent the remaining amount invested in the CD.

Organize the data.

Table $1.7.7$

Interest earned in the mutual fund:	$I = prt$ $= x \cdot 0.07 \cdot 1$ $= 0.07x$
Interest earned in the CD:	$I = prt$ $= (12,500-x) \cdot 0.045 \cdot 1$ $= 0.045(12,500-x)$
Total interest:	$\$670$

The total interest is the sum of the interest earned from each account.

$$\begin{array}{ccccc} mutual\ fund\ interest & + & CD\ interest & = & total\ interest \\ 0.07x & + & 0.045(12,500-x) & = & 670 \end{array}$$

This equation models the problem with one variable. Solve for \(*x*\).

$$\begin{aligned} 0.07x + 0.045(12,500-x) &= 670 \\ 0.07x + 562.5 - 0.045x &= 670 \\ 0.025x + 562.5 &= 670 \\ 0.025x &= 107.5 \\ x &= \frac{107.5}{0.025} \\ x &= 4,300 \end{aligned}$$

Use $12,500-x$ to find the amount in the CD.

$$12,500 - x = 12,500 - 4,300 = 8,200$$

Answer:

Susan invested $\$4,300$ at 7% in a mutual fund and $\$8,200$ at 4.5% in a CD.

Key Takeaways

- Solving general linear equations involves isolating the variable, with coefficient 1, on one side of the equal sign. To do this, first use the appropriate equality property of addition or subtraction to isolate the variable term on one side of the equal sign. Next, isolate the variable using the equality property of multiplication or division. Finally, check to verify that your solution solves the original equation.
- If solving a linear equation leads to a true statement like $0=0$, then the equation is an identity and the solution set consists of all real numbers, $\mathbb{R}$.
- If solving a linear equation leads to a false statement like $0=5$, then the equation is a contradiction and there is no solution, $\emptyset$.
- Clear fractions by multiplying both sides of an equation by the least common multiple of all the denominators. Distribute and multiply all terms by the LCD to obtain an equivalent equation with integer coefficients.
- Simplify the process of solving real-world problems by creating mathematical models that describe the relationship between unknowns. Use algebra to solve the resulting equations.

Exercise $1.7.3$

Determine whether or not the given value is a solution.

1. $-5x+4=-1; x=-1$
2. $4x-3=-7; x=-1$
3. $3y-4=5; y=\frac{9}{3}$
4. $-2y+7=12; y=-\frac{5}{2}$
5. $3a-6=18-a; a=-3$
6. $5(2t-1)=2-t; t=2$
7. $ax-b=0; x=\frac{b}{a}$
8. $ax+b=2b; x=\frac{b}{a}$

Answer

1. No

3. Yes

5. No

7. Yes

Exercise $1.7.4$

Solve.

1. $5x-3=27$
2. $6x-7=47$
3. $4x+13=35$
4. $6x-9=18$
5. $9a+10=10$
6. $5-3a=5$
7. $-8t+5=15$
8. $-9t+12=33$
9. $\frac{2}{3}x+\frac{1}{2}=1$
10. $\frac{3}{8}x+\frac{5}{4}=\frac{3}{2}$
11. $\frac{1-3y}{5}=2$
12. $\frac{2-5y}{6}=-8$
13. $7-y=22$
14. $6-y=12$
15. Solve for $x: ax-b=c$
16. Solve for $x: ax+b=0$

Answer

1. 6
3. $\frac{11}{2}$
5. 0
7. $-\frac{5}{4}$
9. $\frac{3}{4}$
11. -3
13. -15
15. $x=\frac{b+c}{a}$

Exercise 1.7.5

Solve.

1. $6x-5+2x=19$
2. $7-2x+9=24$
3. $12x-2-9x=5x+8$
4. $16-3x-22=8-4x$
5. $5y-6-9y=3-2y+8$
6. $7-9y+12=3y+11-11y$
7. $3+3a-11=5a-8-2a$
8. $2-3a=5a+7-8a$
9. $\frac{1}{3}x-\frac{3}{2}+\frac{5}{2}x=\frac{5}{6}x+\frac{1}{4}$
10. $\frac{5}{8}+\frac{1}{5}x-\frac{3}{4}=\frac{3}{10}x-\frac{1}{4}$
11. $1.2x-0.5-2.6x=2-2.4x$
12. $1.59-3.87x=3.48-4.1x-0.51$
13. $5-10x=2x+8-12x$
14. $8x-3-3x=5x-3$
15. $5(y+2)=3(2y-1)+10$
16. $7(y-3)=4(2y+1)-21$
17. $7-5(3t-9)=22$
18. $10-5(3t+7)=20$
19. $5-2x=4-2(x-4)$
20. $2(4x-5)+7x=5(3x-2)$
21. $4(4a-1)=5(a-3)+2(a-2)$
22. $6(2b-1)+24b=8(3b-1)$
23. $\frac{2}{3}(x+18)+2=\frac{1}{3}x-13$
24. $\frac{2}{5}x-\frac{1}{2}(6x-3)=\frac{4}{3}$
25. $1.2(2x+1)+0.6x=4x$
26. $6+0.5(7x-5)=2.5x+0.3$
27. $5(y+3)=15(y+1)-10y$
28. $3(4-y)-2(y+7)=-5y$
29. $\frac{1}{5}(2a+3)-\frac{1}{2}=\frac{1}{3}a+\frac{1}{10}$
30. $\frac{3}{2}a=\frac{3}{4}(1+2a)-\frac{1}{5}(a+5)$
31. $6-3(7x+1)=7(4-3x)$
32. $6(x-6)-3(2x-9)=-9$
33. $\frac{3}{4}(y-2)+\frac{2}{3}(2y+3)=3$
34. $\frac{5}{4}-\frac{1}{2}(4y-3)=\frac{2}{5}(y-1)$
35. $-2(3x+1)-(x-3)=-7x+1$
36. $6(2x+1)-(10x+9)=0$
37. Solve for $w: P=2l+2w$

38. Solve for $a : P = a + b + c$
39. Solve for $t : D = rt$
40. Solve for $w : V = lwh$
41. Solve for $b : A = \frac{1}{2}bh$
42. Solve for $a : s = \frac{1}{2}at^2$
43. Solve for $a : A = \frac{1}{2}h(a + b)$
44. Solve for $h : V = \frac{1}{3}\pi r^2 h$
45. Solve for $F : C = \frac{5}{9}(F - 32)$
46. Solve for $x : ax + b = c$

Answer

1. 3

3. -5

5. $-\frac{17}{2}$

7. $\mathbb{R}$

9. $\frac{7}{8}$

11. 2.5

13. $\varnothing$

15. 3

17. 2

19. $\varnothing$

21. $-\frac{5}{3}$

23. -81

25. 1.2

27. $\mathbb{R}$

29. 0

31. $\varnothing$

33. $\frac{6}{5}$

35. $\mathbb{R}$

37. $w = \frac{P-2l}{2}$

39. $t = \frac{D}{r}$

41. $b = \frac{2A}{h}$

43. $a = \frac{2A}{h} - b$

45. $F = \frac{9}{5}C + 32$

Exercise 1.7.6

Set up an algebraic equation then solve.

Number Problems

1. When 3 is subtracted from the sum of a number and 10 the result is 2. Find the number.
2. The sum of 3 times a number and 12 is equal to 3. Find the number.
3. Three times the sum of a number and 6 is equal to 5 times the number. Find the number.
4. Twice the sum of a number and 4 is equal to 3 times the sum of the number and 1. Find the number.

5. A larger integer is 1 more than 3 times another integer. If the sum of the integers is 57, find the integers.
6. A larger integer is 5 more than twice another integer. If the sum of the integers is 83, find the integers.
7. One integer is 3 less than twice another integer. Find the integers if their sum is 135.
8. One integer is 10 less than 4 times another integer. Find the integers if their sum is 100.
9. The sum of three consecutive integers is 339. Find the integers.
10. The sum of four consecutive integers is 130. Find the integers.
11. The sum of three consecutive even integers is 174. Find the integers.
12. The sum of four consecutive even integers is 116. Find the integers.
13. The sum of three consecutive odd integers is 81. Find the integers.
14. The sum of four consecutive odd integers is 176. Find the integers.

Answer

1. -5

3. 9

5. $14, 43$

7. $46, 89$

9. $112, 113, 114$

11. $56, 58, 60$

13. $25, 27, 29$

Exercise 1.7.7

Geometry Problems

1. The length of a rectangle is 5 centimeters less than twice its width. If the perimeter is 134 centimeters, find the length and width.
2. The length of a rectangle is 4 centimeters more than 3 times its width. If the perimeter is 64 centimeters, find the length and width.
3. The width of a rectangle is one-half that of its length. If the perimeter measures 36 inches, find the dimensions of the rectangle.
4. The width of a rectangle is 4 inches less than its length. If the perimeter measures 72 inches, find the dimensions of the rectangle.
5. The perimeter of a square is 48 inches. Find the length of each side.
6. The perimeter of an equilateral triangle is 96 inches. Find the length of each side.
7. The circumference of a circle measures 80π units. Find the radius.
8. The circumference of a circle measures 25 centimeters. Find the radius rounded off to the nearest hundredth.

Answer

1. Width: 24 centimeters; length: 43 centimeters

3. Width: 6 inches; length: 12 inches

5. 12 inches

7. 40 units

Exercise 1.7.8

Simple Interest Problems

1. For how many years must $\$1,000$ be invested at $5\frac{1}{2}$ % to earn $\$165$ in simple interest?
2. For how many years must $\$20,000$ be invested at $6\frac{1}{4}$ % to earn $\$3,125$ in simple interest?
3. At what annual interest rate must $\$6500$ be invested for 2 years to yield $\$1,040$ in simple interest?

4. At what annual interest rate must $\$5,750$ be invested for 1 year to yield $\$333.50$ in simple interest?
5. If the simple interest earned for 5 years was $\$1,860$ and the annual interest rate was 6%, what was the principal?
6. If the simple interest earned for 2 years was $\$543.75$ and the annual interest rate was $3\frac{3}{4}$ %, what was the principal?
7. How many years will it take $\$600$ to double earning simple interest at a 5% annual rate? (Hint: To double, the investment must earn $\$600$ in simple interest.)
8. How many years will it take $\$10,000$ to double earning simple interest at a 5% annual rate? (Hint: To double, the investment must earn $\$10,000$ in simple interest.)
9. Jim invested $\$4,200$ in two accounts. One account earns 3% simple interest and the other earns 6%. If the interest after 1 year was $\$159$, how much did he invest in each account?
10. Jane has her $\$6,500$ savings invested in two accounts. She has part of it in a CD at 5% annual interest and the rest in a savings account that earns 4% annual interest. If the simple interest earned from both accounts is $\$303$ for the year, then how much does she have in each account?
11. Jose put last year's bonus of $\$8,400$ into two accounts. He invested part in a CD with 2.5% annual interest and the rest in a money market fund with 1.5% annual interest. His total interest for the year was $\$198$. How much did he invest in each account?
12. Mary invested her total savings of $\$3,300$ in two accounts. Her mutual fund account earned 6.2% last year and her CD earned 2.4%. If her total interest for the year was $\$124.80$, how much was in each account?
13. Alice invests money into two accounts, one with 3% annual interest and another with 5% annual interest. She invests 3 times as much in the higher yielding account as she does in the lower yielding account. If her total interest for the year is $\$126$, how much did she invest in each account?
14. James invested an inheritance in two separate banks. One bank offered $5\frac{1}{2}$ % annual interest rate and the other $6\frac{1}{4}\%$. He invested twice as much in the higher yielding bank account than he did in the other. If his total simple interest for 1 year was $\$5,760$, then what was the amount of his inheritance?

Answer

1. 3 years

3. 8%

5. $\$6,200$

7. 20 years

9. He invested $\$3,100$ at 3% and $\$1,100$ at 6%.

11. Jose invested $\$7,200$ in the CD and $\$1,200$ in the money market fund.

13. Alice invested $\$700$ at 3% and $\$2,100$ at 5%.

Exercise 1.7.9

Uniform Motion Problems

1. If it takes Jim $1\frac{1}{4}$ hours to drive the 40 miles to work, then what is Jim's average speed?
2. It took Jill $3\frac{1}{2}$ hours to drive the 189 miles home from college. What was her average speed?
3. At what speed should Jim drive if he wishes to travel 176 miles in $2\frac{3}{4}$ hours?
4. James and Martin were able to drive the $1,140$ miles from Los Angeles to Seattle. If the total trip took 19 hours, then what was their average speed?

Answer

1. 32 miles per hour

3. 64 miles per hour

Exercise 1.7.10

1. What is regarded as the main business of algebra? Explain.
2. What is the origin of the word *algebra*?
3. Create an identity or contradiction of your own and share it on the discussion board. Provide a solution and explain how you found it.
4. Post something you found particularly useful or interesting in this section. Explain why.
5. Conduct a web search for “solving linear equations.” Share a link to website or video tutorial that you think is helpful.

Answer

1. Answer may vary
3. Answer may vary
5. Answer may vary

Footnotes

[129] Statement indicating that two algebraic expressions are equal.

[130] An equation that can be written in the standard form $ax+b=0$, where a and b are real numbers and $a \neq 0$.

[131] Any value that can replace the variable in an equation to produce a true statement.

[132] Equations with the same solution set.

[133] Properties that allow us to obtain equivalent equations by adding, subtracting, multiplying, and dividing both sides of an equation by nonzero real numbers.

[134] Allows you to solve for the variable on either side of the equal sign, because $x=5$ is equivalent to $5=x$.

[135] Equations that are true for particular values.

[136] An equation that is true for all possible values.

[137] An equation that is never true and has no solution.

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1.8: Solving Linear Inequalities with One Variable

Learning Objectives

- Identify linear inequalities and check solutions.
- Solve linear inequalities and express the solutions graphically on a number line and in interval notation.
- Solve compound linear inequalities and express the solutions graphically on a number line and in interval notation.
- Solve applications involving linear inequalities and interpret the results.

Linear Inequalities

A *linear inequality*[138] is a mathematical statement that relates a linear expression as either less than or greater than another. The following are some examples of linear inequalities, all of which are solved in this section:

Table $1.8.1$

$5x+7<22$	$-2(x+8)+6\geq 20$	$-2(4x-5)<9-2(x-2)$

A *solution to a linear inequality*[139] is a real number that will produce a true statement when substituted for the variable. Linear inequalities have either infinitely many solutions or no solution. If there are infinitely many solutions, graph the solution set on a number line and/or express the solution using interval notation.

Example $1.8.1$:

Are $x=-4$ and $x=6$ solutions to $5x+7<22$?

Solution

Substitute the values in for x, simplify, and check to see if we obtain a true statement.

Table $1.8.2$

Check $x=-4$	**Check** $x=6$
$5(-4)+7<22$ $-20+7<22$ $-13<22$ ✓	$5(6)+7<22$ $30+7<22$ $37<22$ ✗

Answer:

$x=-4$ is a solution and $x=6$ is not

All but one of the techniques learned for solving linear equations apply to solving linear inequalities. You may add or subtract any real number to both sides of an inequality, and you may multiply or divide both sides by any *positive* real number to create equivalent inequalities. For example:

$$\begin{aligned} 10 &> -5 \\ 10-7 &> -5-7 \quad \textit{Subtract 7 on both sides.} \\ 3 &> -12 \quad ✓ \quad \textit{True.} \end{aligned}$$

$$\begin{aligned} 10 &> -5 \\ \frac{10}{5} &> \frac{-5}{5} \quad \textit{Divide both sides by } 5. \\ 2 &> -1 \quad ✓ \quad \textit{True} \end{aligned}$$

Subtracting 7 from each side and dividing each side by positive 5 results in an inequality that is true.

Example 1.8.1:

Solve and graph the solution set: $5x+7<22$.

Solution

$$\begin{aligned}5x+7&<22\\ 5x+7-7&<22-7\\ 5x&<15\\ \frac{5x}{5}&<\frac{15}{5}\\ x&<3\end{aligned}$$

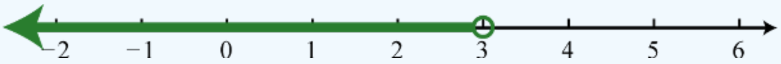


Figure 1.8.1

It is helpful to take a minute and choose a few values in and out of the solution set, substitute them into the original inequality, and then verify the results. As indicated, you should expect $x=0$ to solve the original inequality and that $x=5$ should not.

Table 1.8.3

Check $x=0$	**Check** $x=5$
$5(0)+7<22$ $7<22$ ✓	$5(5)+7<22$ $25+7<22$ $32<22$ ✗

Checking in this manner gives us a good indication that we have solved the inequality correctly.

We can express this solution in two ways: using set notation and interval notation.

$$\begin{array}{rl}\{x \mid x<3\} & \textit{Set notation}\\ (-\infty,3) & \textit{Interval notation}\end{array}$$

In this text we will choose to present answers using interval notation.

Answer

$(-\infty,\ 3)$

When working with linear inequalities, a different rule applies when multiplying or dividing by a negative number. To illustrate the problem, consider the true statement $10>-5$ and divide both sides by -5.

$$\begin{array}{ll}10>-5 & \\ \frac{10}{-5}>\frac{-5}{-5} & \textit{Divide both sides by } -5.\\ -2>1 & ✗\ \textit{False}\end{array}$$

Dividing by -5 results in a false statement. To retain a true statement, the inequality must be reversed.

$$\begin{array}{ll}10>-5 & \\ \frac{10}{-5}<\frac{-5}{-5} & \textit{Reverse the inequality.}\\ -2<1 & \checkmark\ \textit{True}\end{array}$$

The same problem occurs when multiplying by a negative number. This leads to the following new rule: *when multiplying or dividing by a negative number, reverse the inequality*. It is easy to forget to do this so take special care to watch for negative coefficients. In general, given algebraic expressions A and B, where c is a positive nonzero real number, we have the following **properties of inequalities**[140]:

Table 1.8.4

Addition property of inequalities:	If $A<B$ then, $A+c<B+c$
Subtraction property of inequalities:	If $A<B$ then, $A-c<B-c$
Multiplication property of inequalities:	If $A<B$, then $cA<cB$ If $A<B$, then $-cA>-cB$

Division property of inequalities:	If $A < B$, then $\frac{A}{c} < \frac{B}{c}$ If $A < B$, then $\frac{A}{-c} > \frac{B}{-c}$

We use these properties to obtain an **equivalent inequality**[141], one with the same solution set, where the variable is isolated. The process is similar to solving linear equations.

Example 1.8.3:

Solve and graph the solution set: $-2(x+8)+6 \geq 20$.

Solution

$$\begin{aligned} -2(x+8)+6 &\geq 20 && \textit{Distribute.} \\ -2x-16+6 &\geq 20 && \textit{Combine like terms.} \\ -2x-10 &\geq 20 && \textit{Solve for } x. \\ -2x &\geq 30 && \textit{Divide both sides by } -2. \\ \frac{-2x}{-2} &\leq \frac{30}{-2} && \textit{Reverse the inequality.} \\ x &\leq -15 \end{aligned}$$

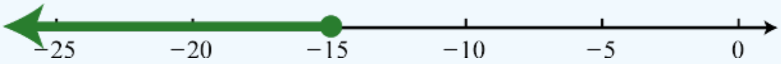


Figure 1.8.2

Answer:

Interval notation $(-\infty, \ -15]$

Example 1.8.4:

Solve and graph the solution set: $-2(4x-5) < 9-2(x-2)$.

Solution

$$\begin{gathered} -2(4x-5) < 9-2(x-2) \\ -8x+10 < 9-2x+4 \\ -8x+10 < 13-2x \\ -6x < 3 \\ \frac{-6x}{-6} > \frac{3}{-6} \textit{Reverse the inequality.} \\ x > -\frac{1}{2} \end{gathered}$$

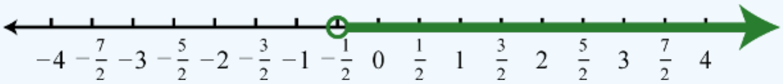


Figure 1.8.3

Answer:

Interval notation $(-\frac{1}{2}, \infty)$

Example 1.8.5:

Solve and graph the solution set: $\frac{1}{2}x-2 \geq \frac{1}{2}(\frac{7}{4}x-9)+1$.

Solution

$$\begin{gathered} \frac{1}{2}x-2 \geq \frac{1}{2}\left(\frac{7}{4}x-9\right)+1 \\ \frac{1}{2}x-2 \geq \frac{7}{8}x-\frac{9}{2}+1 \\ \frac{1}{2}x-\frac{7}{8}x \geq -\frac{7}{2}+2 \\ -\frac{3}{8}x \geq -\frac{3}{2} \\ \left(-\frac{8}{3}\right)\left(-\frac{3}{8}x\right) \leq \left(-\frac{8}{3}\right)\left(-\frac{3}{2}\right) \quad \textit{Reverse the inequality.} \\ x \leq 4 \end{gathered}$$

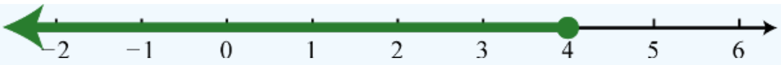


Figure $1.8.4$

Answer:

Interval notation: $(-\infty,\ 4]$

Exercise $1.8.1$

Solve and graph the solution set: $10-5(2x+3)\leq 25$

Answer

$[-3,\infty)$;

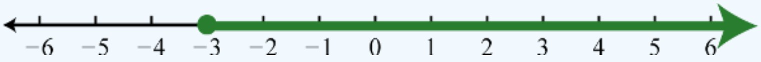


Figure $1.8.5$

www.youtube.com/v/COLLNtwYFm8

Compound Inequalities

Following are some examples of compound linear inequalities:

Table $1.8.5$

$-13<3x-7<17$	$4x+5\leq -15 \text{ or } 6x-11>7$

These **compound inequalities**[142] are actually two inequalities in one statement joined by the word *and* or by the word *or*. For example,

$$-13<3x-7<17$$

is a compound inequality because it can be decomposed as follows:

$$-13<3x-7 \text{ and } 3x-7<17$$

We can solve each inequality individually; the intersection of the two solution sets solves the original compound inequality. While this method works, there is another method that usually requires fewer steps. Apply the properties of this section to all three parts of the compound inequality with the goal of *isolating the variable in the middle* of the statement to determine the bounds of the solution set.

Example $1.8.6$:

Solve and graph the solution set: $-13<3x-7<17$.

Solution

$$\begin{gathered}-13<3x-7<17\\ -13+7<3x-7+7<17+7\\ -6<3x<24\\ \frac{-6}{3}<\frac{3x}{3}<\frac{24}{3}\\ -2<x<8\end{gathered}$$

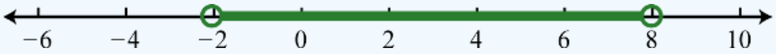


Figure $1.8.6$

Answer:

Interval notation: $(-2,8)$

Example 1.8.7:

Solve and graph the solution set: $\frac{5}{6} \leq \frac{1}{3}(\frac{1}{2}x+4) < 2$.

Solution

$$\begin{gathered}\frac{5}{6} \leq \frac{1}{3}\left(\frac{1}{2}x+4\right) < 2 \\ \frac{5}{6} \leq \frac{1}{6}x+\frac{4}{3} < 2 \\ 6\cdot\left(\frac{5}{6}\right) \leq 6\cdot\left(\frac{1}{6}x+\frac{4}{3}\right) < 6\cdot(2) \\ 5 \leq x+8 < 12 \\ 5-8 \leq x+8-8 < 12-8 \\ -3 \leq x < 4\end{gathered}$$

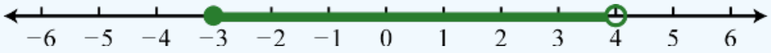


Figure 1.8.7

Answer:

Interval notation $[-3,4)$

It is important to note that when multiplying or dividing all three parts of a compound inequality by a negative number, you must reverse all of the inequalities in the statement. For example:

$$\begin{gathered}-10 < -2x < 20 \\ \frac{-10}{-2} > \frac{-2x}{-2} > \frac{20}{-2} \\ 5 > x > -10\end{gathered}$$

The answer above can be written in an equivalent form, where smaller numbers lie to the left and the larger numbers lie to the right, as they appear on a number line.

$$-10 < x < 5$$

Use interval notation, write: $(-10,5)$.

Exercise 1.8.2

Solve and graph the solution set: $-3 \leq -3(2x-3) < 15$.

Answer

$(-1,2]$;

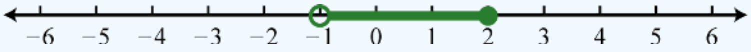


Figure 1.8.8

www.youtube.com/v/BEXeGm_FGDc

For compound inequalities with the word "*or*" you work both inequalities separately and then consider the union of the solution sets. Values in this union solve either inequality.

Example 1.8.8:

Solve and graph the solution set: $4x+5 \leq -15$ or $6x-11 > 7$.

Solution

Solve each inequality and form the union by combining the solution sets.

Table 1.8.6

$4x+5 \leq -15$ $4x \leq -20$ $x \leq -5$	or	$6x-11 > 7$ $6x > 18$ $x > 3$

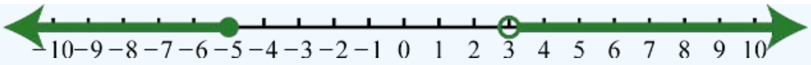


Figure 1.8.9

Answer:

Interval notation $(-\infty,-5]\cup(3,\infty)$

Exercise 1.8.3

Solve and graph the solution set: $5(x-3)<-20$ or $2(5-3x)<1$.

Answer

$(-\infty,-1)\cup\left(\frac{3}{2},\infty\right)$

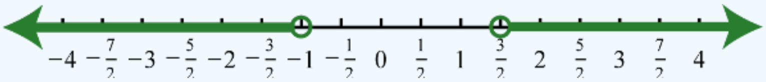

-4 $-\frac{7}{2}$ -3 $-\frac{5}{2}$ -2 $-\frac{3}{2}$ -1 $-\frac{1}{2}$ 0 $\frac{1}{2}$ 1 $\frac{3}{2}$ 2 $\frac{5}{2}$ 3 $\frac{7}{2}$ 4

Figure 1.8.10

www.youtube.com/v/HdXp6Dabl4I

Applications of Linear Inequalities

Some of the key words and phrases that indicate inequalities are summarized below:

Table 1.8.7

Key Phrases	Translation
A number is **at least** 5.	$x\geq 5$
A number is 5 **or more inclusive.**	$x\geq 5$
A number is **at most** 3.	$x\leq 3$
A number is 3 **or less inclusive.**	$x\leq 3$
A number is **strictly less than** 4.	$x<4$
A number is **less than** 4, **noninclusive.**	$x<4$
A number **is greater than** 7.	$x>7$
A number is **more than** 7, **noninclusive**.	$x>7$
A number is **in between** 2 and 10.	$2<x<10$
A number is at least 5 **and** at most 15.	$5\leq x\leq 15$
A number may **range** from 5 to 15.	$5\leq x\leq 15$

As with all applications, carefully read the problem several times and look for key words and phrases. Identify the unknowns and assign variables. Next, translate the wording into a mathematical inequality. Finally, use the properties you have learned to solve the inequality and express the solution graphically or in interval notation.

Example 1.8.9:

Seven less than 3 times the sum of a number and 5 is at most 11. Find all numbers that satisfy this condition.

Solution

First, choose a variable for the unknown number and identify the key words and phrases.

Let *n* represent the unknown indicated by “*a number*.”

$$\overbrace{3(n+5)}^{\text{three times the sum of a number and 5}}\quad\overbrace{-\quad 7}^{\text{seven less than}}\quad\overbrace{\leq}^{\text{is at most}}\quad 11$$

Figure 1.8.11

Solve for *n*.

$$\begin{aligned}3(n+5)-7&\leq 11\\3n+15-7&\leq 11\\3n+8&\leq 11\\3n&\leq 3\\n&\leq 1\end{aligned}$$

Answer:

Any number less than or equal to 1 will satisfy the statement.

Example 1.8.10:

To earn a B in a mathematics course the test average must be at least 80% and less than 90%. If a student earned 92%, 96%, 79%, and 83% on the first four tests, what must she score on the fifth test to earn a B?

Solution

Set up a compound inequality where the test average is between 80% and 90%. In this case, include the lower bound, 80.

Let *x* represent the score on the fifth test.

$$\begin{aligned}80 &\leq \textit{test average} && < 90\\80 &\leq \frac{92+96+79+83+x}{5} && < 90\\5\cdot 80 &\leq 5\cdot\frac{350+x}{5} && < 5\cdot 90\\400 &\leq 350+x && < 45\\50 &\leq x && < 100\end{aligned}$$

Answer:

She must earn a score of at least 50% and less than 100%.

In the previous example, the upper bound 100% was not part of the solution set. What would happen if she did earn a 100% on the fifth test?

$$\begin{aligned}\text{average} &= \frac{92+96+79+83+100}{5}\\&=\frac{450}{5}\\&=90\end{aligned}$$

As we can see, her average would be 90%, which would earn her an A.

Key Takeaways

- Inequalities typically have infinitely many solutions. The solutions are presented graphically on a number line or using interval notation or both.
- All but one of the rules for solving linear inequalities are the same as solving linear equations. If you divide or multiply an inequality by a negative number, reverse the inequality to obtain an equivalent inequality.
- Compound inequalities involving the word “or” require us to solve each inequality and form the union of each solution set. These are the values that solve at least one of the given inequalities.
- Compound inequalities involving the word “and” require the intersection of the solution sets for each inequality. These are the values that solve both or all of the given inequalities.
- The general guidelines for solving word problems apply to applications involving inequalities. Be aware of a new list of key words and phrases that indicate a mathematical setup involving inequalities.

Exercise 1.8.4

Determine whether or not the given value is a solution.

1. $5x-1<-2; x=-1$
2. $-3x+1>-10; x=1$
3. $2x-3<-5; x=1$
4. $5x-7<0; x=2$
5. $9y-4\geq 5; y=1$
6. $-6y+1\leq 3; y=-1$
7. $12a+3\leq -2; a=-\frac{1}{3}$
8. $25a-2\leq -22; a=-\frac{4}{5}$
9. $-10<2x-5<-5; x=-\frac{1}{2}$
10. $3x+8<-2$ or $4x-2>5; x=2$

Answer

1. Yes

3. No

5. Yes

7. No

9. Yes

Exercise 1.8.5

Graph all solutions on a number line and provide the corresponding interval notation.

1. $3x+5>-4$
2. $2x+1>-1$
3. $5-6y<-1$
4. $7-9y>43$
5. $6-a\leq 6$
6. $-2a+5>5$
7. $\frac{5x+6}{3}\leq 7$
8. $\frac{4x+11}{6}\leq\frac{1}{2}$
9. $\frac{1}{2}y+\frac{5}{4}\geq\frac{1}{4}$
10. $\frac{1}{12}y+\frac{2}{3}\leq\frac{5}{6}$
11. $2(3x+14)<-2$
12. $5(2y+9)>-15$
13. $5-2(4+3y)\leq 45$
14. $-12+5(5-2x)<83$
15. $6(7-2a)+6a\leq 12$
16. $2a+10(4-a)\geq 8$
17. $9(2t-3)-3(3t+2)<30$
18. $-3(t-3)-(4-t)>1$
19. $\frac{1}{2}(5x+4)+\frac{5}{6}x>-\frac{4}{3}$
20. $\frac{2}{5}+\frac{1}{6}(2x-3)\geq\frac{1}{15}$
21. $5x-2(x-3)<3(2x-1)$
22. $3(2x-1)-10>4(3x-2)-5x$
23. $-3y\geq 3(y+8)+6(y-1)$
24. $12\leq 4(y-1)+2(2y+1)$
25. $-2(5t-3)-4>5(-2t+3)$
26. $-7(3t-4)>2(3-10t)-t$
27. $\frac{1}{2}(x+5)-\frac{1}{3}(2x+3)>\frac{7}{6}x+\frac{3}{2}$

28. $-\frac{1}{3}(2x-3)+\frac{1}{4}(x-6)\geq\frac{1}{12}x-\frac{5}{4}$
29. $4(3x+4)\geq 3(6x+5)-6x$
30. $1-4(3x+7)<-3(x+9)-9x$
31. $6-3(2a-1)\leq 4(3-a)+1$
32. $12-5(2a+6)\geq 2(5-4a)-a$

Answer

1. $(-3,\infty)$;

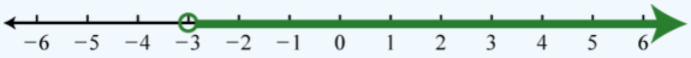

−6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6

Figure 1.8.12

3. $(1,\infty)$;

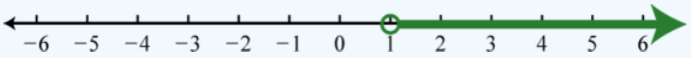

−6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6

Figure 1.8.13

5. $[0,\infty)$;

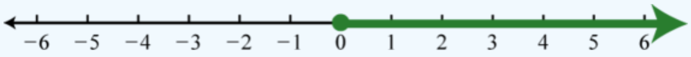

−6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6

Figure 1.8.14

7. $(-\infty,3]$;

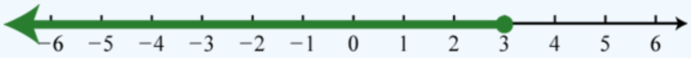

−6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6

Figure 1.8.15

9. $[-2,\infty)$;

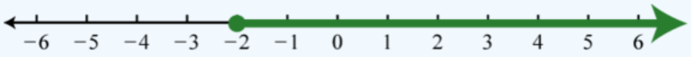

−6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6

Figure 1.8.16

11. $(-\infty,-5)$;

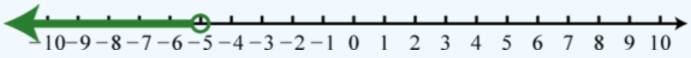

−10 −9 −8 −7 −6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7 8 9 10

Figure 1.8.17

13. $[-8,\infty)$;

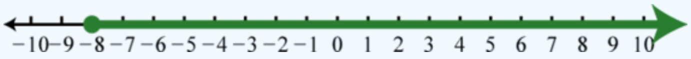

−10 −9 −8 −7 −6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7 8 9 10

Figure 1.8.18

15. $[5,\infty)$;

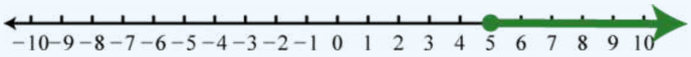

−10 −9 −8 −7 −6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7 8 9 10

Figure 1.8.19

17. $(-\infty,7)$;

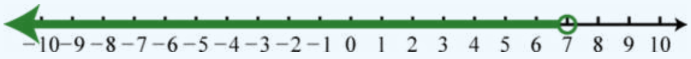

−10 −9 −8 −7 −6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7 8 9 10

Figure 1.8.20

19. $(-1,\infty)$;

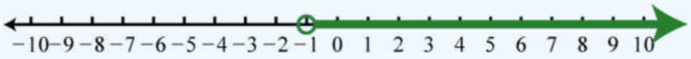

−10 −9 −8 −7 −6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7 8 9 10

Figure 1.8.21

21. $(3,\infty)$;

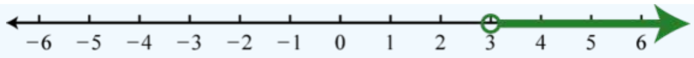

−6
−5
−4
−3
−2
−1
0
1
2
3
4
5
6

Figure 1.8.22

23. $(-\infty,-\frac{3}{2}]$;

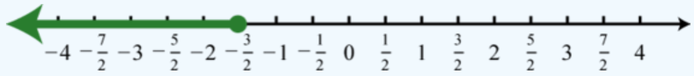
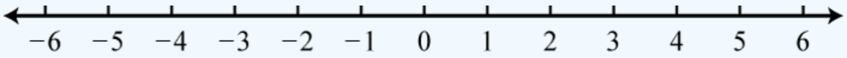

−4
$-\frac{7}{2}$
−3
$-\frac{5}{2}$
−2
$-\frac{3}{2}$
−1
$-\frac{1}{2}$
0
$\frac{1}{2}$
1
$\frac{3}{2}$
2
$\frac{5}{2}$
3
$\frac{7}{2}$
4

Figure 1.8.23

25. $\emptyset$;


−6
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3
4
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6

Figure 1.8.24

27. $(-\infty,0)$;

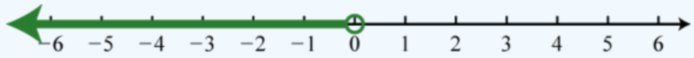
−6
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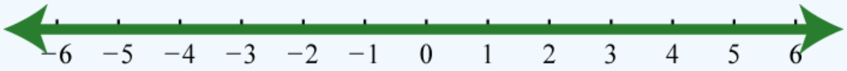
Figure 1.8.25

29. $\mathbb{R}$;


−6
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6

Figure 1.8.26

31. $[-2,\infty)$;

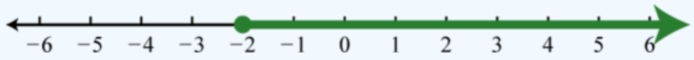
−6
−5
−4
−3
−2
−1
0
1
2
3
4
5
6

Figure 1.8.27

Exercise 1.8.6

Graph all solutions on a number line and provide the corresponding interval notation.

1. $-1 < 2x+1 < 9$
2. $-4 < 5x+11 < 16$
3. $-7 \le 6y-7 \le 17$
4. $-7 \le 3y+5 \le 2$
5. $-7 < \frac{3x+1}{2} \le 8$
6. $-1 \le \frac{2x+7}{3} < 1$
7. $-4 \le 11-5t < 31$
8. $15 < 12-t \le 16$
9. $-\frac{1}{3} \le \frac{1}{6}a+\frac{1}{3} \le \frac{1}{2}$
10. $-\frac{1}{6} < \frac{1}{3}a+\frac{5}{6} < \frac{3}{2}$
11. $5x+2 < -3$ or $7x-6 > 15$
12. $4x+15 \le -1$ or $3x-8 \ge -11$
13. $8x-3 \le 1$ or $6x-7 \ge 8$
14. $6x+1 < -3$ or $9x-20 > -5$
15. $8x-7 < 1$ or $4x+11 > 3$
16. $10x-21 < 9$ or $7x+9 \ge 30$
17. $7+2y < 5$ or $20-3y > 5$
18. $5-y < 5$ or $7-8y \le 23$
19. $15+2x < -15$ or $10-3x > 40$
20. $10-\frac{1}{3}x \le 5$ or $5-\frac{1}{2}x \le 15$
21. $9-2x \le 15$ and $5x-3 \le 7$
22. $5-4x > 1$ and $15+2x \ge 5$
23. $7y-18 < 17$ and $2y-15 < 25$

24. $13y + 20 \geq 7$ and $8 + 15y > 8$
25. $5 - 4x \leq 9$ and $3x + 13 \leq 1$
26. $17 - 5x \geq 7$ and $4x - 7 > 1$
27. $9y + 20 \leq 2$ and $7y + 15 \geq 1$
28. $21 - 6y \leq 3$ and $-7 + 2y \leq -1$
29. $-21 < 6(x - 3) < -9$
30. $0 \leq 2(2x + 5) < 8$
31. $-15 \leq 5 + 4(2y - 3) < 17$
32. $5 < 8 - 3(3 - 2y) \leq 29$
33. $5 < 5 - 3(4 + t) < 17$
34. $-3 \leq 3 - 2(5 + 2t) \leq 21$
35. $-40 < 2(x + 5) - (5 - x) \leq -10$
36. $-60 \leq 5(x - 4) - 2(x + 5) \leq 15$
37. $-\frac{1}{2} < \frac{1}{30}(x - 10) < \frac{1}{3}$
38. $-\frac{1}{5} \leq \frac{1}{15}(x - 7) \leq \frac{1}{3}$
39. $-1 \leq \frac{a+2(a-2)}{5} \leq 0$
40. $0 < \frac{5+2(a-1)}{6} < 2$

Answer

1. $(-1, 4)$;

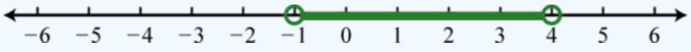
−6
−5
−4
−3
−2
−1
0
1
2
3
4
5
6

Figure 1.8.28

3. $[0, 4]$;

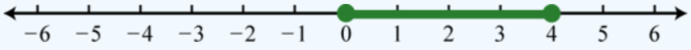
−6
−5
−4
−3
−2
−1
0
1
2
3
4
5
6

Figure 1.8.29

5. $(-5, 5]$;

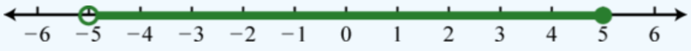
−6
−5
−4
−3
−2
−1
0
1
2
3
4
5
6

Figure 1.8.30

7. $(-4, 3]$;

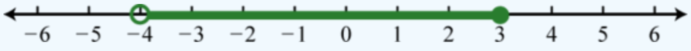
−6
−5
−4
−3
−2
−1
0
1
2
3
4
5
6

Figure 1.8.31

9. $[-4, 1]$;

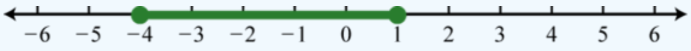
−6
−5
−4
−3
−2
−1
0
1
2
3
4
5
6

Figure 1.8.32

11. $(-\infty, -1) \cup (3, \infty)$;

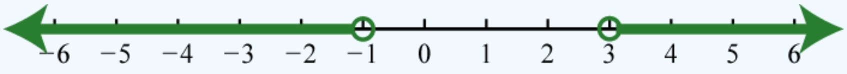
−6
−5
−4
−3
−2
−1
0
1
2
3
4
5
6

Figure 1.8.33

13. $(-\infty, 12] \cup [52, \infty)$;

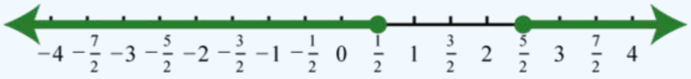
−4
−7/2
−3
−5/2
−2
−3/2
−1
−1/2
0
1/2
1
3/2
2
5/2
3
7/2
4

Figure 1.8.34

15. $\mathbb{R}$;

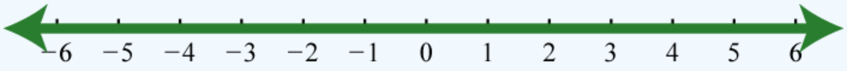
−6
−5
−4
−3
−2
−1
0
1
2
3
4
5
6

Figure 1.8.35

17. $(-\infty, 5)$;

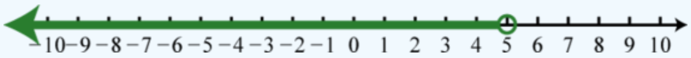
−10 −9 −8 −7 −6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7 8 9 10

Figure 1.8.36

19. $(-\infty, -10)$;

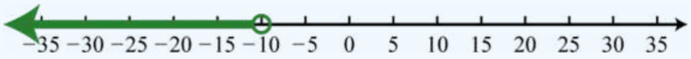
−35 −30 −25 −20 −15 −10 −5 0 5 10 15 20 25 30 35

Figure 1.8.37

21. $[-3, 2]$;

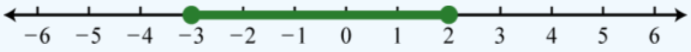
−6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6

Figure 1.8.38

23. $(-\infty, 5)$;

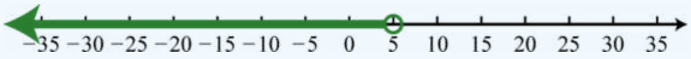
−35 −30 −25 −20 −15 −10 −5 0 5 10 15 20 25 30 35

Figure 1.8.39

25. $\varnothing$;

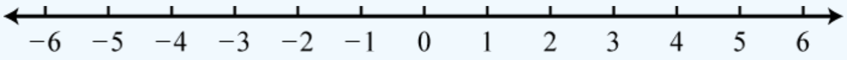
−6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6

Figure 1.8.40

27. -2;

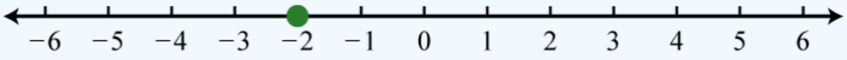
−6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6

Figure 1.8.41

29. $(-12, 32)$;

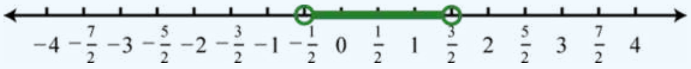
−4 −7/2 −3 −5/2 −2 −3/2 −1 −1/2 0 1/2 1 3/2 2 5/2 3 7/2 4

Figure 1.8.42

31. $[-1, 3)$;

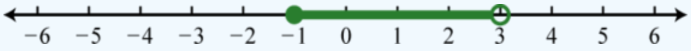
−6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6

Figure 1.8.43

33. $(-8, -4)$;

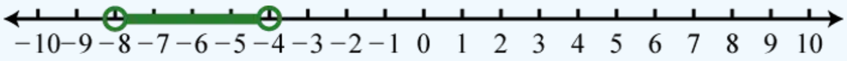
−10 −9 −8 −7 −6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7 8 9 10

Figure 1.8.44

35. $(-15, -5]$;

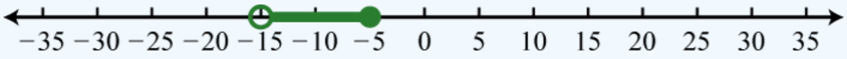
−35 −30 −25 −20 −15 −10 −5 0 5 10 15 20 25 30 35

Figure 1.8.45

37. $(-5, 20)$;

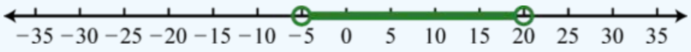
−35 −30 −25 −20 −15 −10 −5 0 5 10 15 20 25 30 35

Figure 1.8.46

39. $[-13, 43]$;

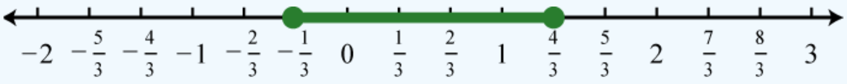
−2 −5/3 −4/3 −1 −2/3 −1/3 0 1/3 2/3 1 4/3 5/3 2 7/3 8/3 3

Figure 1.8.47

Exercise 1.8.7

Find all numbers that satisfy the given condition.

1. Three less than twice the sum of a number and 6 is at most 13.
2. Five less than 3 times the sum of a number and 4 is at most 10.
3. Five times the sum of a number and 3 is at least 5.
4. Three times the difference between a number and 2 is at least 12.
5. The sum of 3 times a number and 8 is between 2 and 20.
6. Eight less than twice a number is between -20 and -8.
7. Four subtracted from three times some number is between -4 and 14.
8. Nine subtracted from 5 times some number is between 1 and 11.

Answer

1. $(-\infty, 2]$

3. $[-2, \infty)$

5. $(-2, 4)$

7. $(0, 6)$

Exercise 1.8.8

Set up an algebraic inequality and then solve.

1. With a golf club membership, costing $\$120$ per month, each round of golf costs only $\$35.00$. How many rounds of golf can a member play if he wishes to keep his costs $\$270$ per month at most?
2. A rental truck costs $\$95$ per day plus $\$0.65$ per mile driven. How many miles can be driven on a one-day rental to keep the cost at most $\$120$?
3. Mark earned $6, 7$, and 10 points out of 10 on the first three quizzes. What must he score on the fourth quiz to average at least 8?
4. Joe earned scores of $78, 82, 88$ and 70 on his first four algebra exams. What must he score on the fifth exam to average at least 80?
5. A gymnast scored $13.2, 13.0, 14.3, 13.8$, and 14.6 on the first five events. What must he score on the sixth event to average at least 14.0?
6. A dancer scored 7.5 and 8.2 from the first two judges. What must her score from the third judge come in as if she is to average 8.4 or higher?
7. If two times an angle is between 180 degrees and 270 degrees, then what are the bounds of the original angle?
8. The perimeter of a square must be between 120 inches and 460 inches. Find the length of all possible sides that satisfy this condition.
9. A computer is set to shut down if the temperature exceeds 45°C. Give an equivalent statement using degrees Fahrenheit. Hint: $C = \frac{5}{9}(F - 32)$.
10. A certain antifreeze is effective for a temperature range of -35°C to 120°C. Find the equivalent range in degrees Fahrenheit.

Answer

1. Members may play 4 rounds or fewer.

3. Mark must earn at least 9 points on the fourth quiz.

5. He must score a 15.1 on the sixth event.

7. The angle is between 90 degrees and 135 degrees.

9. The computer will shut down when the temperature exceeds 113°F.

Exercise 1.8.9

1. Often students reverse the inequality when solving $5x+2<-18$? Why do you think this is a common error? Explain to a beginning algebra student why we do not.
2. Conduct a web search for "solving linear inequalities." Share a link to website or video tutorial that you think is helpful.
3. Write your own 5 key takeaways for this entire chapter. What did you find to be review and what did you find to be new? Share your thoughts on the discussion board.

Answer

1. Answer may vary

3. Answer may vary

Footnotes

[138] Linear expressions related with the symbols $\leq, <, \geq$, and $>$.

[139] A real number that produces a true statement when its value is substituted for the variable.

[140] Properties used to obtain equivalent inequalities and used as a means to solve them.

[141] Inequalities that share the same solution set.

[142] Two or more inequalities in one statement joined by the word "and" or by the word "or."

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1.E: Algebra Fundamentals (Exercises)

Exercise $1.E.1$

Reduce to lowest terms.

1. $\frac{56}{120}$
2. $\frac{54}{60}$
3. $\frac{155}{90}$
4. $\frac{315}{120}$

Answer

1. $\frac{7}{15}$

3. $\frac{31}{18}$

Exercise $1.E.2$

Simplify.

1. $-\left(-\frac{1}{2}\right)$
2. $-\left(-\left(-\frac{5}{8}\right)\right)$
3. $-(-(-a))$
4. $-(-(-(-a)))$

Answer

1. $\frac{1}{2}$

3. $-a$

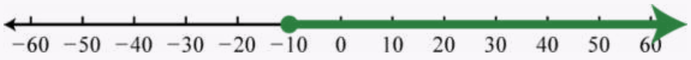
Exercise $1.E.3$

Graph the solution set and give the interval notation equivalent.

1. $x \geq -10$
2. $x < 0$
3. $-8 \leq x < 0$
4. $-10 < x \leq 4$
5. $x < 3$ and $x \geq -1$
6. $x < 0$ and $x > 1$
7. $x < -2$ or $x > -6$
8. $x \leq -1$ or $x > 3$

Answer

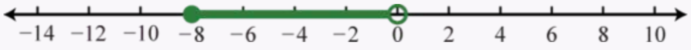
1. $[-10, \infty)$;



−60
−50
−40
−30
−20
−10
0
10
20
30
40
50
60

Figure 1.E.1

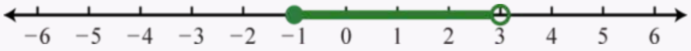
3. $[-8, 0)$;



−14
−12
−10
−8
−6
−4
−2
0
2
4
6
8
10

Figure 1.E.2

5. $[-1, 3)$;



−6
−5
−4
−3
−2
−1
0
1
2
3
4
5
6

Figure 1.E.3

7. $\mathbb{R}$

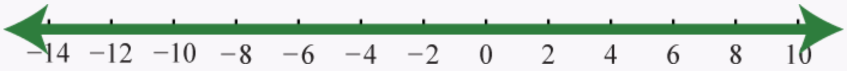


Figure 1.E.4

Exercise $1.E.4$

Determine the inequality that corresponds to the set expressed using interval notation.

1. $[-8, \infty)$
2. $(-\infty, -7)$
3. $[12, 32]$
4. $[-10, 0)$
5. $(-\infty, 1] \cup (5, \infty)$
6. $(-\infty, -10) \cup (-5, \infty)$
7. $(-4, \infty)$
8. $(-\infty, 0)$

Answer

1. $x \geq -8$

3. $12 \leq x \leq 32$

5. $x \leq 1$ or $x > 5$

7. $x > -4$

Exercise $1.E.5$

Simplify.

1. $-\left|-\frac{3}{4}\right|$
2. $-\left|-\left(-\frac{2}{3}\right)\right|$
3. $-(-|-4|)$
4. $-(-(-|-3|))$

Answer

1. $-\frac{3}{4}$

3. 4

Exercise $1.E.6$

Determine the values represented by a.

1. $|a| = 6$
2. $|a| = 1$
3. $|a| = -5$
4. $|a| = a$

Answer

1. $a = \pm 6$

2. $\varnothing$

Exercise $1.E.7$

Perform the operations.

1. $\frac{1}{4} - \frac{1}{5} + \frac{3}{20}$

2. $\frac{2}{3}-\left(-\frac{3}{4}\right)-\frac{5}{12}$
3. $\frac{5}{3}\left(-\frac{6}{7}\right)\div\left(\frac{5}{14}\right)$
4. $\left(-\frac{8}{9}\right)\div\frac{16}{27}\left(\frac{2}{15}\right)$
5. $\left(-\frac{2}{3}\right)^3$
6. $\left(-\frac{3}{4}\right)^2$
7. $(-7)^2-8^2$
8. $-4^2+(-4)^3$
9. $10-8\left((3-5)^2-2\right)$
10. $4+5\left(3-(2-3)^2\right)$
11. $-3^2-\left(7-(-4+2)^3\right)$
12. $(-4+1)^2-(3-6)^3$
13. $\frac{10-3(-2)^3}{3^2-(-4)^2}$
14. $\frac{6\left[(-5)^2-(-3)^2\right]}{4-6(-2)^2}$
15. $7-3\left|6-(-3-2)^2\right|$
16. $-6^2+5\left|3-2(-2)^2\right|$
17. $\frac{12-\left|6-2(-4)^2\right|}{3-|-4|}$
18. $\frac{-(5-2|-3|)^3}{|4-(-3)^2|-3^2}$

Answer

1. $\frac{1}{5}$

3. -4

5. $-\frac{8}{27}$

7. -15

9. -6

11. -24

13. $-\frac{34}{7}$

15. -50

17. 14

Exercise $1.E.8$

Simplify.

1. $3\sqrt{8}$
2. $5\sqrt{18}$
3. $6\sqrt{0}$
4. $\sqrt{-6}$
5. $\sqrt{\frac{75}{16}}$
6. $\sqrt{\frac{80}{49}}$
7. $\sqrt[3]{40}$
8. $\sqrt[3]{81}$
9. $\sqrt[3]{-81}$
10. $\sqrt[3]{-32}$
11. $\sqrt[3]{\frac{250}{27}}$
12. $\sqrt[3]{\frac{1}{125}}$

Answer

1. $6\sqrt{2}$

3. 0

5. $\frac{5\sqrt{3}}{4}$

7. $2\sqrt[3]{5}$

9. $-3\sqrt[3]{3}$

11. $\frac{5\sqrt[3]{2}}{3}$

Exercise $1.E.9$

Use a calculator to approximate the following to the nearest thousandth.

1. $\sqrt{12}$
2. $3\sqrt{14}$
3. $\sqrt[3]{18}$
4. $7\sqrt[3]{25}$
5. Find the length of the diagonal of a square with sides measuring 8 centimeters.
6. Find the length of the diagonal of a rectangle with sides measuring 6 centimeters and 12 centimeters.

Answer

1. 3.464

3. 2.621

5. $8\sqrt{2}$ centimeters

Exercise $1.E.10$

Multiply

1. $\frac{2}{3}\left(9x^2+3x-6\right)$
2. $-5\left(\frac{1}{5}y^2-\frac{3}{5}y+\frac{1}{2}\right)$
3. $\left(a^2-5ab-2b^2\right)(-3)$
4. $\left(2m^2-3mn+n^2\right)\cdot 6$

Answer

1. $6x^2+2x-4$

3. $-3a^2+15ab+6b^2$

Exercise $1.E.11$

Combine like terms.

1. $5x^2y-3xy^2-4x^2y-7xy^2$
2. $9x^2y^2+8xy+3-5x^2y^2-8xy-2$
3. $a^2b^2-7ab+6-a^2b^2+12ab-5$
4. $5m^2n-3mn+2mn^2-2nm-4m^2n+mn^2$

Answer

1. $x^2y-10xy^2$

3. $5ab+1$

Exercise $1.E.12$

Simplify.

1. $5x^2+4x-3\left(2x^2-4x-1\right)$
2. $\left(6x^2y^2+3xy-1\right)-\left(7x^2y^2-3xy+2\right)$
3. $a^2-b^2-\left(2a^2+ab-3b^2\right)$
4. $m^2+mn-6\left(m^2-3n^2\right)$

Answer

1. $-x^2+16x+3$

3. $-a^2-ab+2b^2$

Exercise $1.E.13$

Evaluate.

1. x^2-3x+1 where $x=-\frac{1}{2}$
2. x^2-x-1 where $x=-\frac{2}{3}$
3. a^4-b^4 where $a=-3$ and $b=-1$
4. $a^2-3ab+5b^2$ where $a=4$ and $b=-2$
5. $(2x+1)(x-3)$ where $x=-3$
6. $(3x+1)(x+5)$ where $x=-5$
7. $\sqrt{b^2-4ac}$ where $a=2, b=-4,$ and $c=-1$
8. $\sqrt{b^2-4ac}$ where $a=3, b=-6,$ and $c=-2$
9. $\pi r^2 h$ where $r=2\sqrt{3}$ and $h=5$
10. $\frac{4}{3}\pi r^3$ where $r=2\sqrt[3]{6}$
11. What is the simple interest earned on a 4 year investment of $\$4,500$ at an annual interest rate of $4\frac{3}{4}\%$?
12. James traveled at an average speed of 48 miles per hour for $2\frac{1}{4}$ hours. How far did he travel?
13. The period of a pendulum T in seconds is given by the formula $T=2\pi\sqrt{\frac{L}{32}}$ where L represents its length in feet. Approximate the period of a pendulum with length 2 feet. Round off to the nearest tenth of a foot.
14. The average distance d, in miles, a person can see an object is given by the formula $d=\frac{\sqrt{6h}}{2}$ where h represents the person's height above the ground, measured in feet. What average distance can a person see an object from a height of 10 feet? Round off to the nearest tenth of a mile.

Answer

1. $\frac{11}{4}$

3. 80

5. 30

7. $2\sqrt{6}$

9. 60π

11. $\$855$

13. 1.6 seconds

Exercise $1.E.14$

Multiply.

1. $\frac{x^{10}\cdot x^2}{x^5}$
2. $\frac{x^6\left(x^2\right)^4}{x^3}$
3. $-7x^2yz^3\cdot 3x^4y^2z$
4. $3a^2b^3c\left(-4a^2bc^4\right)^2$
5. $\frac{-10a^5b^0c^{-4}}{25a^{-2}b^2c^{-3}}$
6. $\frac{-12x^{-6}y^{-2}z}{36x^{-3}y^4z^6}$
7. $\left(-2x^{-5}y^{-3}z\right)^{-4}$
8. $\left(3x^6y^{-3}z^0\right)^{-3}$
9. $\left(\frac{-5a^2b^3}{c^5}\right)^2$
10. $\left(\frac{-3m^5}{5n^2}\right)^3$
11. $\left(\frac{-2a^{-2}b^3c}{3ab^{-2}c^0}\right)^{-3}$
12. $\left(\frac{6a^3b^{-3}c}{2a^7b^0c^{-4}}\right)^{-2}$

Answer

1. x^7

3. $-21x^6y^3z^4$

5. $-\frac{2a^7}{5b^2c}$

7. $\frac{x^{20}y^{12}}{16z^4}$

9. $\frac{25a^4b^6}{c^{10}}$

11. $-\frac{27a^9}{8b^{15}c^3}$

Exercise $1.E.15$

Perform the operations.

1. $\left(4.3\times 10^{22}\right)\left(3.1\times 10^{-8}\right)$
2. $\left(6.8\times 10^{-33}\right)\left(1.6\times 10^{7}\right)$
3. $\frac{1.4\times 10^{-32}}{2\times 10^{-10}}$
4. $\frac{1.15\times 10^{26}}{2.3\times 10^{-7}}$
5. The value of a new tablet computer in dollars can be estimated using the formula $v=450(t+1)^{-1}$ where t represents the number of years after it is purchased. Use the formula to estimate the value of the tablet computer $2\frac{1}{2}$ years after it was purchased.
6. The speed of light is approximately 6.7×10^8 miles per hour. Express this speed in miles per minute and determine the distance light travels in 4 minutes.

Answer

1. 1.333×10^{15}

3. 7×10^{-23}

5. $\$128.57$

Exercise $1.E.16$

Simplify.

1. $\left(x^2+3x-5\right)-\left(2x^2+5x-7\right)$
2. $\left(6x^2-3x+5\right)+\left(9x^2+3x-4\right)$
3. $\left(a^2b^2-ab+6\right)-(ab+9)+\left(a^2b^2-10\right)$
4. $\left(x^2-2y^2\right)-\left(x^2+3xy-y^2\right)-\left(3xy+y^2\right)$
5. $-\frac{3}{4}\left(16x^2+8x-4\right)$
6. $6\left(\frac{4}{3}x^2-\frac{3}{2}x+\frac{5}{6}\right)$
7. $(2x+5)(x-4)$
8. $(3x-2)\left(x^2-5x+2\right)$
9. $\left(x^2-2x+5\right)\left(2x^2-x+4\right)$
10. $\left(a^2+b^2\right)\left(a^2-b^2\right)$
11. $(2a+b)\left(4a^2-2ab+b^2\right)$
12. $(2x-3)^2$
13. $(3x-1)^3$
14. $(2x+3)^4$
15. $\left(x^2-y^2\right)^2$
16. $\left(x^2y^2+1\right)^2$
17. $\frac{27a^2b-9ab+81ab^2}{3ab}$
18. $\frac{125x^3y^3-25x^2y^2+5xy^2}{5xy^2}$
19. $\frac{2x^3-7x^2+7x-2}{2x-1}$
20. $\frac{12x^3+5x^2-7x-3}{4x+3}$
21. $\frac{5x^3-21x^2+6x-3}{x-4}$

22. $\frac{x^4+x^3-3x^2+10x-1}{x+3}$
23. $\frac{a^4-a^3+4a^2-2a+4}{a^2+2}$
24. $\frac{8a^4-10}{a^2-2}$

Answer

1. $-x^2-2x+2$
3. $2a^2b^2-2ab-13$
5. $-12x^2-6x+3$
7. $2x^2-3x-20$
9. $2x^4-5x^3+16x^2-13x+20$
11. $8a^3+b^3$
13. $27x^3-27x^2+9x-1$
15. $x^4-2x^2y^2+y^4$
17. $9a+27b-3$
19. x^2-3x+2
21. $5x^2-x+2+\frac{5}{x-4}$
23. a^2-a+2

Exercise $1.E.17$

Solve.

1. $6x-8=2$
2. $12x-5=3$
3. $\frac{5}{4}x-3=\frac{1}{2}$
4. $\frac{5}{6}x-\frac{1}{4}=\frac{3}{2}$
5. $\frac{9x+2}{3}=\frac{5}{6}$
6. $\frac{3x-8}{10}=\frac{5}{2}$
7. $3a-5-2a=4a-6$
8. $8-5y+2=4-7y$
9. $5x-6-8x=1-3x$
10. $17-6x-10=5x+7-11x$
11. $5(3x+3)-(10x-4)=4$
12. $6-2(3x-1)=-4(1-3x)$
13. $9-3(2x+3)+6x=0$
14. $-5(x+2)-(4-5x)=1$
15. $\frac{5}{9}(6y+27)=2-\frac{1}{3}(2y+3)$
16. $4-\frac{4}{5}(3a+10)=\frac{1}{10}(4-2a)$
17. Solve for $s: A=\pi r^2+\pi rs$
18. Solve for $x: y=mx+b$
19. A larger integer is 3 more than twice another. If their sum divided by 2 is 9, find the integers.
20. The sum of three consecutive odd integers is 171. Find the integers.
21. The length of a rectangle is 3 meters less than twice its width. If the perimeter measures 66 meters, find the length and width.
22. How long will it take $\$500$ to earn $\$124$ in simple interest earning 6.2% annual interest?
23. It took Sally $3\frac{1}{2}$ hours to drive the 147 miles home from her grandmother's house. What was her average speed?
24. Jeannine invested her bonus of $\$8,300$ in two accounts. One account earned $3\frac{1}{2}$ % simple interest and the other earned $4\frac{3}{4}$ % simple interest. If her total interest for one year was $\$341.75$, how much did she invest in each account?

Answer

1. $\frac{5}{3}$

3. $\frac{14}{5}$

5. $\frac{1}{18}$

7. $\frac{1}{3}$

9. $\varnothing$

11. -3

13. $\mathbb{R}$

15. $-\frac{7}{2}$

17. $s=\frac{A-\pi r^{2}}{\pi r}$

19. $5, 13$

21. Length: 21 meters; Width: 12 meters

23. 42 miles per hour

Exercise $1.E.18$

Solve. Graph all solutions on a number line and provide the corresponding interval notation.

1. $5x-7<18$
2. $2x-1>2$
3. $9-x\leq 3$
4. $3-7x\geq 10$
5. $61-3(x+3)>13$
6. $7-3(2x-1)\geq 6$
7. $\frac{1}{3}(9x+15)-\frac{1}{2}(6x-1)<0$
8. $\frac{2}{3}(12x-1)+\frac{1}{4}(1-32x)<0$
9. $20+4(2a-3)\geq\frac{1}{2}a+2$
10. $\frac{1}{3}\left(2x+\frac{3}{2}\right)-\frac{1}{4}x<\frac{1}{2}\left(1-\frac{1}{2}x\right)$
11. $-4\leq 3x+5<11$
12. $5<2x+15\leq 13$
13. $-1<4(x+1)-1<9$
14. $0\leq 3(2x-3)+1\leq 10$
15. $-1<\frac{2x-5}{4}<1$
16. $-2\leq\frac{3-x}{3}<1$
17. $2x+3<13$ and $4x-1>10$
18. $3x-1\leq 8$ and $2x+5\geq 23$
19. $5x-3<-2$ or $5x-3>2$
20. $1-3x\leq -1$ or $1-3x\geq 1$
21. $5x+6<6$ or $9x-2>-11$
22. $2(3x-1)<-16$ or $3(1-2x)<-15$
23. Jerry scored $90, 85, 92$, and 76 on the first four algebra exams. What must he score on the fifth exam so that his average is at least 80?
24. If 6 degrees less than 3 times an angle is between 90 degrees and 180 degrees, then what are the bounds of the original angle?

Answer

1. $(-\infty, 5)$;

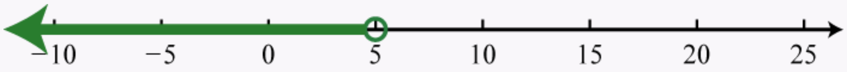


Figure 1.E.5

3. $[6, \infty)$;

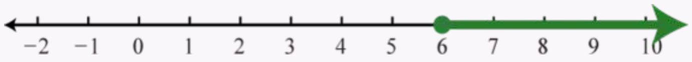

−2
−1
0
1
2
3
4
5
6
7
8
9
10

Figure 1.E.6

5. $(-\infty, 13)$;

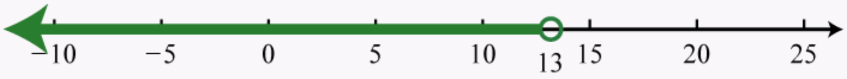

−10
−5
0
5
10
13
15
20
25

Figure 1.E.7

7. $\varnothing$;

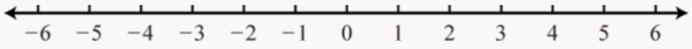

−6
−5
−4
−3
−2
−1
0
1
2
3
4
5
6

Figure 1.E.8

9. $[-\frac{4}{5}, \infty)$;

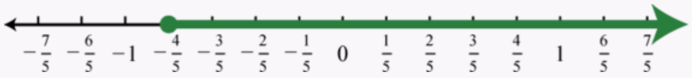


Figure 1.E.9

11. $[-3, 2)$;

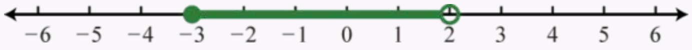

−6
−5
−4
−3
−2
−1
0
1
2
3
4
5
6

Figure 1.E.10

13. $(-1, \frac{3}{2})$;

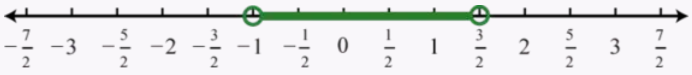


Figure 1.E.11

15. $(\frac{1}{2}, \frac{9}{2})$;

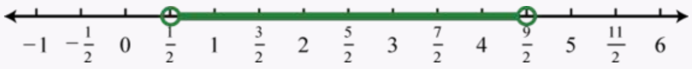


Figure 1.E.12

17. $(\frac{11}{4}, 5)$;

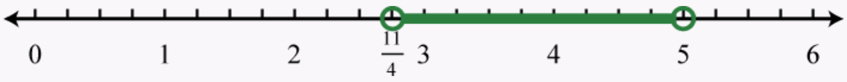

0
1
2
11/4
3
4
5
6

Figure 1.E.13

19. $(-\infty, \frac{1}{5}) \cup (1, \infty)$;

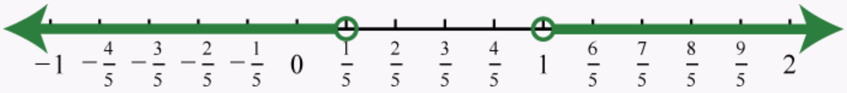


Figure 1.E.14

21. $\mathbb{R}$;

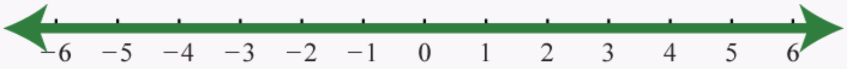

−6
−5
−4
−3
−2
−1
0
1
2
3
4
5
6

Figure 1.E.15

23. Jerry must score at least 57 on the fifth exam.

Sample Exam

Exercise $1.E.19$

Simplify.

1. $5-3\left(12-\left|2-5^{2}\right|\right)$
2. $\left(-\frac{1}{2}\right)^{2}-\left(3-2\left|-\frac{3}{4}\right|\right)^{3}$
3. $-7\sqrt{60}$
4. $5\sqrt[3]{-32}$
5. Find the diagonal of a square with sides measuring 6 centimeters.

Answer

1. 38

3. $-14\sqrt{15}$

5. $6\sqrt{2}$ centimeters

Exercise $1.E.20$

Simplify

1. $-5x^{2}yz^{-1}\left(3x^{3}y^{-2}z\right)$
2. $\left(\frac{-2a^{-4}b^{2}c}{a^{-3}b^{0}c^{2}}\right)^{-3}$
3. $2\left(3a^{2}b^{2}+2ab-1\right)-a^{2}b^{2}+2ab-1$
4. $\left(x^{2}-6x+9\right)-\left(3x^{2}-7x+2\right)$
5. $(2x-3)^{3}$
6. $(3a-b)\left(9a^{2}+3ab+b^{2}\right)$
7. $\frac{6x^{4}-17x^{3}+16x^{2}-18x+13}{2x-3}$

Answer

2. $-\frac{a^{3}c^{3}}{8b^{6}}$

4. $-2x^{2}+x+7$

6. $27a^{3}-b^{3}$

Exercise $1.E.21$

Solve.

1. $\frac{4}{5}x-\frac{2}{15}=2$
2. $\frac{3}{4}(8x-12)-\frac{1}{2}(2x-10)=16$
3. $12-5(3x-1)=2(4x+3)$
4. $\frac{1}{2}(12x-2)+5=4\left(\frac{3}{2}x-8\right)$
5. Solve for $y: ax+by=c$

Answer

1. $\frac{8}{3}$

3. $\frac{11}{23}$

5. $y=\frac{c-ax}{b}$

Exercise $1.E.22$

Solve. Graph the solutions on a number line and give the corresponding interval notation.

1. $2(3x-5)-(7x-3)\geq 0$

2. $2(4x-1)-4(5+2x)<-10$
3. $-6 \leq \frac{1}{4}(2x-8)<4$
4. $3x-7>14$ or $3x-7<-14$

Answer

2. $\mathbb{R}$;

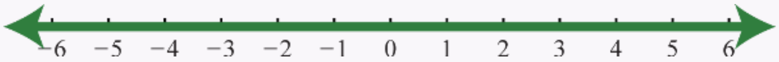


Figure 1.E.16

4. $\left(-\infty,-\frac{7}{3}\right) \cup (7,\infty)$;

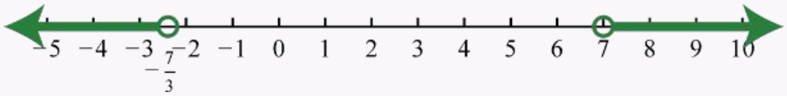


Figure 1.E.17

Exercise $1.E.23$

Use algebra to solve the following.

1. Degrees Fahrenheit F is given by the formula $F=\frac{9}{5}C+32$ where C represents degrees Celsius. What is the Fahrenheit equivalent to $35°$ Celsius?
2. The length of a rectangle is 5 inches less than its width. If the perimeter is 134 inches, find the length and width of the rectangle.
3. Melanie invested $4,500$ in two separate accounts. She invested part in a CD that earned 3.2% simple interest and the rest in a savings account that earned 2.8% simple interest. If the total simple interest for one year was $\$138.80$, how much did she invest in each account?
4. A rental car costs $\$45.00$ per day plus $\$0.48$ per mile driven. If the total cost of a one-day rental is to be at most $\$105$, how many miles can be driven?

Answer

2. Length: 31 inches; width: 36 inches

4. The car can be driven at most 125 miles.

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02: Graphing Functions and Inequalities

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CHAPTER OVERVIEW

Functions

Absolute Value Functions
Composition of Functions
Domain and Range
Functions and Function Notation
Inverse Functions
Prelude to Functions
Rates of Change and Behavior of Graphs
Transformation of Functions

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Absolute Value Functions

Learning Objectives

- Graph an absolute value function.
- Solve an absolute value equation.

Until the 1920s, the so-called spiral nebulae were believed to be clouds of dust and gas in our own galaxy, some tens of thousands of light years away. Then, astronomer Edwin Hubble proved that these objects are galaxies in their own right, at distances of millions of light years. Today, astronomers can detect galaxies that are billions of light years away. Distances in the universe can be measured in all directions. As such, it is useful to consider distance as an absolute value function. In this section, we will investigate **absolute value functions**.

Figure 1: Distances in deep space can be measured in all directions. As such, it is useful to consider distance in terms of absolute values. (credit: "s58y"/Flickr)

Understanding Absolute Value

Recall that in its basic form $f(x) = |x|$, the **absolute value** function, is one of our toolkit functions. The absolute value function is commonly thought of as providing the distance the number is from zero on a number line. Algebraically, for whatever the input value is, the output is the value without regard to sign.

Absolute Value Function

The absolute value function can be defined as a piecewise function

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \tag{1}$$

Example 1: Determine a Number within a Prescribed Distance

Describe all values x within or including a distance of 4 from the number 5.

Solution

We want the distance between x and 5 to be less than or equal to 4. We can draw a number line, such as the one in , to represent the condition to be satisfied.

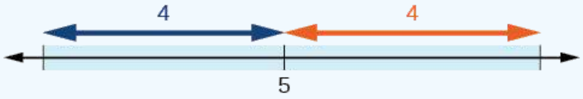


Figure 2: Number line describing the difference of the distance of 4 away from 5.

The distance from x to 5 can be represented using the absolute value as $|x-5|$. We want the values of x that satisfy the condition $|x-5| \leq 4$.

Analysis

Note that

$$-4 \leq x - 5 \qquad x - 5 \leq 4$$
$$1 \leq x \qquad x \leq 9$$

So $|x-5| \leq 4$ is equivalent to $1 \leq x \leq 9$.

However, mathematicians generally prefer absolute value notation.

Exercise 1

Describe all values x within a distance of 3 from the number 2.

Answer

$|x-2| \leq 3$

Example 2: Resistance of a Resistor

Electrical parts, such as resistors and capacitors, come with specified values of their operating parameters: resistance, capacitance, etc. However, due to imprecision in manufacturing, the actual values of these parameters vary somewhat from piece to piece, even when they are supposed to be the same. The best that manufacturers can do is to try to guarantee that the variations will stay within a specified range, often ±1%, ±5%, or ±10%.

Suppose we have a resistor rated at 680 ohms, ±5%. Use the absolute value function to express the range of possible values of the actual resistance.

Solution

5% of 680 ohms is 34 ohms. The absolute value of the difference between the actual and nominal resistance should not exceed the stated variability, so, with the resistance R in ohms,

$$|R-680| \leq 34$$

Exercise 2

Students who score within 20 points of 80 will pass a test. Write this as a distance from 80 using absolute value notation.

Answer

Using the variable p for passing, $|p-80| \leq 20$

Graphing an Absolute Value Function

The most significant feature of the absolute value graph is the corner point at which the graph changes direction. This point is shown at the origin in Figure 3.

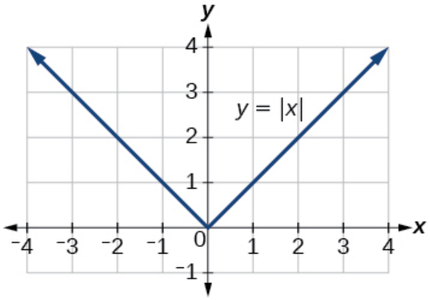


Figure 3: Graph of an absolute function.

Figure 3 shows the graph of $y=2|x-3|+4$. The graph of $y=|x|$ has been shifted right 3 units, vertically stretched by a factor of 2, and shifted up 4 units. This means that the corner point is located at $(3,4)$ for this transformed function.

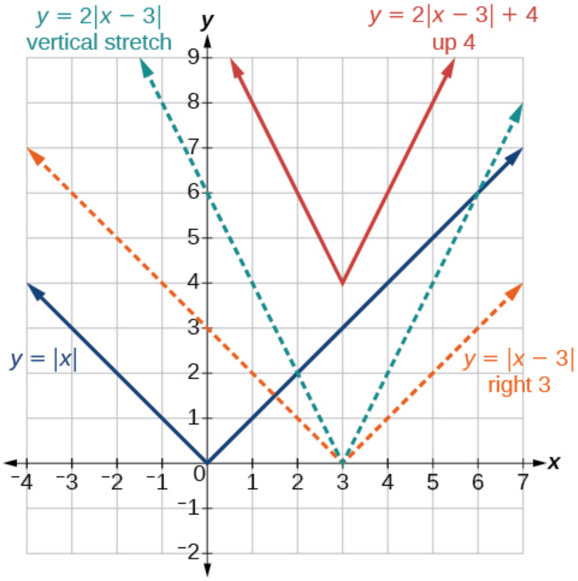


Figure 4: Graph of the different types of transformations for an absolute function.

Example 3: Writing an Equation for an Absolute Value Function

Write an equation for the function graphed in Figure 5.

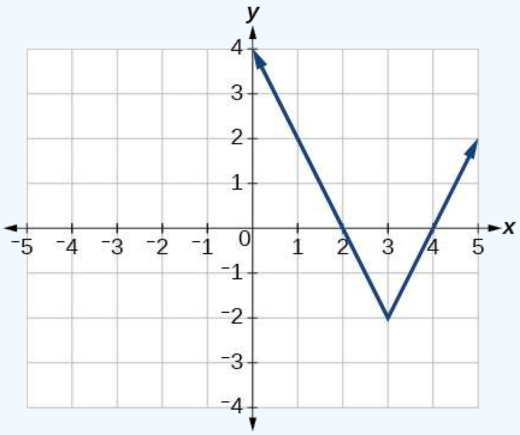


Figure 5: Graph of an absolute function.

Solution

The basic absolute value function changes direction at the origin, so this graph has been shifted to the right 3 units and down 2 units from the basic toolkit function. See Figure 6.

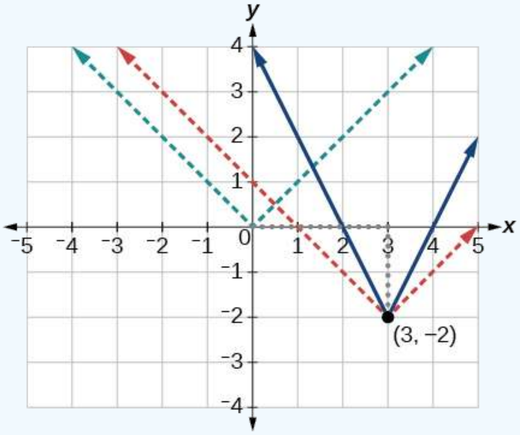


Figure 6: Graph of two transformations for an absolute function at $(3,-2)$.

We also notice that the graph appears vertically stretched, because the width of the final graph on a horizontal line is not equal to 2 times the vertical distance from the corner to this line, as it would be for an unstretched absolute value function. Instead, the width is equal to 1 times the vertical distance as shown in Figure 7.

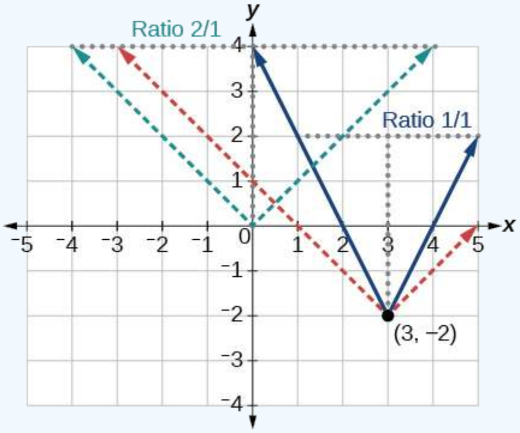


Figure 7: Graph of two transformations for an absolute function at $(3,-2)$ and the ratios between the two different transformations.

From this information we can write the equation

$$\begin{aligned} f(x) &= 2|x-3|-2, && \text{treating the stretch as a vertial stretch, or} \\ f(x) &= |2(x-3)|-2, && \text{treating the stretch as a horizontal compression.} \end{aligned}$$

Analysis

Note that these equations are algebraically equivalent—the stretch for an absolute value function can be written interchangeably as a vertical or horizontal stretch or compression.

Q & A

If we couldn't observe the stretch of the function from the graphs, could we algebraically determine it?

Answer

Yes. If we are unable to determine the stretch based on the width of the graph, we can solve for the stretch factor by putting in a known pair of values for x and $f(x)$.

$$f(x) = a|x-3|-2$$

Now substituting in the point $(1,2)$

$$\begin{aligned} 2 &= a|1-3|-2 \\ 4 &= 2a \\ a &= 2 \end{aligned}$$

Exercise 3

Write the equation for the absolute value function that is horizontally shifted left 2 units, is vertically flipped, and vertically shifted up 3 units.

Answer

$f(x) = -|x+2| + 3$

Q & A

Do the graphs of absolute value functions always intersect the vertical axis? The horizontal axis?

Answer

Yes, they always intersect the vertical axis. The graph of an absolute value function will intersect the vertical axis when the input is zero.

No, they do not always intersect the horizontal axis. The graph may or may not intersect the horizontal axis, depending on how the graph has been shifted and reflected. It is possible for the absolute value function to intersect the horizontal axis at zero, one, or two points (Figure 8).

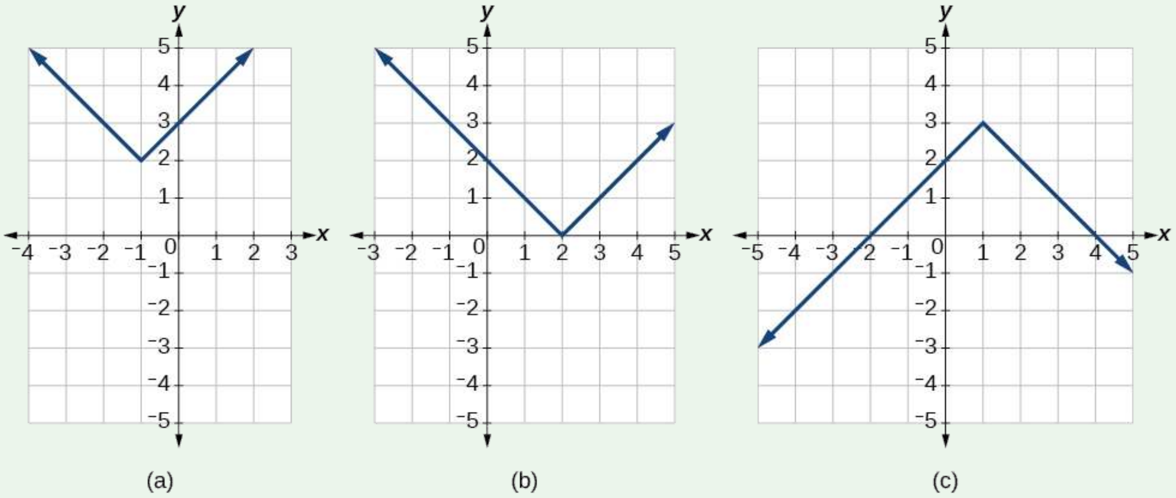


Figure 8: (a) The absolute value function does not intersect the horizontal axis. (b) The absolute value function intersects the horizontal axis at one point. (c) The absolute value function intersects the horizontal axis at two points.

Solving an Absolute Value Equation

Now that we can graph an absolute value function, we will learn how to solve an absolute value equation. To solve an equation such as $8 = |2x-6|$, we notice that the absolute value will be equal to 8 if the quantity inside the absolute value is 8 or -8. This leads to two different equations we can solve independently.

$$2x-6=8 \quad \text{or} \quad 2x-6=-8$$

$$\begin{array}{ll} 2x=14 & 2x=-2 \\ x=7 & x=-1 \end{array}$$

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

An **absolute value equation** is an equation in which the unknown variable appears in absolute value bars. For example,

$$\begin{aligned} |x| &= 4, \\ |2x-1| &= 3, \\ |5x+2|-4 &= 9. \end{aligned}$$

Solutions to Absolute Value Equations

For real numbers A and B, an equation of the form $|A| = B$, with $B \geq 0$, will have solutions when $A = B$ or $A = -B$. If $B < 0$, the equation $|A| = B$ has no solution.

How To ...

Given the formula for an absolute value function, find the horizontal intercepts of its graph.

1. Isolate the absolute value term.
2. Use $|A| = B$ to write $A = B$ or $-A = B$, assuming $B > 0$.
3. Solve for x.

Example 4: Finding the Zeros of an Absolute Value Function

For the function $f(x) = |4x+1| - 7$, find the values of x such that $f(x) = 0$.

Solution

$$\begin{array}{lll} 0 = |4x+1| - 7 & & \text{Substitute 0 for f(x).} \\ & & \text{Isolate the absolute value on one} \\ 7 = |4x+1| & & \text{side of the equation.} \\ 7 = 4x+1 & \text{or} -7 = 4x+1 & \text{Break into two separate equations and solve.} \\ 6 = 4x & \quad -8 = 4x & \\ x = \dfrac{6}{4} = 1.5 & \quad x = \dfrac{-8}{4} = -2 & \end{array}$$

The function outputs 0 when $x = 1.5$ or $x = -2$ (Figure 9).

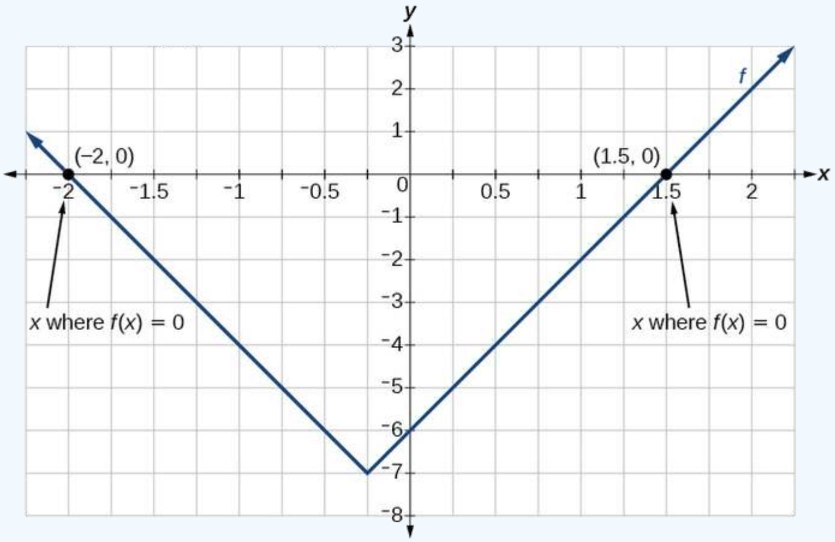


Figure 9: Graph of an absolute function with x-intercepts at -2 and 1.5.

Exercise 4

For the function $f(x) = |2x-1| - 3$, find the values of x such that $f(x) = 0$.

Solution

$x = -1$ or $x = 2$

Q & A

Should we always expect two answers when solving $|A| = B$?

Answer

No. We may find one, two, or even no answers. For example, there is no solution to $2 + |3x - 5| = 1$.

How To ...

Given an absolute value equation, solve it.

1. Isolate the absolute value term.
2. Use $|A| = B$ to write $A = B$ or $A = -B$.
3. Solve for x.

Example 5: Solving an Absolute Value Equation

Solve $1 = 4|x - 2| + 2$.

Solution

Isolating the absolute value on one side of the equation gives the following.

$$\begin{aligned} 1 &= 4|x-2|+2 \\ -1 &= 4|x-2| \\ -\frac{1}{4} &= |x-2| \end{aligned}$$

The absolute value always returns a positive value, so it is impossible for the absolute value to equal a negative value. At this point, we notice that this equation has no solutions.

Q & A

In Example 5, if $f(x) = 1$ and $g(x) = 4|x - 2| + 2$ were graphed on the same set of axes, would the graphs intersect?

Answer

No. The graphs of f and g would not intersect, as shown in Figure 10. This confirms, graphically, that the equation $1 = 4|x - 2| + 2$ has no solution.

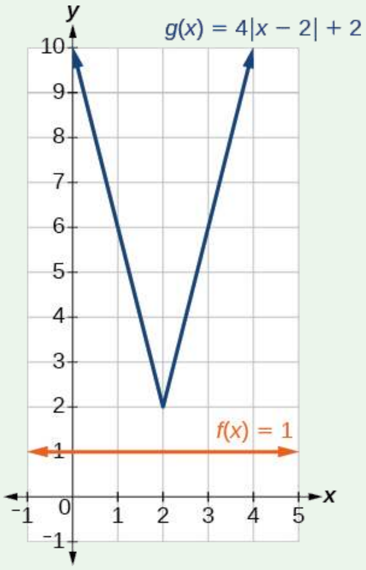


Figure 10: Graph of $g(x) = 4|x - 2| + 2$ and $f(x) = 1$.

Find where the graph of the function $f(x) = -|x + 2| + 3$ intersects the horizontal and vertical axes.

$f(0)=1$, so the graph intersects the vertical axis at $(0,1)$. $f(x)=0$ when $x=-5$ and $x=1$ so the graph intersects the horizontal axis at $(-5,0)$ and $(1,0)$.

Solving an Absolute Value Inequality

Absolute value equations may not always involve equalities. Instead, we may need to solve an equation within a range of values. We would use an **absolute value inequality** to solve such an equation. An absolute value inequality is an equation of the form

$$|A|<B,\ |A|\leq B, |A|>B,$$

or

$$|A|\geq B,$$

where an expression A (and possibly but not usually B) depends on a variable x. Solving the inequality means finding the set of all x that satisfy the inequality. Usually this set will be an interval or the union of two intervals.

There are two basic approaches to solving absolute value inequalities: graphical and algebraic. The advantage of the graphical approach is we can read the solution by interpreting the graphs of two functions. The advantage of the algebraic approach is it yields solutions that may be difficult to read from the graph.

For example, we know that all numbers within 200 units of 0 may be expressed as

$$|x|<200$$

or

$$-200<x<200$$

Suppose we want to know all possible returns on an investment if we could earn some amount of money within \$200 of \$600. We can solve algebraically for the set of values x such that the distance between x and 600 is less than 200. We represent the distance between x and 600 as $|x-600|$.

$$|x-600|<200 \tag{2}$$

or

$$-200<x-600<200 \tag{3}$$

$$\begin{aligned} -200+600<&x-600+600<200+600 \\ 400<&x<800 \end{aligned}$$

This means our returns would be between \$400 and \$800.

Sometimes an absolute value inequality problem will be presented to us in terms of a shifted and/or stretched or compressed absolute value function, where we must determine for which values of the input the function's output will be negative or positive.

How To ...

Given an absolute value inequality of the form $|x-A|\leq B$ for real numbers a and b where b is positive, solve the absolute value inequality algebraically.

1. Find boundary points by solving $|x-A|=B$.
2. Test intervals created by the boundary points to determine where $|x-A|\leq B$.
3. Write the interval or union of intervals satisfying the inequality in interval, inequality, or set-builder notation.

Example 6: Solving an Absolute Value Inequality

Solve $|x-5|\leq 4$.

Solution

With both approaches, we will need to know first where the corresponding equality is true. In this case we first will find where $|x-5|=4$. We do this because the absolute value is a function with no breaks, so the only way the function values can switch from being less than 4 to being greater than 4 is by passing through where the values equal 4. Solve $|x-5|=4$.

$$\begin{array}{lcl} x-5=4 & \text{or} & x=9 \\ x-5=-4 & & x=1 \end{array}$$

After determining that the absolute value is equal to 4 at $x=1$ and $x=9$, we know the graph can change only from being less than 4 to greater than 4 at these values. This divides the number line up into three intervals:

$$x<1,\ 1<x<9,\ \text{and}\ x>9.$$

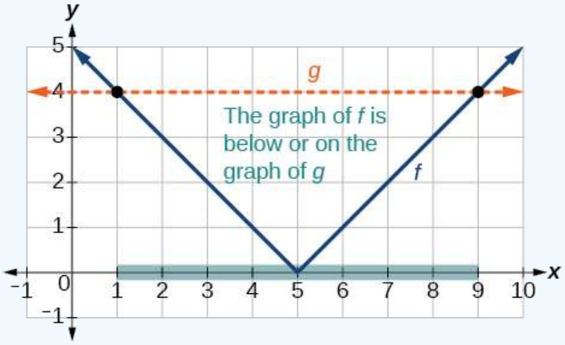
To determine when the function is less than 4, we could choose a value in each interval and see if the output is less than or greater than 4, as shown in Table 1.

Table 1

Interval test x	$f(x)$	<4 or >4	
$x<1$	0	$\lvert 0-5\rvert=5$	Greater than
$1<x<9$	6	$\lvert 6-5\rvert=1$	Less than
$x>9$	11	$\lvert 11-5\rvert=6$	Greater than

Because $1 \leq x \leq 9$ is the only interval in which the output at the test value is less than 4, we can conclude that the solution to $|x-5| \leq 4$ is $1 \leq x \leq 9$, or $[1,9]$.

To use a graph, we can sketch the function $f(x)=|x-5|$. To help us see where the outputs are 4, the line $g(x)=4$ could also be sketched as in Figure 11.



The graph of f is below or on the graph of g

Figure 11: Graph to find the points satisfying an absolute value inequality.

We can see the following:

- The output values of the absolute value are equal to 4 at $x=1$ and $x=9$.
- The graph of f is below the graph of g on $1<x<9$. This means the output values of $f(x)$ are less than the output values of $g(x)$.
- The absolute value is less than or equal to 4 between these two points, when $1 \leq x \leq 9$. In interval notation, this would be the interval $[1,9]$.

Analysis

For absolute value inequalities,

$$\begin{array}{cc} |x-A|<C, & |x-A|>C, \\ -C<x-A<C, & x-A<-C \text{ or } x-A>C. \end{array}$$

The $<$ or $>$ symbol may be replaced by $\leq$ or $\geq$.

So, for this example, we could use this alternative approach.

$$\begin{array}{ll} |x-5| \leq 4 & \\ -4 \leq x-5 \leq 4 & \text{Rewrite by removing the absolute value bars.} \\ -4+5 \leq x-5+5 \leq 4+5 & \text{Isolate the x.} \\ 1 \leq x \leq 9 & \end{array}$$

Exercise 5

Solve $|x+2| \leq 6$.

Answer

$-8 \leq x \leq 4$

How To ...

Given an absolute value function, solve for the set of inputs where the output is positive (or negative).

1. Set the function equal to zero, and solve for the boundary points of the solution set.
2. Use test points or a graph to determine where the function's output is positive or negative.

Example 7: Using a Graphical Approach to Solve Absolute Value Inequalities

Given the function $f(x) = -\frac{1}{2}|4x-5|+3$, determine the x-values for which the function values are negative.

Solution

We are trying to determine where $f(x) < 0$, which is when $-\frac{1}{2}|4x-5|+3 < 0$. We begin by isolating the absolute value.

$$\begin{aligned} -\frac{1}{2}|4x-5| &< -3 \quad \text{Multiply both sides by } -2\text{, and reverse the inequality.} \\ |4x-5| &> 6 \end{aligned}$$

Next we solve for the equality $|4x-5| = 6$.

$$\begin{aligned} 4x-5 &= 6 \qquad 4x-5=-6 \\ 4x-6 &= 6 \end{aligned}$$

or

$$\begin{aligned} 4x &= -1 \\ x &= \frac{11}{4} \qquad x = -\frac{1}{4} \end{aligned}$$

Now, we can examine the graph of f to observe where the output is negative. We will observe where the branches are below the x-axis. Notice that it is not even important exactly what the graph looks like, as long as we know that it crosses the horizontal axis at $x = -\frac{1}{4}$ and $x = \frac{11}{4}$ and that the graph has been reflected vertically. See Figure 12.

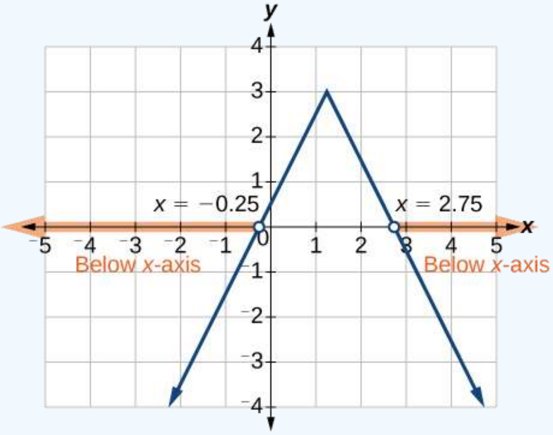


Figure 12: Graph of an absolute function with x-intercepts at -0.25 and 2.75.]

We observe that the graph of the function is below the x-axis left of $x = -\frac{1}{4}$ and right of $x = \frac{11}{4}$. This means the function values are negative to the left of the first horizontal intercept at $x = -\frac{1}{4}$, and negative to the right of the second intercept at $x = \frac{11}{4}$. This gives us the solution to the inequality.

$$x < -\frac{1}{4} \text{ or } x > 1\frac{1}{4}$$

In interval notation, this would be $(-\infty, -0.25) \cup (2.75, \infty)$.

Exercise 6

Solve $-2|k-4| \leq -6$.

Answer

$k \leq 1$ or $k \geq 7$; in interval notation, this would be $(-\infty, 1] \cup [7, \infty)$

Key Concepts

- The absolute value function is commonly used to measure distances between points.
- Applied problems, such as ranges of possible values, can also be solved using the absolute value function.
- The graph of the absolute value function resembles a letter V. It has a corner point at which the graph changes direction.
- In an absolute value equation, an unknown variable is the input of an absolute value function.
- If the absolute value of an expression is set equal to a positive number, expect two solutions for the unknown variable.
- An absolute value equation may have one solution, two solutions, or no solutions.
- An absolute value inequality is similar to an absolute value equation but takes the form | A |<B, | A |≤B, | A |>B, or | A |≥B.It can be solved by determining the boundaries of the solution set and then testing which segments are in the set.
- Absolute value inequalities can also be solved graphically.

Glossary

absolute value equation
an equation of the form $|A| = B$, with $B \geq 0$; it will have solutions when $A = B$ or $A = -B$

absolute value inequality
a relationship in the form $|A| < B$, $|A| \leq B$, $|A| > B$, or $|A| \geq B$

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Composition of Functions

Learning Objectives

- Combine functions using algebraic operations.
- Create a new function by composition of functions.
- Evaluate composite functions.
- Find the domain of a composite function.
- Decompose a composite function into its component functions.

Suppose we want to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The cost depends on the temperature, and the temperature depends on the day.

Using descriptive variables, we can notate these two functions. The function $C(T)$ gives the cost C of heating a house for a given average daily temperature in T degrees Celsius. The function $T(d)$ gives the average daily temperature on day d of the year. For any given day, $Cost = C(T(d))$ means that the cost depends on the temperature, which in turns depends on the day of the year. Thus, we can evaluate the cost function at the temperature $T(d)$. For example, we could evaluate $T(5)$ to determine the average daily temperature on the 5th day of the year. Then, we could evaluate the **cost function** at that temperature. We would write $C(T(5))$.

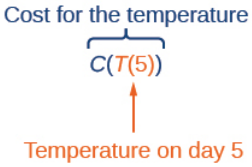


Figure 1: Explanation of $C(T(5))$, which is the cost for the temperature and $T(5)$ is the temperature on day 5.

By combining these two relationships into one function, we have performed function composition, which is the focus of this section.

Combining Functions Using Algebraic Operations

Function composition is only one way to combine existing functions. Another way is to carry out the usual algebraic operations on functions, such as addition, subtraction, multiplication and division. We do this by performing the operations with the function outputs, defining the result as the output of our new function.

Suppose we need to add two columns of numbers that represent a husband and wife's separate annual incomes over a period of years, with the result being their total household income. We want to do this for every year, adding only that year's incomes and then collecting all the data in a new column. If $w(y)$ is the wife's income and $h(y)$ is the husband's income in year y, and we want T to represent the total income, then we can define a new function.

$$T(y) = h(y) + w(y)$$

If this holds true for every year, then we can focus on the relation between the functions without reference to a year and write

$$T = h + w$$

Just as for this sum of two functions, we can define difference, product, and ratio functions for any pair of functions that have the same kinds of inputs (not necessarily numbers) and also the same kinds of outputs (which do have to be numbers so that the usual operations of algebra can apply to them, and which also must have the same units or no units when we add and subtract). In this way, we can think of adding, subtracting, multiplying, and dividing functions.

For two functions $f(x)$ and $g(x)$ with real number outputs, we define new functions $f+g$, $f-g$, fg, and $\frac{f}{g}$ by the relations.

$$\begin{gathered}(f+g)(x) = f(x) + g(x) \\ (f-g)(x) = f(x) - g(x) \\ (fg)(x) = f(x)g(x) \\ \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}\end{gathered}$$

Example 1: Performing Algebraic Operations on Functions

Find and simplify the functions $(g-f)(x)$ and $\left(\frac{g}{f}\right)(x)$, given $f(x)=x-1$ and $g(x)=x^2-1$. Are they the same function?

Solution

Begin by writing the general form, and then substitute the given functions.

$$\begin{aligned}(g-f)(x)&=g(x)-f(x)\\(g-f)(x)&=x^2-1-(x-1)\\&=x^2-x\\&=x(x-1)\end{aligned}$$

$$\begin{aligned}\left(\frac{g}{f}\right)(x)&=g(x)f(x)\\\left(\frac{g}{f}\right)(x)&=\frac{x^2-1}{x-1}\\&=\frac{(x+1)(x-1)}{x-1}\\&=x+1\end{aligned}$$

No, the functions are not the same.

Note: For $\left(\frac{g}{f}\right)(x)$, the condition $x\neq 1$ is necessary because when $x=1$, the denominator is equal to 0, which makes the function undefined.

Exercise 1

Find and simplify the functions $(fg)(x)$ and $(f-g)(x)$.

$$f(x)=x-1$$

and

$$g(x)=x^2-1$$

Are they the same function?

Answer

$(fg)(x)=f(x)g(x)=(x-1)(x2-1)=x^3-x^2-x+1$
$(f-g)(x)=f(x)-g(x)=(x-1)-(x^2-1)=x-x^2$

No, the functions are not the same.

Create a Function by Composition of Functions

Performing algebraic operations on functions combines them into a new function, but we can also create functions by composing functions. When we wanted to compute a heating cost from a day of the year, we created a new function that takes a day as input and yields a cost as output. The process of **combining functions** so that the output of one function becomes the input of another is known as a **composition of functions**. The resulting function is known as a **composite function**. We represent this combination by the following notation:

$$f\circ g(x)=f(g(x)) \tag{1}$$

We read the left-hand side as“f composed with g at x,” and the right-hand side as“f of g of x.”The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the composition operator. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value. Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication takes two numbers and gives a new number. However, it is important not to confuse function composition with multiplication because, as we learned above, in most cases $f(g(x)) \neq f(x)g(x)$.

It is also important to understand the order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function g takes the input x first and yields an output $g(x)$. Then the function f takes $g(x)$ as an input and yields an output $f(g(x))$.

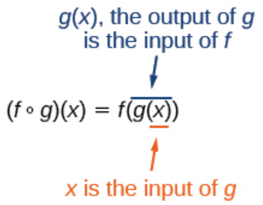


Figure 2: Explanation of the composite function.

In general, $f \circ g$ and $g \circ f$ are different functions. In other words, in many cases $f(g(x)) \neq g(f(x))$ for all x. We will also see that sometimes two functions can be composed only in one specific order.

For example, if $f(x) = x^2$ and $g(x) = x + 2$, then

$$\begin{aligned} f(g(x)) &= f(x+2) \\ &= (x+2)^2 \\ &= x^2 + 4x + 4 \end{aligned}$$

but

$$\begin{aligned} g(f(x)) &= g(x^2) \\ &= x^2 + 2 \end{aligned}$$

These expressions are not equal for all values of x, so the two functions are not equal. It is irrelevant that the expressions happen to be equal for the single input value $x = -\frac{1}{2}$.

Note that the range of the inside function (the first function to be evaluated) needs to be within the domain of the outside function. Less formally, the composition has to make sense in terms of inputs and outputs.

Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a composition of functions. For any input x and functions f and g, this action defines a **composite function**, which we write as $f \circ g$ such that

$$(f \circ g)(x) = f(g(x)) \tag{2}$$

The domain of the composite function $f \circ g$ is all x such that x is in the domain of g and $g(x)$ is in the domain of f.

It is important to realize that the product of functions fg is not the same as the function composition $f(g(x))$, because, in general, $f(x)g(x) \neq f(g(x))$.

Example 2: Determining whether Composition of Functions is Commutative

Using the functions provided, find $f(g(x))$ and $g(f(x))$. Determine whether the composition of the functions is **commutative**.

$$f(x) = 2x + 1 \quad g(x) = 3 - x$$

Solution

Let’s begin by substituting $g(x)$ into $f(x)$.

$$\begin{aligned} f(g(x)) &= 2(3-x) + 1 \\ &= 6 - 2x + 1 \\ &= 7 - 2x \end{aligned}$$

Now we can substitute $f(x)$ into $g(x)$.

$$\begin{aligned} g(f(x)) &= 3-(2x+1) \\ &= 3-2x-1 \\ &= 2-2x \end{aligned}$$

We find that $g(f(x)) \neq f(g(x))$, so the operation of function composition is not commutative.

Example 3: Interpreting Composite Functions

The function $c(s)$ gives the number of calories burned completing s sit-ups, and $s(t)$ gives the number of sit-ups a person can complete in t minutes. Interpret $c(s(3))$.

Solution

The inside expression in the composition is $s(3)$. Because the input to the s-function is time, $t=3$ represents 3 minutes, and $s(3)$ is the number of sit-ups completed in 3 minutes.

Using $s(3)$ as the input to the function $c(s)$ gives us the number of calories burned during the number of sit-ups that can be completed in 3 minutes, or simply the number of calories burned in 3 minutes (by doing sit-ups).

Example 4: Investigating the Order of Function Composition

Suppose $f(x)$ gives miles that can be driven in x hours and $g(y)$ gives the gallons of gas used in driving y miles. Which of these expressions is meaningful: $f(g(y))$ or $g(f(x))$?

Solution

The function $y=f(x)$ is a function whose output is the number of miles driven corresponding to the number of hours driven.

$$\text{number of miles } = f(\text{number of hours})$$

The function $g(y)$ is a function whose output is the number of gallons used corresponding to the number of miles driven. This means:

$$\text{number of gallons } = g(\text{number of miles})$$

The expression $g(y)$ takes miles as the input and a number of gallons as the output. The function $f(x)$ requires a number of hours as the input. Trying to input a number of gallons does not make sense. The expression $f(g(y))$ is meaningless.

The expression $f(x)$ takes hours as input and a number of miles driven as the output. The function $g(y)$ requires a number of miles as the input. Using $f(x)$ (miles driven) as an input value for $g(y)$, where gallons of gas depends on miles driven, does make sense. The expression $g(f(x))$ makes sense, and will yield the number of gallons of gas used, g, driving a certain number of miles, $f(x)$, in x hours.

Question/Answer

Are there any situations where $f(g(y))$ and $g(f(x))$ would both be meaningful or useful expressions?

Yes. For many pure mathematical functions, both compositions make sense, even though they usually produce different new functions. In real-world problems, functions whose inputs and outputs have the same units also may give compositions that are meaningful in either order

Exercise 2

The gravitational force on a planet a distance r from the sun is given by the function $G(r)$. The acceleration of a planet subjected to any force F is given by the function $a(F)$. Form a meaningful composition of these two functions, and explain what it means.

Answer

> A gravitational force is still a force, so $a(G(r))$ makes sense as the acceleration of a planet at a distance r from the Sun (due to gravity), but $G(a(F))$ does not make sense.

Evaluating Composite Functions

Once we compose a new function from two existing functions, we need to be able to evaluate it for any input in its domain. We will do this with specific numerical inputs for functions expressed as tables, graphs, and formulas and with variables as inputs to functions expressed as formulas. In each case, we evaluate the inner function using the starting input and then use the inner function's output as the input for the outer function.

Evaluating Composite Functions Using Tables

When working with functions given as tables, we read input and output values from the table entries and always work from the inside to the outside. We evaluate the inside function first and then use the output of the inside function as the input to the outside function.

Example 5: Using a Table to Evaluate a Composite Function

Using Table 1, evaluate $f(g(3))$ and $g(f(3))$.

Table 1

x	$f(x)$	$g(x)$
1	6	3
2	8	5
3	3	2
4	1	7

Solution

To evaluate $f(g(3))$, we start from the inside with the input value 3. We then evaluate the inside expression $g(3)$ using the table that defines the function $g: g(3)=2$. We can then use that result as the input to the function f, so $g(3)$ is replaced by 2 and we get $f(2)$. Then, using the table that defines the function f, we find that $f(2)=8$.

$$g(3)=2$$

$$f(g(3))=f(2)=8$$

To evaluate $g(f(3))$, we first evaluate the inside expression $f(3)$ using the first table: $f(3)=3$. Then, using the table for g, we can evaluate

$$g(f(3))=g(3)=2$$

Table 2 shows the composite functions $f \circ g$ and $g \circ f$ as tables.

Table 2

x	$g(x)$	$f(g(x))$	$f(x)$	$g(f(x))$
3	2	8	3	2

Exercise 3

Using Table 1, evaluate $f(g(1))$ and $g(f(4))$.

Answer

$f(g(1))=f(3)=3$ and $g(f(4))=g(1)=3$

Evaluating Composite Functions Using Graphs

When we are given individual functions as graphs, the procedure for evaluating composite functions is similar to the process we use for evaluating tables. We read the input and output values, but this time, from the x- and y-axes of the graphs.

How To ...

Given a composite function and graphs of its individual functions, evaluate it using the information provided by the graphs.

1. Locate the given input to the inner function on the x-axis of its graph.
2. Read off the output of the inner function from the y-axis of its graph.
3. Locate the inner function output on the x-axis of the graph of the outer function.
4. Read the output of the outer function from the y-axis of its graph. This is the output of the composite function.

Example 6: Using a Graph to Evaluate a Composite Function

Using Figure 3, evaluate $f(g(1))$.

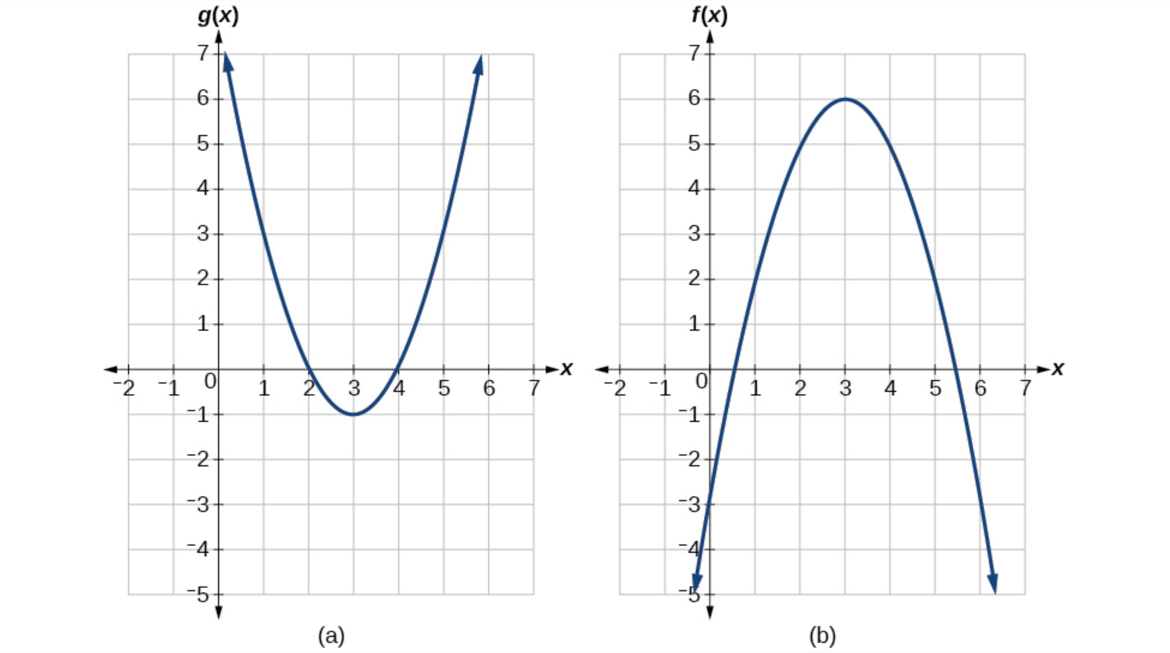
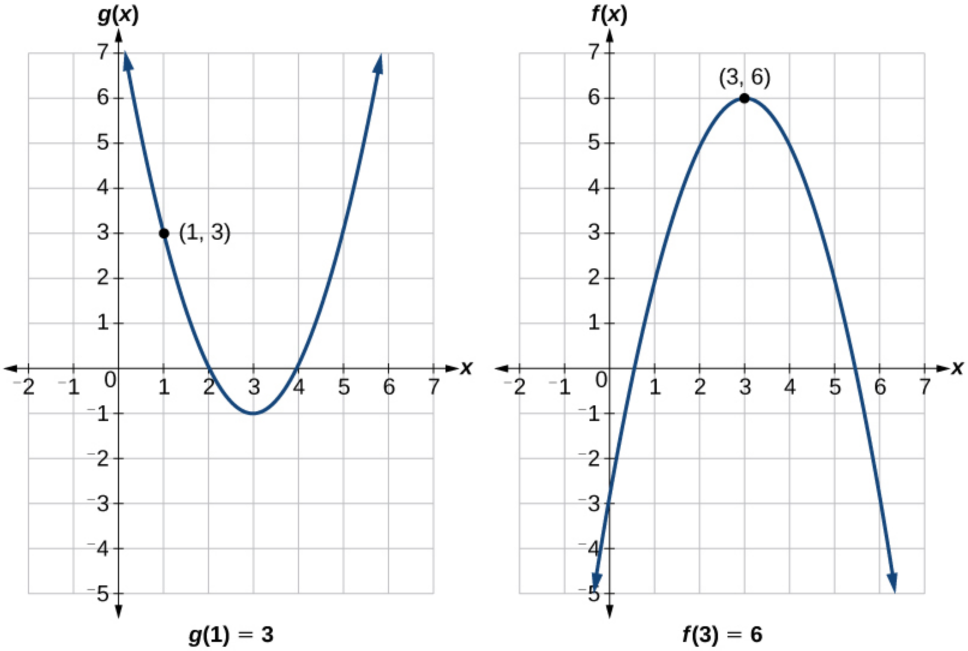


Figure 3: Two graphs of a positive and negative parabola.

Solution

To evaluate $f(g(1))$, we start with the inside evaluation. See Figure 4.



g(1) = 3

f(3) = 6

Figure 4: Two graphs of a positive parabola $g(x)$ and a negative parabola $f(x)$. The following points are plotted: $g(1)=3$ and $f(3)=6$.

We evaluate $g(1)$ using the graph of $g(x)$, finding the input of 1 on the x-axis and finding the output value of the graph at that input. Here, $g(1)=3$. We use this value as the input to the function f.

$$f(g(1))=f(3)$$

We can then evaluate the composite function by looking to the graph of $f(x)$, finding the input of 3 on the x-axis and reading the output value of the graph at this input. Here, $f(3)=6$, so $f(g(1))=6$.

Analysis

Figure 5 shows how we can mark the graphs with arrows to trace the path from the input value to the output value.

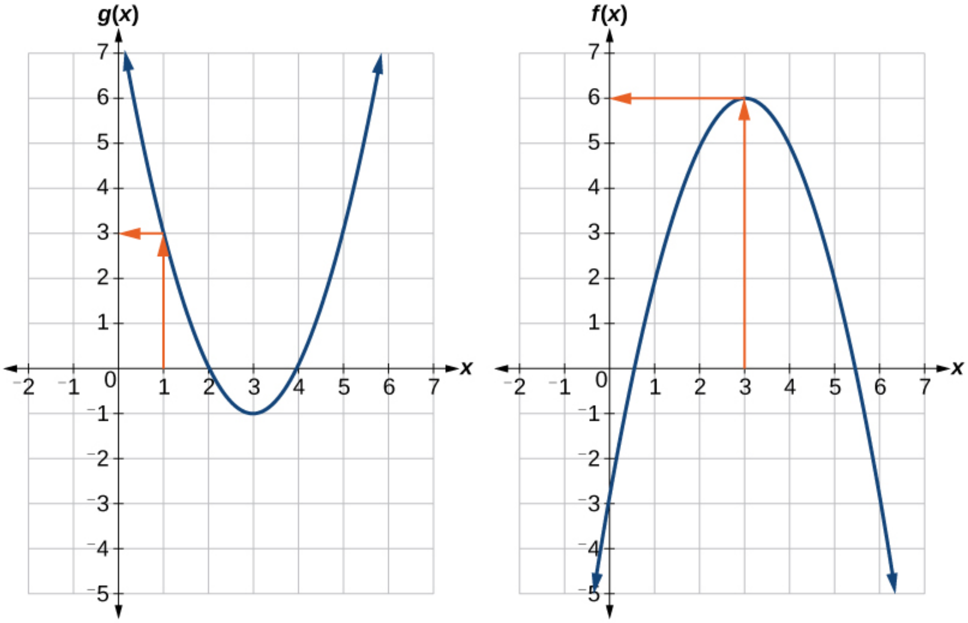


Figure 5: Two graphs of a positive and negative parabola.

Exercise 4

Using Figure 3, evaluate $g(f(2))$.

Answer

$g(f(2))=g(5)=3$

Evaluating Composite Functions Using Formulas

When evaluating a composite function where we have either created or been given formulas, the rule of working from the inside out remains the same. The input value to the outer function will be the output of the inner function, which may be a numerical value, a variable name, or a more complicated expression.

While we can compose the functions for each individual input value, it is sometimes helpful to find a single formula that will calculate the result of a composition $f(g(x))$. To do this, we will extend our idea of function evaluation. Recall that, when we evaluate a function like $f(t)=t^2-t$, we substitute the value inside the parentheses into the formula wherever we see the input variable.

How To...

Given a formula for a composite function, evaluate the function.

1. Evaluate the inside function using the input value or variable provided.
2. Use the resulting output as the input to the outside function.

Example 7: Evaluating a Composition of Functions Expressed as Formulas with a Numerical Input

Given $f(t)=t^2-t$ and $h(x)=3x+2$, evaluate $f(h(1))$.

Solution

Because the inside expression is $h(1)$, we start by evaluating $h(x)$ at 1.

$$\begin{aligned} h(1)&=3(1)+2 \\ h(1)&=5 \end{aligned}$$

Then $f(h(1))=f(5)$, so we evaluate $f(t)$ at an input of 5.

$$\begin{aligned} f(h(1))&=f(5) \\ f(h(1))&=5^2-5 \\ f(h(1))&=20 \end{aligned}$$

Analysis

It makes no difference what the input variables t and x were called in this problem because we evaluated for specific numerical values.

Exercise 5

Given $f(t)=t^2-t$ and $h(x)=3x+2$, evaluate

a. $h(f(2))$
b. $h(f(-2))$

Answer a

8

Answer b

20

Finding the Domain of a Composite Function

As we discussed previously, the **domain of a composite function** such as $f∘g$ is dependent on the domain of g and the domain of f. It is important to know when we can apply a composite function and when we cannot, that is, to know the domain of a function such as $f∘g$. Let us assume we know the domains of the functions f and g separately. If we write the composite function for an input x as $f(g(x))$, we can see right away that x must be a member of the domain of g in order for the expression to be meaningful, because otherwise we cannot complete the inner function evaluation. However, we also see that $g(x)$ must be a member of the domain of f, otherwise the second function evaluation in $f(g(x))$ cannot be completed, and the expression is still undefined. Thus the domain of $f∘g$ consists of only those inputs in the domain of g that produce outputs from g belonging to the domain of f. Note that the domain of f composed with g is the set of all x such that x is in the domain of g and g(x)\) is in the domain of f.

Definition: Domain of a Composite Function

The **domain of a composite function** $f(g(x))$ is the set of those inputs x in the domain of g for which $g(x)$ is in the domain of f.

How To...

Given a function composition $f(g(x))$, determine its domain.

1. Find the domain of g.
2. Find the domain of f.
3. Find those inputs x in the domain of g for which $g(x)$ is in the domain of f. That is, exclude those inputs x from the domain of g for which $g(x)$ is not in the domain of f. The resulting set is the domain of $f∘g$.

Example $8A$: Finding the Domain of a Composite Function

Find the domain of

$$(f\circ g)(x) \text{ where } f(x)=\frac{5}{x-1} \text{ and } g(x)=\frac{4}{3x-2}$$

Solution

The domain of $g(x)$ consists of all real numbers except $x=\frac{2}{3}$, since that input value would cause us to divide by 0. Likewise, the domain of f consists of all real numbers except 1. So we need to exclude from the domain of $g(x)$ that value of x for which $g(x)=1$.

$$\begin{aligned}\frac{4}{3x-2}&=1\\4&=3x-2\\6&=3x\\x&=2\end{aligned}$$

So the domain of $f∘g$ is the set of all real numbers except $\frac{2}{3}$ and 2. This means that

$$x{\neq}\frac{2}{3} \text{ or } x\neq 2$$

We can write this in interval notation as

$$\left(-\infty,\frac{2}{3}\right)\cup\left(\frac{2}{3},2\right)\cup(2,\infty)$$

Example $8B$: Finding the Domain of a Composite Function Involving Radicals

Find the domain of

$$(f∘g)(x) \text{ where } f(x)=\sqrt{x+2} \text{ and } g(x)=\sqrt{3-x}$$

Solution

Because we cannot take the square root of a negative number, the domain of g is $(-\infty, 3]$. Now we check the domain of the composite function

$$(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}$$

For $(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}, \sqrt{3-x}+2 \geq 0$, since the radicand of a square root must be positive. Since square roots are positive, $\sqrt{3-x} \geq 0$,or, $3-x \geq 0$, which gives a domain of $(-\infty, 3]$.

Analysis

This example shows that knowledge of the range of functions (specifically the inner function) can also be helpful in finding the domain of a composite function. It also shows that the domain of $f \circ g$ can contain values that are not in the domain of f, though they must be in the domain of g.

Exercise 6

Find the domain of

$$(f \circ g)(x) \text{ where } f(x) = \frac{1}{x-2} \text{ and } g(x) = \sqrt{x+4}$$

Answer

$[-4, 0) \cup (0, \infty)$

Decomposing a Composite Function into its Component Functions

In some cases, it is necessary to decompose a complicated function. In other words, we can write it as a composition of two simpler functions. There may be more than one way to **decompose a composite function**, so we may choose the decomposition that appears to be most expedient.

Example 9: Decomposing a Function

Write $f(x) = \sqrt{5-x^2}$ as the composition of two functions.

Solution

We are looking for two functions, g and h, so $f(x) = g(h(x))$. To do this, we look for a function inside a function in the formula for $f(x)$. As one possibility, we might notice that the expression $5-x^2$ is the inside of the square root. We could then decompose the function as

$$h(x) = 5-x^2 \text{ and } g(x) = \sqrt{x}$$

We can check our answer by recomposing the functions.

$$g(h(x)) = g(5-x^2) = \sqrt{5-x^2}$$

Exercise 7

Write $f(x) = \dfrac{4}{3-\sqrt{4+x^2}}$ as the composition of two functions.

Answer

Possible answers:

$g(x) = \sqrt{4+x^2}$

$h(x) = \dfrac{4}{3-x}$

$f = h \circ g$

Access these online resources for additional instruction and practice with composite functions.

- Composite Functions (http://openstaxcollege.org/l/compfunction)
- Composite Function Notation Application (http://openstaxcollege.org/l/compfuncnot)
- Composite Functions Using Graphs (http://openstaxcollege.org/l/compfuncgraph)
- Decompose Functions (http://openstaxcollege.org/l/decompfunction)
- Composite Function Values (http://openstaxcollege.org/l/compfuncvalue)

Key Equation

- Composite function $(f \circ g)(x) = f(g(x))$

Key Concepts

- We can perform algebraic operations on functions. See Example.
- When functions are combined, the output of the first (inner) function becomes the input of the second (outer) function.
- The function produced by combining two functions is a composite function. See Example and Example.
- The order of function composition must be considered when interpreting the meaning of composite functions. See Example.
- A composite function can be evaluated by evaluating the inner function using the given input value and then evaluating the outer function taking as its input the output of the inner function.
- A composite function can be evaluated from a table. See Example.
- A composite function can be evaluated from a graph. See Example.
- A composite function can be evaluated from a formula. See Example.
- The domain of a composite function consists of those inputs in the domain of the inner function that correspond to outputs of the inner function that are in the domain of the outer function. See Example and Example.
- Just as functions can be combined to form a composite function, composite functions can be decomposed into simpler functions.
- Functions can often be decomposed in more than one way. See Example.

Glossary

composite function

the new function formed by function composition, when the output of one function is used as the input of another

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Domain and Range

Learning Objectives

- Find the domain of a function defined by an equation.
- Graph piecewise-defined functions.

If you’re in the mood for a scary movie, you may want to check out one of the five most popular horror movies of all time—I am Legend, Hannibal, The Ring, The Grudge, and The Conjuring. Figure 1 shows the amount, in dollars, each of those movies grossed when they were released as well as the ticket sales for horror movies in general by year. Notice that we can use the data to create a function of the amount each movie earned or the total ticket sales for all horror movies by year. In creating various functions using the data, we can identify different independent and dependent variables, and we can analyze the data and the functions to determine the domain and range. In this section, we will investigate methods for determining the domain and range of functions such as these.

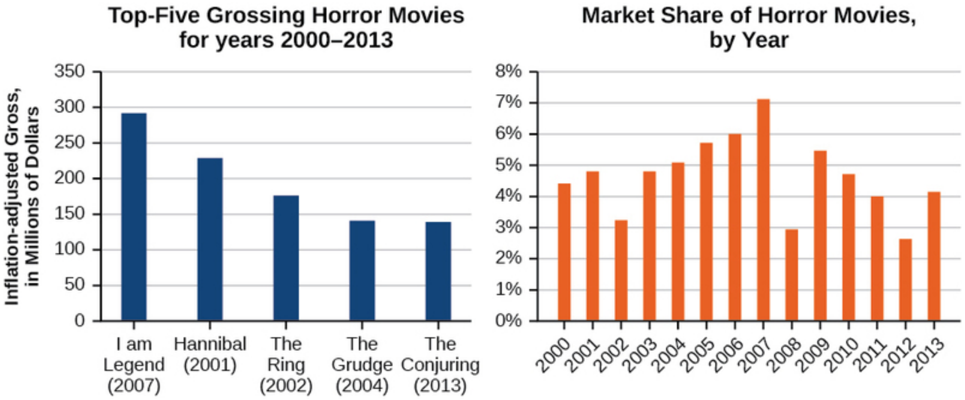


Figure 1: Graph of the Top-Five Grossing Horror Movies for years 2000-2003, and a Graph of the Market Share of Horror Movies by Year. Based on data compiled by www.the-numbers.com.

Finding the Domain of a Function Defined by an Equation

In Functions and Function Notation, we were introduced to the concepts of **domain and range**. In this section, we will practice determining domains and ranges for specific functions. Keep in mind that, in determining domains and ranges, we need to consider what is physically possible or meaningful in real-world examples, such as tickets sales and year in the horror movie example above. We also need to consider what is mathematically permitted. For example, we cannot include any input value that leads us to take an even root of a negative number if the domain and range consist of real numbers. Or in a function expressed as a formula, we cannot include any input value in the domain that would lead us to divide by 0.

We can visualize the domain as a “holding area” that contains “raw materials” for a “function machine” and the range as another “holding area” for the machine’s products (Figure 2).

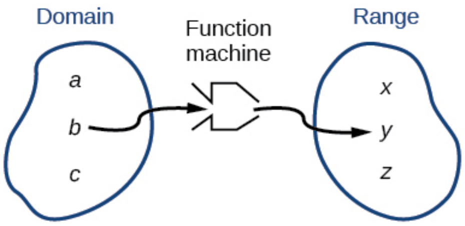


Figure 2: Diagram of how a function relates two relations

We can write the **domain and range** in **interval notation**, which uses values within brackets to describe a set of numbers. In interval notation, we use a square bracket [when the set includes the endpoint and a parenthesis (to indicate that the endpoint is either not included or the interval is unbounded. For example, if a person has $100 to spend, he or she would need to express the interval that is more than 0 and less than or equal to 100 and write $(0,100]$. We will discuss interval notation in greater detail later.

Let’s turn our attention to finding the domain of a function whose equation is provided. Oftentimes, finding the domain of such functions involves remembering three different forms. First, if the function has no denominator or an even root, consider whether the domain could be all real numbers. Second, if there is a denominator in the function’s equation, exclude values in the domain that force the denominator to be zero. Third, if there is an even root, consider excluding values that would make the radicand negative.

Before we begin, let us review the conventions of interval notation:

- The smallest term from the interval is written first.
- The largest term in the interval is written second, following a comma.
- Parentheses, (or), are used to signify that an endpoint is not included, called exclusive.
- Brackets, [or], are used to indicate that an endpoint is included, called inclusive.

See Figure 3 for a summary of interval notation.

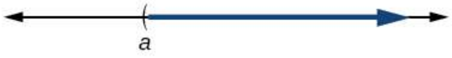
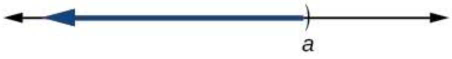
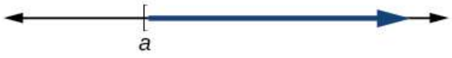
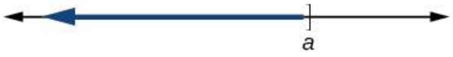
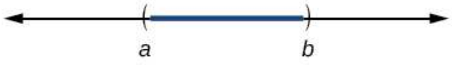
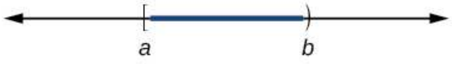
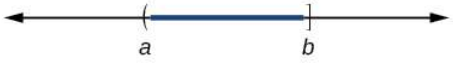
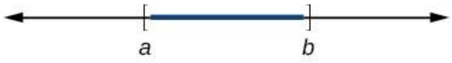
Inequality	Interval Notation	Graph on Number Line	Description
$x > a$	(a, ∞)		*x* is greater than *a*
$x < a$	$(-\infty, a)$		*x* is less than *a*
$x \geq a$	$[a, \infty)$		*x* is greater than or equal to *a*
$x \leq a$	$(-\infty, a]$		*x* is less than or equal to *a*
$a < x < b$	(a, b)		*x* is strictly between *a* and *b*
$a \leq x < b$	$[a, b)$		*x* is between *a* and *b*, to include *a*
$a < x \leq b$	$(a, b]$		*x* is between *a* and *b*, to include *b*
$a \leq x \leq b$	$[a, b]$		*x* is between *a* and *b*, to include *a* and *b*

Figure 3: Summary of interval notation.

Example 1: Finding the Domain of a Function as a Set of Ordered Pairs

Find the domain of the following function: $\{(2,10),(3,10),(4,20),(5,30),(6,40)\}$.

Solution

First identify the input values. The input value is the first coordinate in an ordered pair. There are no restrictions, as the ordered pairs are simply listed. The domain is the set of the first coordinates of the ordered pairs.

$$\{2,3,4,5,6\}$$

Exercse 1

Find the domain of the function:

$$\{(-5,4),(0,0),(5,-4),(10,-8),(15,-12)\}$$

Answer

$\{-5,0,5,10,15\}$

How To: Given a function written in equation form, find the domain.

1. Identify the input values.
2. Identify any restrictions on the input and exclude those values from the domain.
3. Write the domain in interval form, if possible.

Example 2: Finding the Domain of a Function

Find the domain of the function $f(x)=x^2-1$.

Solution

The input value, shown by the variable x in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers.

In interval form, the domain of f is $(-\infty,\infty)$.

Exercse 2

Find the domain of the function:

$$f(x)=5-x+x^3$$

Answer

$(-\infty,\infty)$

Howto: Given a function written in an equation form that includes a fraction, find the domain

1. Identify the input values.
2. Identify any restrictions on the input. If there is a denominator in the function's formula, set the denominator equal to zero and solve for x . If the function's formula contains an even root, set the radicand greater than or equal to 0, and then solve.
3. Write the domain in interval form, making sure to exclude any restricted values from the domain.

Example 3: Finding the Domain of a Function Involving a Denominator

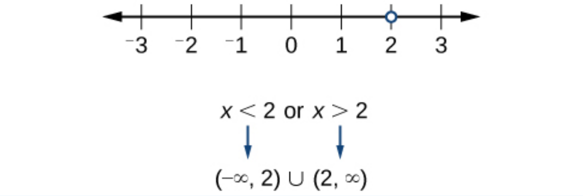
Find the domain of the function $f(x)=\dfrac{x+1}{2-x}$.

Solution

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for x.

$$\begin{aligned}2-x&=0\\-x&=-2\\x&=2\end{aligned}$$

Now, we will exclude 2 from the domain. The answers are all real numbers where $x<2$ or $x>2$. We can use a symbol known as the union, $\cup$,to combine the two sets. In interval notation, we write the solution:$(-\infty,2)\cup(2,\infty)$.



In interval form, the domain of f is $(-\infty, 2) \cup (2, \infty)$.

Exercse 3

Find the domain of the function:

$$f(x) = \frac{1+4x}{2x-1}$$

Answer

$$(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$$

How To: Given a function written in equation form including an even root, find the domain.

1. Identify the input values.
2. Since there is an even root, exclude any real numbers that result in a negative number in the radicand. Set the radicand greater than or equal to zero and solve for x.
3. The solution(s) are the domain of the function. If possible, write the answer in interval form.

Example 4: Finding the Domain of a Function with an Even Root

Find the domain of the function:

$$f(x) = \sqrt{7-x}.$$

Solution

When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand.

Set the radicand greater than or equal to zero and solve for x.

$$\begin{aligned} 7-x &\geq 0 \\ -x &\geq -7 \\ x &\leq 7 \end{aligned}$$

Now, we will exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to 7, or $(-\infty, 7]$.

Exercse 4

Find the domain of the function

$$f(x) = \sqrt{5+2x}.$$

Answer

$$[-2.5, \infty)$$

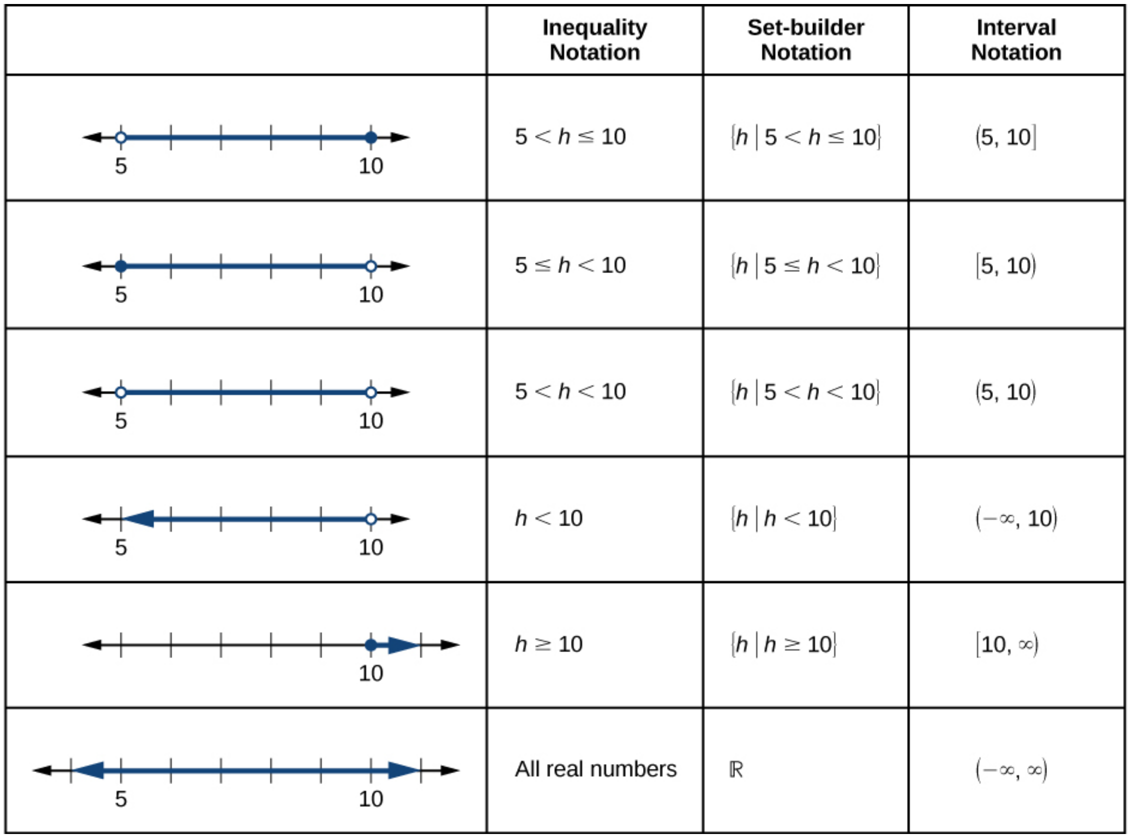
Q&A: Can there be functions in which the domain and range do not intersect at all?

Yes. For example, the function $f(x) = -\frac{1}{\sqrt{x}}$ *has the set of all positive real numbers as its domain but the set of all negative real numbers as its range. As a more extreme example, a function's inputs and outputs can be completely different categories (for example, names of weekdays as inputs and numbers as outputs, as on an attendance chart), in such cases the domain and range have no elements in common.*

Using Notations to Specify Domain and Range

In the previous examples, we used inequalities and lists to describe the domain of functions. We can also use inequalities, or other statements that might define sets of values or data, to describe the behavior of the variable in set-builder notation. For example, $\{x|10 \leq x < 30\}$ describes the behavior of x in set-builder notation. The braces $\{\}$ are read as "the set of," and the vertical bar $|$ is read as "such that," so we would read$\{x|10 \leq x < 30\}$ as "the set of x-values such that 10 is less than or equal to x, and x is less than 30."

Figure 4 compares inequality notation, set-builder notation, and interval notation.



	Inequality Notation	Set-builder Notation	Interval Notation
5 10	$5 < h \leq 10$	$\{h \mid 5 < h \leq 10\}$	$(5, 10]$
5 10	$5 \leq h < 10$	$\{h \mid 5 \leq h < 10\}$	$[5, 10)$
5 10	$5 < h < 10$	$\{h \mid 5 < h < 10\}$	$(5, 10)$
5 10	$h < 10$	$\{h \mid h < 10\}$	$(-\infty, 10)$
10	$h \geq 10$	$\{h \mid h \geq 10\}$	$[10, \infty)$
5 10	All real numbers	$\mathbb{R}$	$(-\infty, \infty)$

Figure 4: Summary of notations for inequalities, set-builder, and interval.

To combine two intervals using inequality notation or set-builder notation, we use the word "or." As we saw in earlier examples, we use the union symbol, $\cup$,to combine two unconnected intervals. For example, the union of the sets$\{2,3,5\}$ and $\{4,6\}$ is the set $\{2,3,4,5,6\}$. It is the set of all elements that belong to one or the other (or both) of the original two sets. For sets with a finite number of elements like these, the elements do not have to be listed in ascending order of numerical value. If the original two sets have some elements in common, those elements should be listed only once in the union set. For sets of real numbers on intervals, another example of a union is

$$\{x||x| \geq 3\} = (-\infty, -3] \cup [3, \infty) \tag{1}$$

Set-Builder Notation and Interval Notation

Set-builder notation is a method of specifying a set of elements that satisfy a certain condition. It takes the form $\{x| \text{ statement about x}\}$ which is read as, “the set of all x such that the statement about x is true.” For example,

$$\{x|4 < x \leq 12\}$$

Interval notation is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set. For example,

$$(4, 12]$$

How To: Given a line graph, describe the set of values using interval notation.

1. Identify the intervals to be included in the set by determining where the heavy line overlays the real line.
2. At the left end of each interval, use [with each end value to be included in the set (solid dot) or (for each excluded end value (open dot).
3. At the right end of each interval, use] with each end value to be included in the set (filled dot) or) for each excluded end value (open dot).
4. Use the union symbol ∪ to combine all intervals into one set.

Example 5: Describing Sets on the Real-Number Line

Describe the intervals of values shown in Figure 5 using inequality notation, set-builder notation, and interval notation.

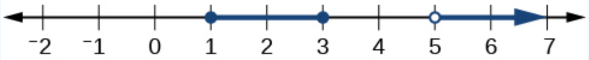


Figure 5: Line graph of $1 \leq x \leq 3$ or $x > 5$.

Solution

To describe the values, x, included in the intervals shown, we would say, “x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5.”

Inequality

$$1 \leq x \leq 3 \text{ or } x > 5$$

Set-builder Notation

$$\{x|1 \leq x \leq 3 \text{ or } x > 5\}$$

Interval notation

$$[1, 3] \cup (5, \infty)$$

Remember that, when writing or reading interval notation, using a square bracket means the boundary is included in the set. Using a parenthesis means the boundary is not included in the set.

Exercse 5

Given Figure 6, specify the graphed set in

a. words
b. set-builder notation
c. interval notation

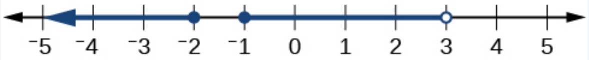


Figure 6: Line graph of $-2 \leq x$, $-1 \leq x < 3$.leq}x\), $-1 \leq x < 3$.

Answer a

Values that are less than or equal to –2, or values that are greater than or equal to –1 and less than 3;

Answer b

$$\{x|x \leq -2 or -1 \leq x < 3\}$$

Answer c

$$(-\infty, -2] \cup [-1, 3)$$

Finding Domain and Range from Graphs

Another way to identify the domain and range of functions is by using graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the x-axis. The range is the set of possible output values, which are shown on the y-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values. See Figure 7.

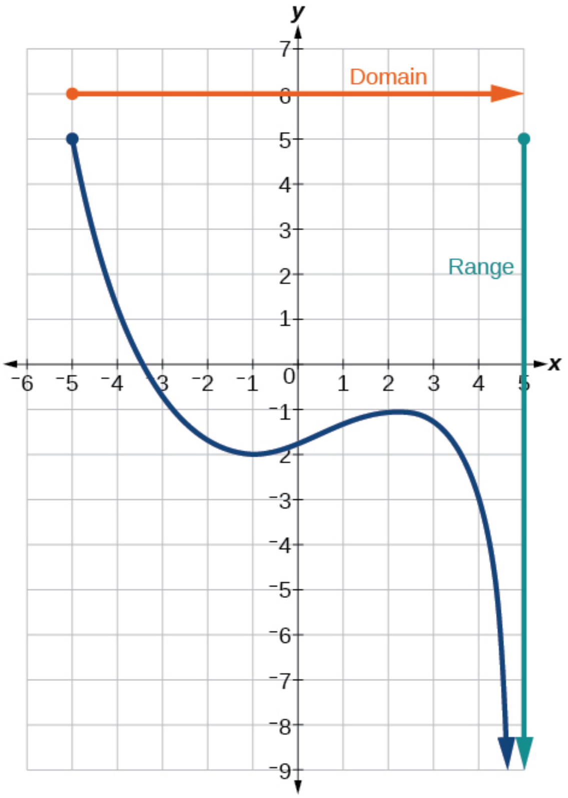


Figure 7: Graph of a polynomial that shows the x-axis is the domain and the y-axis is the range

We can observe that the graph extends horizontally from −5 to the right without bound, so the domain is $[-5, \infty)$. The vertical extent of the graph is all range values 5 and below, so the range is $(-\infty, 5]$. Note that the domain and range are always written from smaller to larger values, or from left to right for domain, and from the bottom of the graph to the top of the graph for range.

Example $6A$: Finding Domain and Range from a Graph

Find the domain and range of the function f whose graph is shown in Figure 1.2.8.

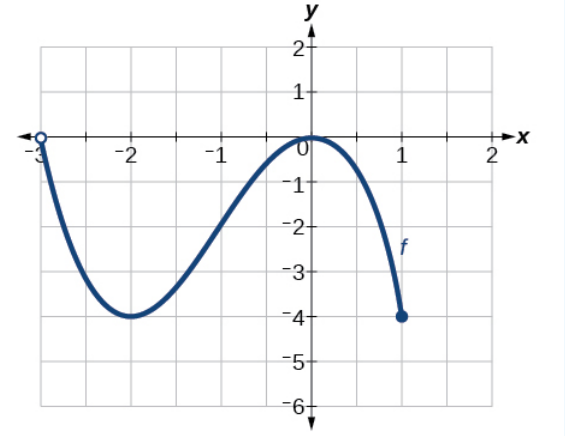


Figure 8: Graph of a function from (-3, 1].

Solution

We can observe that the horizontal extent of the graph is –3 to 1, so the domain of f is $(-3, 1]$.

The vertical extent of the graph is 0 to –4, so the range is $[-4, 0)$. See Figure 9.

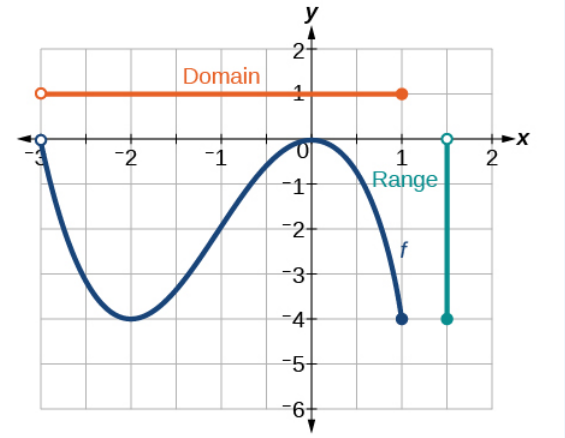


Figure 9: Graph of the previous function shows the domain and range

Example $6B$: Finding Domain and Range from a Graph of Oil Production

Find the domain and range of the function f whose graph is shown in Figure 10.

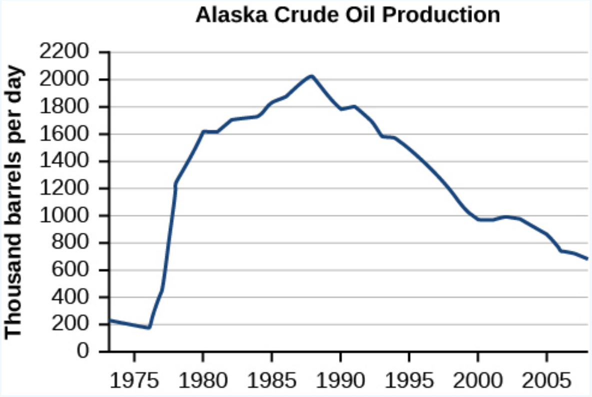


Figure 10: Graph of the Alaska Crude Oil Production where the y-axis is thousand barrels per day and the -axis is the years(credit: modification of work by the U.S. Energy Information Administration)

Solution

The input quantity along the horizontal axis is “years,” which we represent with the variable t for time. The output quantity is “thousands of barrels of oil per day,” which we represent with the variable b for barrels. The graph may continue to the left and right beyond what is viewed, but based on the portion of the graph that is visible, we can determine the domain as $1973 \leq t \leq 2008$ and the range as approximately $180 \leq b \leq 2010$.

In interval notation, the domain is $[1973, 2008]$, and the range is about $[180, 2010]$. For the domain and the range, we approximate the smallest and largest values since they do not fall exactly on the grid lines.

Exercse 6

Given Figure 11, identify the domain and range using interval notation.

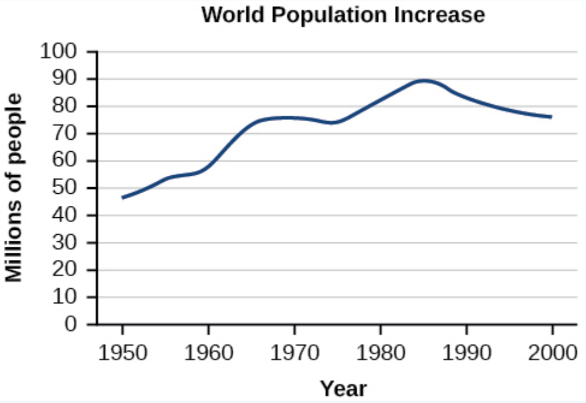


Figure 11: Graph of World Population Increase where the y-axis represents millions of people and the x-axis represents the year.

Answer

domain =$[1950, 2002]$

range = $[47,000,000, 89,000,000]$

Can a function's domain and range be the same?

Yes. For example, the domain and range of the cube root function are both the set of all real numbers.

Finding Domains and Ranges of the Toolkit Functions

We will now return to our set of toolkit functions to determine the domain and range of each.

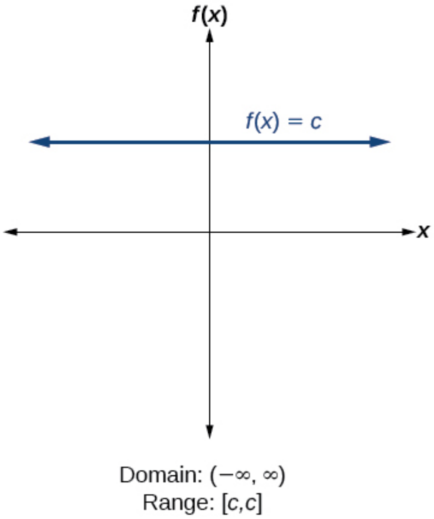


Figure 12: Constant function $f(x) = c$.

For the **constant function**$f(x) = c$, the domain consists of all real numbers; there are no restrictions on the input. The only output value is the constant c, so the range is the set $\{c\}$ that contains this single element. In interval notation, this is written as $[c, c]$, the interval that both begins and ends with c.

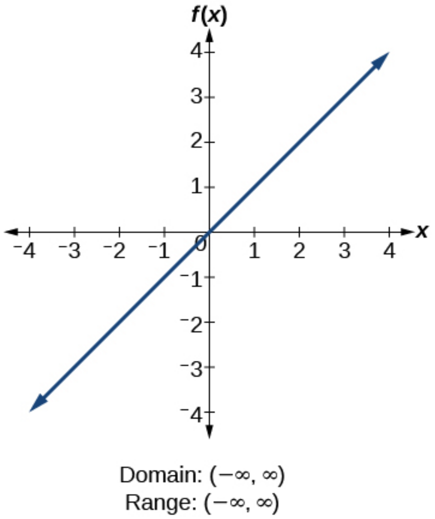


Figure 13***:*** *Identity function f(x)=x.*

For the **identity function** $f(x) = x$, there is no restriction on x. Both the domain and range are the set of all real numbers.

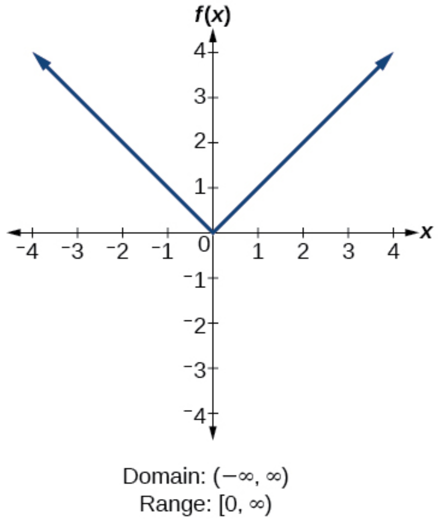


Figure 14: Absolute function $f(x) = |x|$.

For the **absolute value function** $f(x) = |x|$, there is no restriction on x. However, because absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.

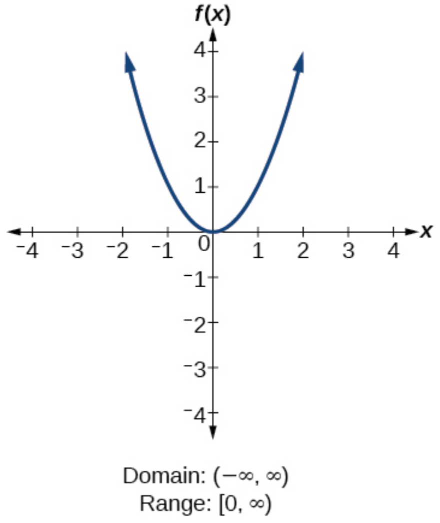


Figure 15: Quadratic function $f(x) = x^2$.

For the **quadratic function** $f(x) = x^2$, the domain is all real numbers since the horizontal extent of the graph is the whole real number line. Because the graph does not include any negative values for the range, the range is only nonnegative real numbers.

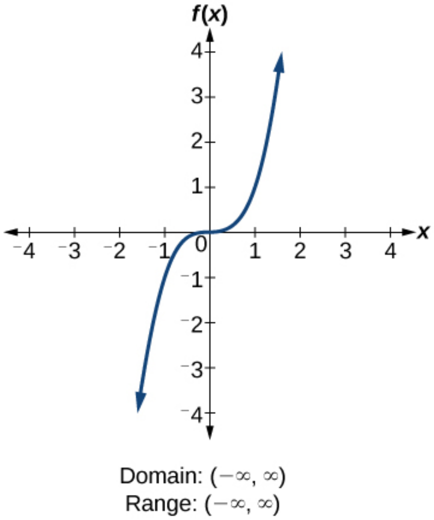


Figure 16: Cubic function $f(x) - x^3$.

For the **cubic function** $f(x) = x^3$, the domain is all real numbers because the horizontal extent of the graph is the whole real number line. The same applies to the vertical extent of the graph, so the domain and range include all real numbers.

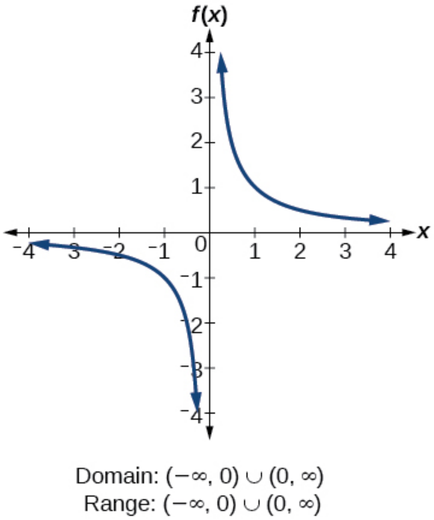


Figure 17: Reciprocal function $f(x) = \frac{1}{x}$.

For the **reciprocal function** $f(x) = \frac{1}{x}$, we cannot divide by 0, so we must exclude 0 from the domain. Further, 1 divided by any value can never be 0, so the range also will not include 0. In set-builder notation, we could also write$\{x|x \neq 0\}$,the set of all real numbers that are not zero.

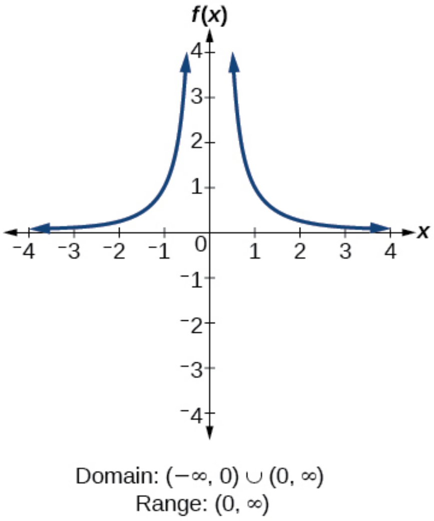


Figure 18: Reciprocal squared function $f(x) = \frac{1}{x^2}$

For the **reciprocal squared function** $f(x) = \frac{1}{x^2}$,we cannot divide by 0, so we must exclude 0 from the domain. There is also no x that can give an output of 0, so 0 is excluded from the range as well. Note that the output of this function is always positive due to the square in the denominator, so the range includes only positive numbers.

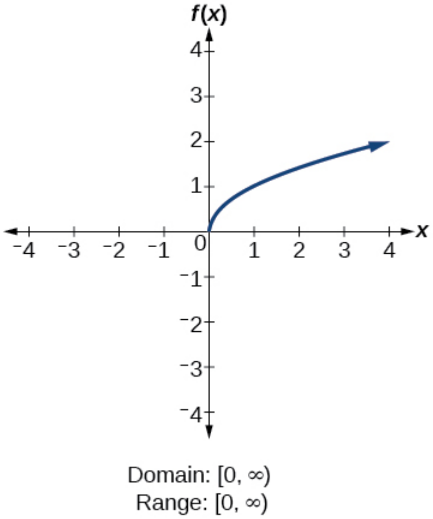


***Figure** 19: Square root function* $f(x) = \sqrt{(x)}$.

For the **square root function** $f(x) = \sqrt{x}$, we cannot take the square root of a negative real number, so the domain must be 0 or greater. The range also excludes negative numbers because the square root of a positive number x is defined to be positive, even though the square of the negative number $-\sqrt{x}$ also gives us x.

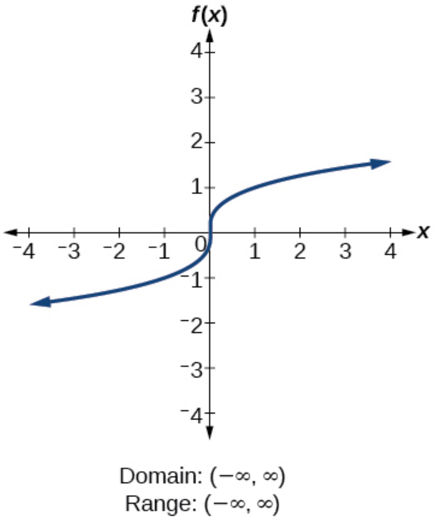


Figure 20: Cube root function $f(x) = \sqrt[3]{x}$.

For the **cube root function** $f(x) = \sqrt[3]{x}$, the domain and range include all real numbers. Note that there is no problem taking a cube root, or any odd-integer root, of a negative number, and the resulting output is negative (it is an odd function).

How To: Given the formula for a function, determine the domain and range.

1. Exclude from the domain any input values that result in division by zero.
2. Exclude from the domain any input values that have nonreal (or undefined) number outputs.
3. Use the valid input values to determine the range of the output values.
4. Look at the function graph and table values to confirm the actual function behavior.

Finding the Domain and Range Using Toolkit Functions

Find the domain and range of $f(x) = 2x^3 - x$.

Solution

There are no restrictions on the domain, as any real number may be cubed and then subtracted from the result.

The domain is $(-\infty, \infty)$ and the range is also $(-\infty, \infty)$.

Example $7B$: Finding the Domain and Range

Find the domain and range of $f(x) = \frac{2}{x+1}$.

Solution

We cannot evaluate the function at -1 because division by zero is undefined. The domain is $(-\infty, -1) \cup (-1, \infty)$. Because the function is never zero, we exclude 0 from the range. The range is $(-\infty, 0) \cup (0, \infty)$.

Example $7C$: Finding the Domain and Range

Find the domain and range of $f(x) = 2\sqrt{x+4}$.

Solution

We cannot take the square root of a negative number, so the value inside the radical must be nonnegative.

$x + 4 \geq 0$ when $x \geq -4$

The domain of $f(x)$ is $[-4, \infty)$.

We then find the range. We know that $f(-4) = 0$, and the function value increases as x increases without any upper limit. We conclude that the range of f is $[0, \infty)$.

Analysis

Figure 19 represents the function f.

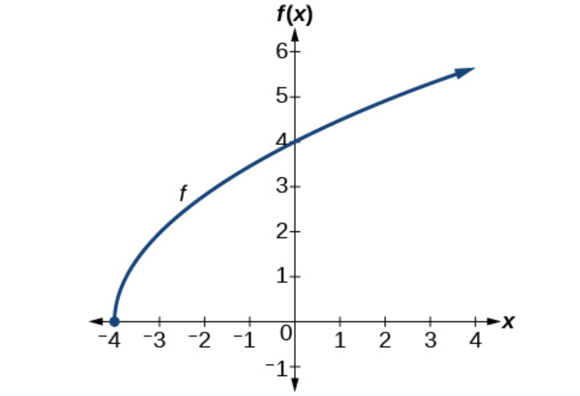


Figure 19: Graph of a square root function at $(-4, 0)$.

Exercise 7

Find the domain and range of

$f(x) = \sqrt{-2-x}$.

Answer

domain: $(-\infty, -2]$

range: $[0, \infty)$

Graphing Piecewise-Defined Functions

Sometimes, we come across a function that requires more than one formula in order to obtain the given output. For example, in the toolkit functions, we introduced the absolute value function $f(x) = |x|$. With a domain of all real numbers and a range of values greater than or equal to 0, **absolute value** can be defined as the **magnitude**, or **modulus**, of a real number value regardless of sign. It is the distance from 0 on the number line. All of these definitions require the output to be greater than or equal to 0.

If we input 0, or a positive value, the output is the same as the input.

$$f(x) = x \text{ if } x \geq 0$$

If we input a negative value, the output is the opposite of the input.

$$f(x) = -x \text{ if } x < 0$$

Because this requires two different processes or pieces, the absolute value function is an example of a **piecewise function**. A piecewise function is a function in which more than one formula is used to define the output over different pieces of the domain.

We use piecewise functions to describe situations in which a rule or relationship changes as the input value crosses certain "boundaries." For example, we often encounter situations in business for which the cost per piece of a certain item is discounted once the number ordered exceeds a certain value. Tax brackets are another real-world example of piecewise functions. For example, consider a simple tax system in which incomes up to \$10,000 are taxed at 10%, and any additional income is taxed at 20%. The tax on a total income S would be $0.1S$ if $S \leq \$10,000$ and $\$1000 + 0.2(S - \$10,000)$ if $S > \$10,000$.

Piecewise Function

A piecewise function is a function in which more than one formula is used to define the output. Each formula has its own domain, and the domain of the function is the union of all these smaller domains. We notate this idea like this:

$$f(x) = \begin{cases} \text{formula 1} & \text{if x is in domain 1} \\ \text{formula 2} & \text{if x is in domain 2} \\ \text{formula 3} & \text{if x is in domain 3} \end{cases}$$

In piecewise notation, the absolute value function is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Given a piecewise function, write the formula and identify the domain for each interval.

1. Identify the intervals for which different rules apply.
2. Determine formulas that describe how to calculate an output from an input in each interval.
3. Use braces and if-statements to write the function.

Example $8A$: Writing a Piecewise Function

A museum charges \$5 per person for a guided tour with a group of 1 to 9 people or a fixed \$50 fee for a group of 10 or more people. Write a **function** relating the number of people, n, to the cost, C.

Solution

Two different formulas will be needed. For n-values under 10, $C = 5n$. For values of n that are 10 or greater, $C = 50$.

$$C(n) = \begin{cases} 5n & \text{if } n < 10 \\ 50 & \text{if } n \geq 10 \end{cases}$$

Analysis

The function is represented in Figure 20. The graph is a diagonal line from $n = 0$ to $n = 10$ and a constant after that. In this example, the two formulas agree at the meeting point where $n = 10$, but not all piecewise functions have this property.

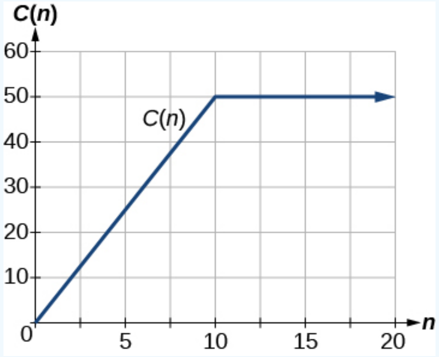


Figure 20

Example $8B$: Working with a Piecewise Function

A cell phone company uses the function below to determine the cost, C, in dollars for g gigabytes of data transfer.

$$C(g) = \begin{cases} 25 & \text{if } 0 < g < 2 \\ 25 + 10(g - 2) & \text{if } g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data and the cost of using 4 gigabytes of data.

Soltuion

To find the cost of using 1.5 gigabytes of data, $C(1.5)$, we first look to see which part of the domain our input falls in. Because 1.5 is less than 2, we use the first formula.

$$C(1.5) = \$25$$

To find the cost of using 4 gigabytes of data, C(4), we see that our input of 4 is greater than 2, so we use the second formula.

$$C(4) = 25 + 10(4 - 2) = \$45$$

Analysis

The function is represented in Figure 21. We can see where the function changes from a constant to a shifted and stretched identity at $g = 2$. We plot the graphs for the different formulas on a common set of axes, making sure each formula is applied on its proper domain.

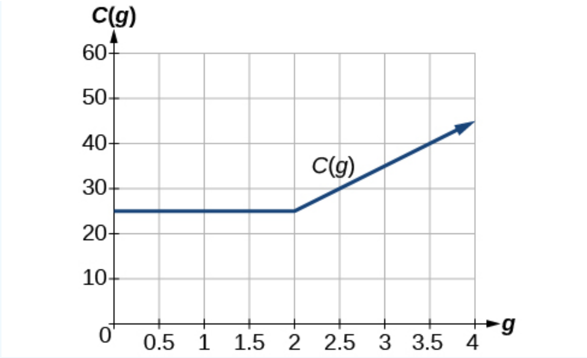


Figure 21

Given a piecewise function, sketch a graph.

1. Indicate on the x-axis the boundaries defined by the intervals on each piece of the domain.
2. For each piece of the domain, graph on that interval using the corresponding equation pertaining to that piece. Do not graph two functions over one interval because it would violate the criteria of a function.

Example $8C$: Graphing a Piecewise Function

Sketch a graph of the function.

$$f(x)=\begin{cases}x^2 & \text{if } x\leq 1\\ 3 & \text{if } 1<x\leq 2\\ x & \text{if } x>2\end{cases}$$

Solution

Each of the component functions is from our library of toolkit functions, so we know their shapes. We can imagine graphing each function and then limiting the graph to the indicated domain. At the endpoints of the domain, we draw open circles to indicate where the endpoint is not included because of a less-than or greater-than inequality; we draw a closed circle where the endpoint is included because of a less-than-or-equal-to or greater-than-or-equal-to inequality.

Figure 20 shows the three components of the piecewise function graphed on separate coordinate systems.

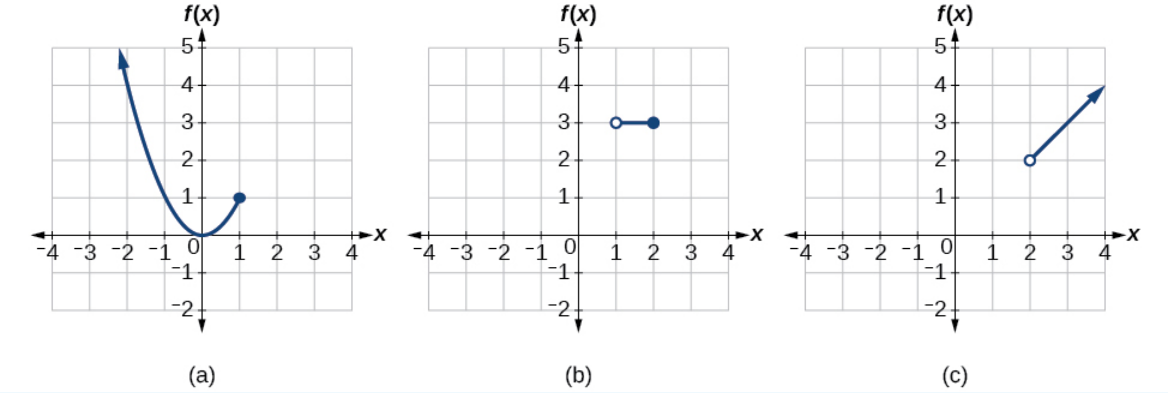


***Figure** 20**:** Graph of each part of the piece-wise function f(x)*

(a)$f(x)=x^2$ if $x\leq 1$; (b) $f(x)=3$ if $1<x\leq 2$; (c) $f(x)=x$ if $x>2$

Now that we have sketched each piece individually, we combine them in the same coordinate plane. See Figure 21.

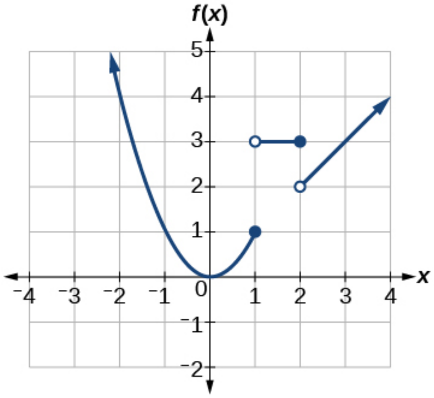


Figure 21: Graph of the entire function.

Analysis

Note that the graph does pass the vertical line test even at $x = 1$ and $x = 2$ because the points $(1, 3)$ and $(2, 2)$ are not part of the graph of the function, though $(1, 1)$ and $(2, 3)$ are.

Exercise 8

Graph the following piecewise function.

$$f(x) = \begin{cases} x^3 & \text{if } x < -1 \\ -2 & \text{if } -1 < x < 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$$

Answer

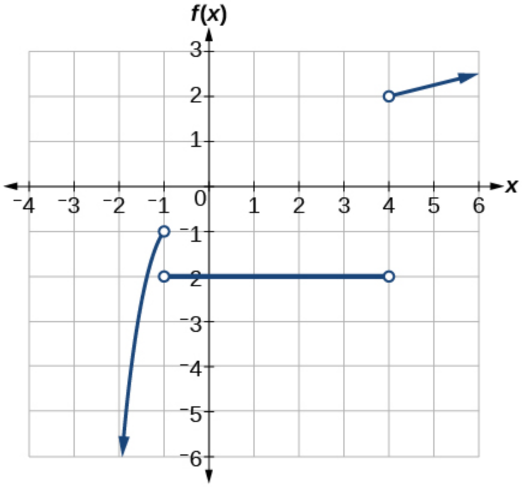


Figure 22

Can more than one formula from a piecewise function be applied to a value in the domain?

No. Each value corresponds to one equation in a piecewise formula.

Key Concepts

- The domain of a function includes all real input values that would not cause us to attempt an undefined mathematical operation, such as dividing by zero or taking the square root of a negative number.
- The domain of a function can be determined by listing the input values of a set of ordered pairs.
- The domain of a function can also be determined by identifying the input values of a function written as an equation.
- Interval values represented on a number line can be described using inequality notation, set-builder notation, and interval notation.
- For many functions, the domain and range can be determined from a graph.

- An understanding of toolkit functions can be used to find the domain and range of related functions.
- A piecewise function is described by more than one formula.
- A piecewise function can be graphed using each algebraic formula on its assigned subdomain.

Footnotes

1 The Numbers: Where Data and the Movie Business Meet. “Box Office History for Horror Movies.” http://www.the-numbers.com/market/genre/Horror. Accessed 3/24/2014
2 www.eia.gov/dnav/pet/hist/Lea...s=MCRFPAK2&f=A.

Glossary

interval notation

a method of describing a set that includes all numbers between a lower limit and an upper limit; the lower and upper values are listed between brackets or parentheses, a square bracket indicating inclusion in the set, and a parenthesis indicating exclusion

piecewise function

a function in which more than one formula is used to define the output

set-builder notation

a method of describing a set by a rule that all of its members obey; it takes the form {x| statement about x}

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Functions and Function Notation

Learning Objectives

- Determine whether a relation represents a function.
- Find the value of a function.
- Determine whether a function is one-to-one.
- Use the vertical line test to identify functions.
- Graph the functions listed in the library of functions.

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

Determining Whether a Relation Represents a Function

A relation is a set of ordered pairs. The set of the first components of each ordered pair is called the domain and the set of the second components of each ordered pair is called the range. Consider the following set of ordered pairs. The first numbers in each pair are the first five natural numbers. The second number in each pair is twice that of the first.

$$\{(1,2),(2,4),(3,6),(4,8),(5,10)\} \tag{1.1.1}$$

The domain is $\{1,2,3,4,5\}$. The range is $\{2,4,6,8,10\}$.

Note that each value in the domain is also known as an **input** value, or **independent variable**, and is often labeled with the lowercase letter x. Each value in the range is also known as an output value, or **dependent variable**, and is often labeled lowercase letter y.

A function f is a relation that assigns a single value in the range to each value in the domain. In other words, no x-values are repeated. For our example that relates the first five **natural numbers** to numbers double their values, this relation is a function because each element in the domain, {1, 2, 3, 4, 5}, is paired with exactly one element in the range, $\{2,4,6,8,10\}$.

Now let’s consider the set of ordered pairs that relates the terms “even” and “odd” to the first five natural numbers. It would appear as

$$\{(\text{odd},1),(\text{even},2),(\text{odd},3),(\text{even},4),(\text{odd},5)\} \tag{1.1.2}$$

Notice that each element in the domain, {even, odd} is not paired with exactly one element in the range, $\{1,2,3,4,5\}$. For example, the term “odd” corresponds to three values from the range, $\{1,3,5\}$, and the term “even” corresponds to two values from the range, $\{2,4\}$. This violates the definition of a function, so this relation is not a function.

Figure 1 compares relations that are functions and not functions.

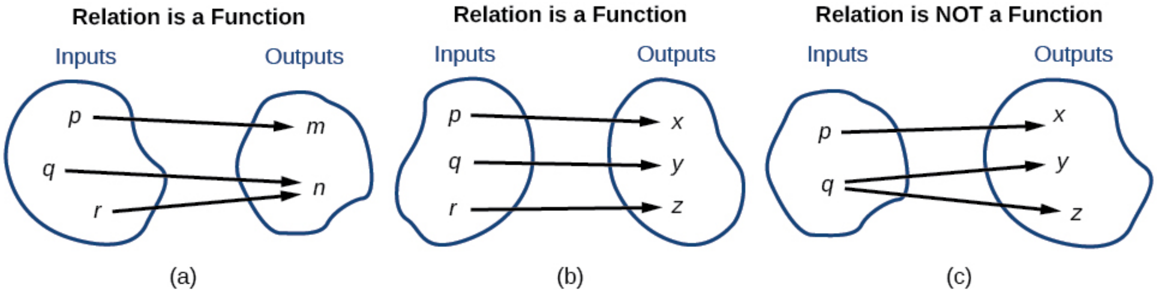


Figure 1: (a) This relationship is a function because each input is associated with a single output. Note that input q and r both give output n. (b) This relationship is also a function. In this case, each input is associated with a single output. (c) This relationship is not a function because input q is associated with two different outputs.

Function

A **function** is a relation in which each possible input value leads to exactly one output value. We say “the output is a function of the input.”

The **input** values make up the **domain**, and the **output** values make up the **range**.

How To: Given a relationship between two quantities, determine whether the relationship is a function

1. Identify the input values.
2. Identify the output values.
3. If each input value leads to only one output value, classify the relationship as a function. If any input value leads to two or more outputs, do not classify the relationship as a function.

Example 1: Determining If Menu Price Lists Are Functions

The coffee shop menu, shown inFigure 2 consists of items and their prices.

a. Is price a function of the item?
b. Is the item a function of the price?

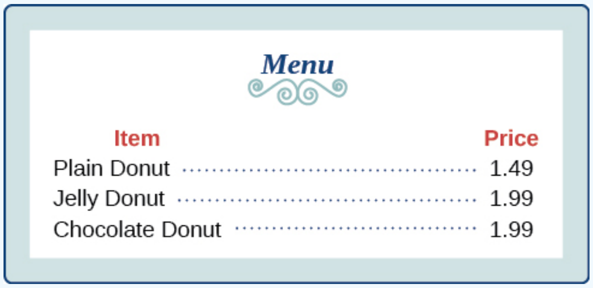

Menu
Item
Price
Plain Donut
1.49
Jelly Donut
1.99
Chocolate Donut
1.99

Figure 2: A menu of donut prices from a coffee shop where a plain donut is $1.49 and a jelly donut and chocolate donut are $1.99.

Solution

a. Let’s begin by considering the input as the items on the menu. The output values are then the prices. See Figure 3.

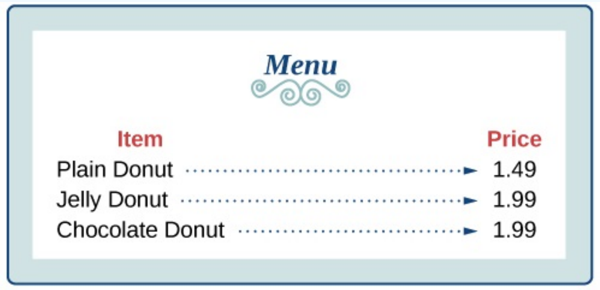

Menu
Item
Price
Plain Donut
1.49
Jelly Donut
1.99
Chocolate Donut
1.99

Figure 3: A menu of donut prices from a coffee shop where a plain donut is $1.49 and a jelly donut and chocolate donut are $1.99.

Each item on the menu has only one price, so the price is a function of the item.

a. Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value could have more than one output associated with it. See Figure 4.

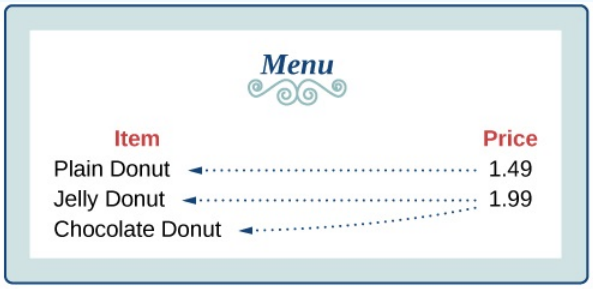

Menu
Item
Price
Plain Donut
1.49
Jelly Donut
1.99
Chocolate Donut

Figure 4: Association of the prices to the donuts.

Therefore, the item is a not a function of price.

Example 2: Determining If Class Grade Rules Are Functions

In a particular math class, the overall percent grade corresponds to a grade point average. Is grade point average a function of the percent grade? Is the percent grade a function of the grade point average? Table 1 shows a possible rule for assigning grade points.

Table 1: Class grade points.

Percent grade	0–56	57–61	62–66	67–71	72–77	78–86	87–91	92–100
Grade point average	0.0	1.0	1.5	2.0	2.5	3.0	3.5	4.0

Solution

For any percent grade earned, there is an associated grade point average, so the grade point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade point average.

In the grading system given, there is a range of percent grades that correspond to the same grade point average. For example, students who receive a grade point average of 3.0 could have a variety of percent grades ranging from 78 all the way to 86. Thus, percent grade is not a function of grade point average.

Exercise 2

Table 2 lists the five greatest baseball players of all time in order of rank.

Table 2: Five greatest baseball players.

Player	Rank
Babe Ruth	1
Willie Mays	2
Ty Cobb	3
Walter Johnson	4
Hank Aaron	5

a. Is the rank a function of the player name?
b. Is the player name a function of the rank?

Answer a

Yes

Answer b

yes. (Note: If two players had been tied for, say, 4th place, then the name would not have been a function of rank.)

Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also so that we can program them into computers. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions.

To represent "height is a function of age," we start by identifying the descriptive variables h for height and a for age. The letters f, g,and h are often used to represent functions just as we use x, y,and z to represent numbers and A, B, and C to represent sets.

$$\begin{array}{ll} h \text{ is } f \text{ of } a & \text{We name the function } f\text{; height is a function of age.} \\ h = f(a) & \text{We use parentheses to indicate the function input.} \\ f(a) & \text{We name the function } f \text{ ; the expression is read as ``} f \text{ of } a\text{.''} \end{array} \tag{1}$$

Remember, we can use any letter to name the function; the notation $h(a)$ shows us that h depends on a. The value a must be put into the function h to get a result. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We can also give an algebraic expression as the input to a function. For example $f(a+b)$ means "first add a and b, and the result is the input for the function f." The operations must be performed in this order to obtain the correct result.

Function Notation

The notation $y = f(x)$ defines a function named f. This is read as "y is a function of x." The letter x represents the input value, or independent variable. The letter y, or $f(x)$, represents the output value, or dependent variable.

Example 3: Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

Solution

Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

The number of days in a month is a function of the name of the month, so if we name the function f, we write $\text{days} = f(\text{month})$ or $d = f(m)$. The name of the month is the input to a "rule" that associates a specific number (the output) with each input.

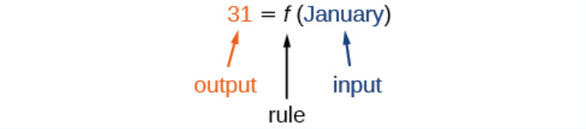


Figure 5: The function $31 = f(January)$ where 31 is the output, f is the rule, and January is the input.

For example, $f(\text{March}) = 31$, because March has 31 days. The notation $d = f(m)$ reminds us that the number of days, d (the output), is dependent on the name of the month, m (the input).

Analysis

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

Example $3B$: Interpreting Function Notation

A function $N = f(y)$ gives the number of police officers, N, in a town in year y. What does $f(2005) = 300$ represent?

Solution

When we read $f(2005) = 300$, we see that the input year is 2005. The value for the output, the number of police officers (N), is 300. Remember, $N = f(y)$. The statement $f(2005) = 300$ tells us that in the year 2005 there were 300 police officers in the town.

Exercise 3

Use function notation to express the weight of a pig in pounds as a function of its age in days d.

Answer

$w = f(d)$

Q&A

Instead of a notation such as $y = f(x)$, could we use the same symbol for the output as for the function, such as $y = y(x)$, meaning “y is a function of x?”

Yes, this is often done, especially in applied subjects that use higher math, such as physics and engineering. However, in exploring math itself we like to maintain a distinction between a function such as f, which is a rule or procedure, and the output y we get by applying f to a particular input x. This is why we usually use notation such as $y = f(x), P = W(d)$, and so on.

Representing Functions Using Tables

A common method of representing functions is in the form of a table. The table rows or columns display the corresponding input and output values. In some cases, these values represent all we know about the relationship; other times, the table provides a few select examples from a more complete relationship.

Table 3 lists the input number of each month ($\text{January} = 1$, $\text{February} = 2$, and so on) and the output value of the number of days in that month. This information represents all we know about the months and days for a given year (that is not a leap year). Note that, in this table, we define a days-in-a-month function f where $D = f(m)$ identifies months by an integer rather than by name.

Table 3: Months and number of days per month.

Month number, m (input)	1	2	3	4	5	6	7	8	9	10	11	12
Days in month, D (output)	31	28	31	30	31	30	31	31	30	31	30	31

Table 4 defines a function $Q = g(n)$ Remember, this notation tells us that g is the name of the function that takes the input n and gives the output Q.

Table 4: Function $Q = g(n)$

n	1	2	3	4	5
Q	8	6	7	6	8

Table 5 displays the age of children in years and their corresponding heights. This table displays just some of the data available for the heights and ages of children. We can see right away that this table does not represent a function because the same input value, 5 years, has two different output values, 40 in. and 42 in.

Table 5: Age of children and their corresponding heights.

Age in years, a (input)	5	5	6	7	8	9	10
Height in inches, h (output)	40	42	44	47	50	52	54

How To: Given a table of input and output values, determine whether the table represents a function

1. Identify the input and output values.
2. Check to see if each input value is paired with only one output value. If so, the table represents a function.

Example 5: Identifying Tables that Represent Functions

Which table, Table 6, Table 7, or Table 8, represents a function (if any)?

Table 6

Input	Output
2	1
5	3
8	6

Table 7

Input	Output
-3	5
0	1
4	5

Table 8

Input	Output
1	0
5	2
5	4

Solution

Table 6 and Table 7 define functions. In both, each input value corresponds to exactly one output value. Table 8 does not define a function because the input value of 5 corresponds to two different output values.

When a table represents a function, corresponding input and output values can also be specified using function notation.

The function represented by Table 6 can be represented by writing

$$f(2)=1,\ f(5)=3,\text{ and } f(8)=6$$

Similarly, the statements

$$g(-3)=5,\ g(0)=1,\text{ and } g(4)=5$$

represent the function in Table 7.

Table 8 cannot be expressed in a similar way because it does not represent a function.

Exercise 5

Does Table 9 represent a function?

Table 9

Input	Output
1	10
2	100
3	1000

Answer

yes

Finding Input and Output Values of a Function

When we know an input value and want to determine the corresponding output value for a function, we evaluate the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

When we know an output value and want to determine the input values that would produce that output value, we set the output equal to the function's formula and solve for the input. Solving can produce more than one solution because different input values can produce the same output value.

Evaluation of Functions in Algebraic Forms

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f(x) = 5$ – $3x^2$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

How To: Given the formula for a function, evaluate.

Given the formula for a function, evaluate.

1. Replace the input variable in the formula with the value provided.
2. Calculate the result.

Example $6A$: Evaluating Functions at Specific Values

1. Evaluate $f(x) = x^2 + 3x - 4$ at

a. 2
b. a
c. $a + h$
d. Evaluate $\frac{f(a+h) - f(a)}{h}$

Solution

Replace the x in the function with each specified value.

a. Because the input value is a number, 2, we can use simple algebra to simplify.

$$\begin{aligned} f(2) &= 2^2 + 3(2) - 4 \\ &= 4 + 6 - 4 \\ &= 6 \end{aligned}$$

b. In this case, the input value is a letter so we cannot simplify the answer any further.

$$f(a) = a^2 + 3a - 4$$

c. With an input value of $a + h$, we must use the distributive property.

$$\begin{aligned} f(a+h) &= (a+h)^2 + 3(a+h) - 4 \\ &= a^2 + 2ah + h^2 + 3a + 3h - 4 \end{aligned}$$

d. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that

$$f(a+h) = a^2 + 2ah + h^2 + 3a + 3h - 4$$

and we know that

$$f(a) = a^2 + 3a - 4$$

Now we combine the results and simplify.

$$\begin{aligned}\frac{f(a+h)-f(a)}{h} &= \frac{(a^2+2ah+h^2+3a+3h-4)-(a^2+3a-4)}{h} \\ &= \frac{(2ah+h^2+3h)}{h} \\ &= \frac{h(2a+h+3)}{h} && \text{Factor out h.} \\ &= 2a+h+3 && \text{Simplify.}\end{aligned}$$

Example $6B$: Evaluating Functions

Given the function $h(p)=p^2+2p$, evaluate $h(4)$.

Solution

To evaluate $h(4)$, we substitute the value 4 for the input variable p in the given function.

$$\begin{aligned}h(p) &= p^2+2p \\ h(4) &= (4)^2+2(4) \\ &= 16+8 \\ &= 24\end{aligned}$$

Therefore, for an input of 4, we have an output of 24.

Exercise 6

Given the function $g(m)=\sqrt{m-4}$, evaluate $g(5)$.

Answer

$g(5)=1$

Example 7: Solving Functions

Given the function $h(p)=p^2+2p$, solve for $h(p)=3$.

Solution

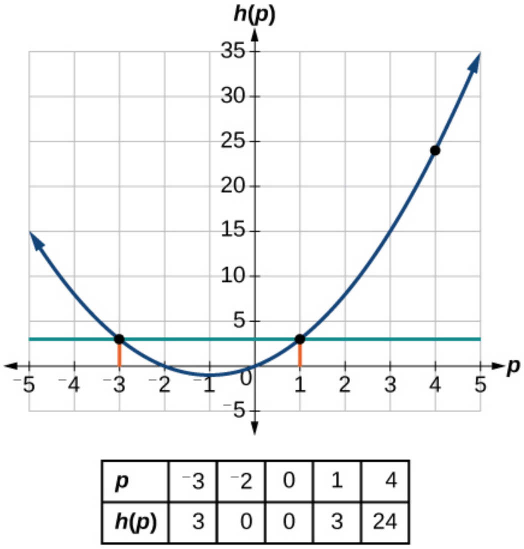
$$\begin{aligned}h(p) &= 3 \\ p^2+2p &= 3 && \text{Substitute the original function} \\ p^2+2p-3 &= 0 && \text{Subtract 3 from each side.} \\ (p+3)(p-1) &= 0 && \text{Factor.}\end{aligned}$$

If $(p+3)(p-1)=0$, either $(p+3)=0$ or $(p-1)=0$ (or both of them equal 0). We will set each factor equal to 0 and solve for p in each case.

$$(p+3)=0,\ \ p=-3$$

$$(p-1)=0,\ p=1$$

This gives us two solutions. The output $h(p)=3$ when the input is either $p=1$ or $p=-3$. We can also verify by graphing as in Figure 6. The graph verifies that $h(1)=h(-3)=3$ and $h(4)=24$.



p	-3	-2	0	1	4
h(p)	3	0	0	3	24

Figure 6: Graph of $h(p) = p^2 + 2p$

Exercise 7

Given the function $g(m) = \sqrt{m-4}$, solve $g(m) = 2$.

Answer

$m = 8$

Evaluating Functions Expressed in Formulas

Some functions are defined by mathematical rules or procedures expressed in **equation** form. If it is possible to express the function output with a formula involving the input quantity, then we can define a function in algebraic form. For example, the equation $2n + 6p = 12$ expresses a functional relationship between n and p. We can rewrite it to decide if p is a function of n.

How to: Given a function in equation form, write its algebraic formula.

1. Solve the equation to isolate the output variable on one side of the equal sign, with the other side as an expression that involves only the input variable.
2. Use all the usual algebraic methods for solving equations, such as adding or subtracting the same quantity to or from both sides, or multiplying or dividing both sides of the equation by the same quantity.

Example $8A$: Finding an Equation of a Function

Express the relationship $2n + 6p = 12$ as a function $p = f(n)$, if possible.

Solution

To express the relationship in this form, we need to be able to write the relationship where p is a function of n, which means writing it as $p = [\text{expression involving } n]$.

$$\begin{aligned} 2n + 6p &= 12 \\ 6p &= 12 - 2n && \text{Subtract 2n from both sides.} \\ p &= \frac{12 - 2n}{6} && \text{Divide both sides by 6 and simplify.} \\ p &= \frac{12}{6} - \frac{2n}{6} \\ p &= 2 - \frac{1}{3}n \end{aligned}$$

Therefore, p as a function of n is written as

$$p = f(n) = 2 - \frac{1}{3}n$$

Analysis

It is important to note that not every relationship expressed by an equation can also be expressed as a function with a formula.

Example $8B$: Expressing the Equation of a Circle as a Function

Does the equation $x^2+y^2=1$ represent a function with x as input and y as output? If so, express the relationship as a function $y = f(x)$.

Solution

First we subtract x^2 from both sides.

$$y^2=1-x^2$$

We now try to solve for y in this equation.

$$y=\pm\sqrt{1-x^2}$$

$$\text{so, } y=\sqrt{1-x^2} \text{ and } y=-\sqrt{1-x^2}$$

We get two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y=f(x)$.

Exercise 8

If $x-8y^3=0$, express y as a function of x.

Answer

$$y=f(x)=\frac{\sqrt[3]{x}}{2}$$

Q & A

Are there relationships expressed by an equation that do represent a function but which still cannot be represented by an algebraic formula?

Yes, this can happen. For example, given the equation $x=y+2^y$, if we want to express y as a function of x, there is no simple algebraic formula involving only x that equals y. However, each x does determine a unique value for y, and there are mathematical procedures by which y can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for y as a function of x, even though the formula cannot be written explicitly.

Evaluating a Function Given in Tabular Form

As we saw above, we can represent functions in tables. Conversely, we can use information in tables to write functions, and we can evaluate functions using the tables. For example, how well do our pets recall the fond memories we share with them? There is an urban legend that a goldfish has a memory of 3 seconds, but this is just a myth. Goldfish can remember up to 3 months, while the beta fish has a memory of up to 5 months. And while a puppy's memory span is no longer than 30 seconds, the adult dog can remember for 5 minutes. This is meager compared to a cat, whose memory span lasts for 16 hours.

The function that relates the type of pet to the duration of its memory span is more easily visualized with the use of a table (Table 10).

Table 10

Pet Memory	span in hours
Puppy	0.008
Adult Dog	0.083
Cat	3
Goldfish	2160

Pet Memory	span in hours
Beta Fish	3600

At times, evaluating a function in table form may be more useful than using equations. Here let us call the function P. The domain of the function is the type of pet and the range is a real number representing the number of hours the pet's memory span lasts. We can evaluate the function P at the input value of "goldfish." We would write $P(goldfish) = 2160$. Notice that, to evaluate the function in table form, we identify the input value and the corresponding output value from the pertinent row of the table. The tabular form for function P seems ideally suited to this function, more so than writing it in paragraph or function form.

How To: Given a function represented by a table, identify specific output and input values

1. Find the given input in the row (or column) of input values.
2. Identify the corresponding output value paired with that input value.
3. Find the given output values in the row (or column) of output values, noting every time that output value appears.
4. Identify the input value(s) corresponding to the given output value.

Example 9: Evaluating and Solving a Tabular Function

Using Table 11,

a. Evaluate $g(3)$.
b. Solve $g(n) = 6$.

Table 11

n	1	2	3	4	5
$g(n)$	8	6	7	6	8

Solution

a. Evaluating $g(3)$ means determining the output value of the function g for the input value of $n = 3$. The table output value corresponding to $n = 3$ is 7, so $g(3) = 7$.
b. Solving $g(n) = 6$ means identifying the input values, n,that produce an output value of 6. Table 12 shows two solutions: 2 and 4.

Table 12

n	1	2	3	4	5
$g(n)$	8	6	7	6	8

When we input 2 into the function g, our output is 6. When we input 4 into the function g, our output is also 6.

Exercise 1

UsingTable 12, evaluate $g(1)$.

Answer

$g(1) = 8$

Finding Function Values from a Graph

Evaluating a function using a graph also requires finding the corresponding output value for a given input value, only in this case, we find the output value by looking at the graph. Solving a function equation using a graph requires finding all instances of the given output value on the graph and observing the corresponding input value(s).

Example 10: Reading Function Values from a Graph

Given the graph in Figure 7,

a. Evaluate $f(2)$.
b. Solve $f(x) = 4$.

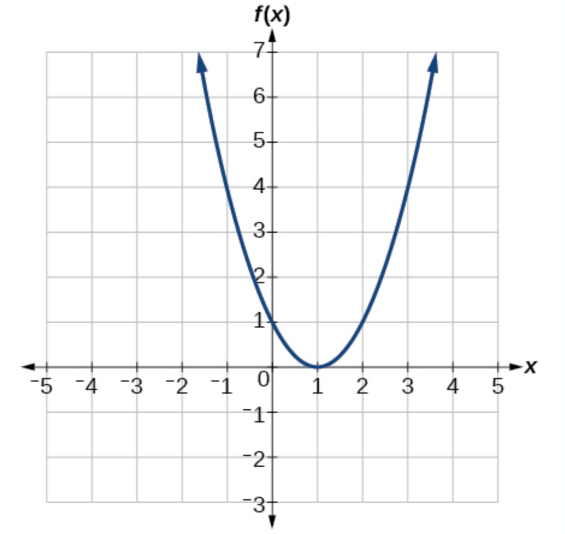
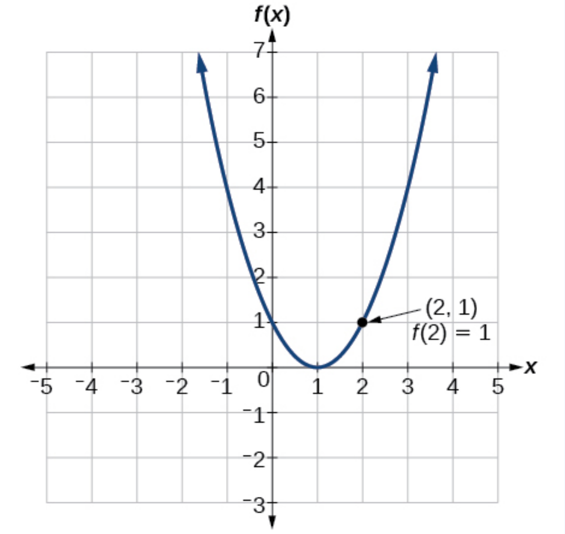


Figure 7: Graph of a positive parabola centered at $(1, 0)$.

Solution

To evaluate $f(2)$, locate the point on the curve where $x = 2$, then read the y-coordinate of that point. The point has coordinates $(2, 1)$, so $f(2) = 1$. See Figure 8.



8: Graph of a positive parabola centered at $(1, 0)$ with the labeled point $(2, 1)$ where $f(2) = 1$.

To solve $f(x) = 4$, we find the output value 4 on the vertical axis. Moving horizontally along the line $y = 4$, we locate two points of the curve with output value 4: $(-1, 4)$ and $(3, 4)$. These points represent the two solutions to $f(x) = 4$: −1 or 3. This means $f(-1) = 4$ and $f(3) = 4$, or when the input is −1 or 3, the output is 4. See Figure 9.

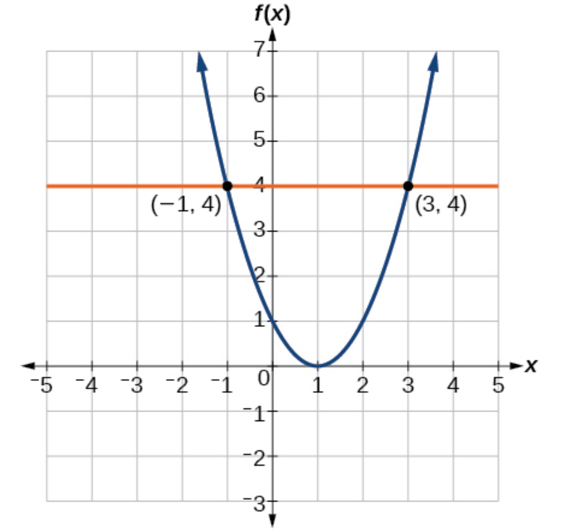


Figure 9: Graph of an upward-facing parabola with a vertex at $(0, 1)$ and labeled points at $(-1, 4)$ and $(3, 4)$. A line at $y = 4$ intersects the parabola at the labeled points.

Exercise 10

Given the graph in Figure 7, solve $f(x) = 1$.

Answer

$x = 0$ or $x = 2$

Determining Whether a Function is One-to-One

Some functions have a given output value that corresponds to two or more input values. For example, in the stock chart shown in the Figure at the beginning of this chapter, the stock price was $1000 on five different dates, meaning that there were five different input values that all resulted in the same output value of $1000.

However, some functions have only one input value for each output value, as well as having only one output for each input. We call these functions one-to-one functions. As an example, consider a school that uses only letter grades and decimal equivalents, as listed in Table 13.

Table 13: Letter grades and decimal equivalents.

Letter Grade	Grade Point Average
A	4.0
B	3.0
C	2.0
D	1.0

This grading system represents a one-to-one function, because each letter input yields one particular grade point average output and each grade point average corresponds to one input letter.

To visualize this concept, let’s look again at the two simple functions sketched in Figures $1a$ and $1b$. The function in part (a) shows a relationship that is not a one-to-one function because inputs q and r both give output n. The function in part (b) shows a relationship that is a one-to-one function because each input is associated with a single output.

One-to-One Functions

A **one-to-one function** is a function in which each output value corresponds to exactly one input value.

Example 11: Determining Whether a Relationship Is a One-to-One Function

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

Solution

A circle of radius r has a unique area measure given by $A = \pi r^2$, so for any input, r, there is only one output, A. The area is a function of radius$r$.

If the function is one-to-one, the output value, the area, must correspond to a unique input value, the radius. Any area measure A is given by the formula $A = \pi r^2$. Because areas and radii are positive numbers, there is exactly one solution:$\sqrt{\frac{A}{\pi}}$. So the area of a circle is a one-to-one function of the circle's radius.

Exercise $11A$

a. Is a balance a function of the bank account number?
b. Is a bank account number a function of the balance?
c. Is a balance a one-to-one function of the bank account number?

Answer

a. yes, because each bank account has a single balance at any given time;

b. no, because several bank account numbers may have the same balance;

c. no, because the same output may correspond to more than one input.

Exercise $11B$

Evaluate the following:

a. If each percent grade earned in a course translates to one letter grade, is the letter grade a function of the percent grade?
b. If so, is the function one-to-one?

Answer

a. Yes, letter grade is a function of percent grade;
b. No, it is not one-to-one. There are 100 different percent numbers we could get but only about five possible letter grades, so there cannot be only one percent number that corresponds to each letter grade.

Using the Vertical Line Test

As we have seen in some examples above, we can represent a function using a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value x and the output y, and we say y is a function of x, or $y = f(x)$ when the function is named f. The graph of the function is the set of all points (x, y) in the plane that satisfies the equation $y = f(x)$. If the function is defined for only a few input values, then the graph of the function is only a few points, where the x-coordinate of each point is an input value and the y-coordinate of each point is the corresponding output value. For example, the black dots on the graph in Figure 10 tell us that $f(0) = 2$ and $f(6) = 1$. However, the set of all points (x, y) satisfying $y = f(x)$ is a curve. The curve shown includes $(0, 2)$ and $(6, 1)$ because the curve passes through those points

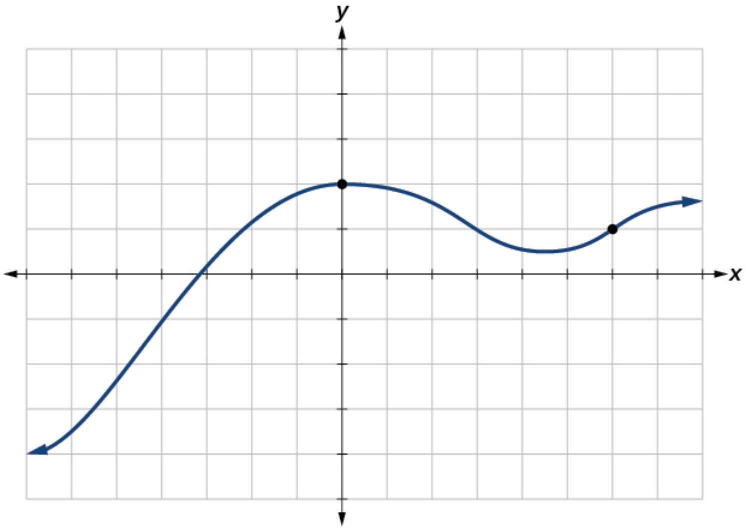


Figure 10: Graph of a polynomial.

The vertical line test can be used to determine whether a graph represents a function. If we can draw any vertical line that intersects a graph more than once, then the graph does not define a function because a function has only one output value for each input value. See Figure 11.

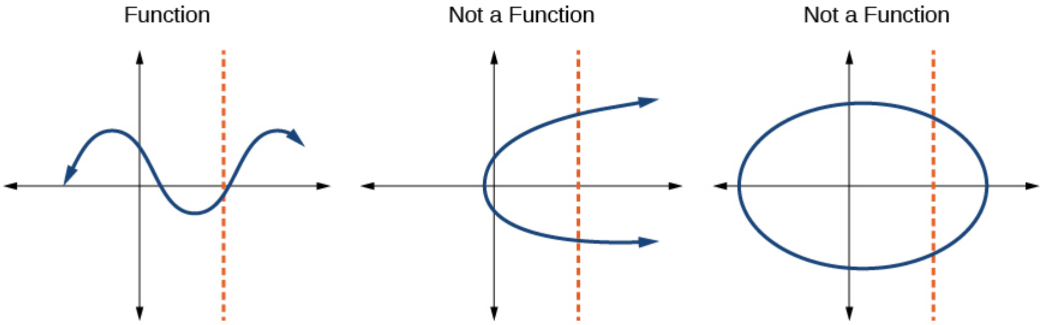


Figure 11: Three graphs visually showing what is and is not a function.

Howto: Given a graph, use the vertical line test to determine if the graph represents a function

1. Inspect the graph to see if any vertical line drawn would intersect the curve more than once.
2. If there is any such line, determine that the graph does not represent a function.

Example 12: Applying the Vertical Line Test

Which of the graphs inFigure 12 represent(s) a function $y = f(x)$?

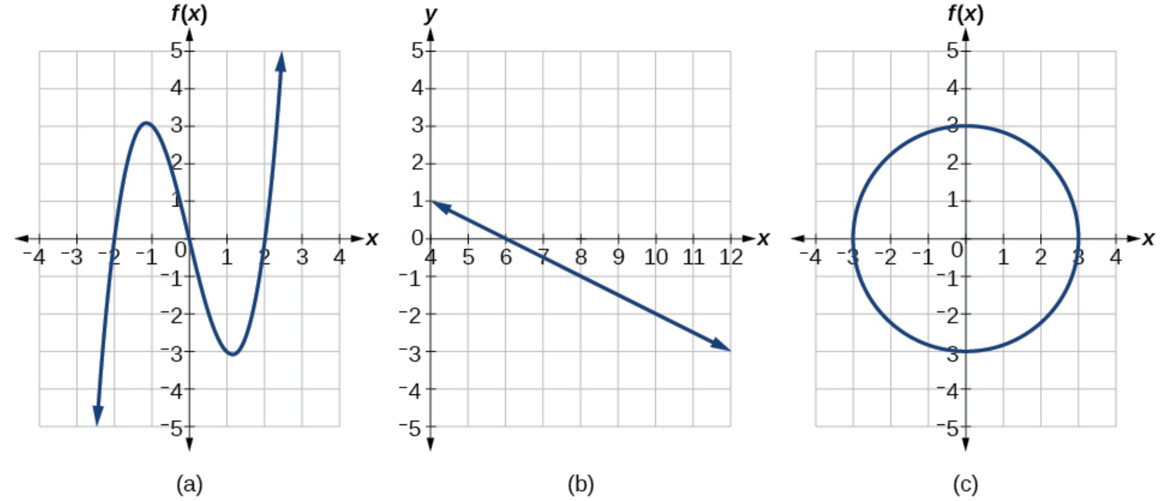


Figure 12: Graph of a polynomial (a), a downward-sloping line (b), and a circle (c).

Solution

If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b) of Figure 12. From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most x-values, a vertical line would intersect the graph at more than one point, as shown in Figure 13.

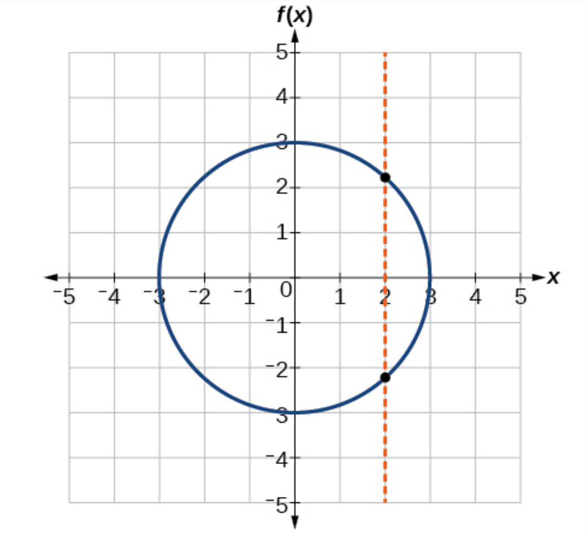


Figure 13: Graph of a circle.

Exercise 12

Does the graph in Figure 14 represent a function?

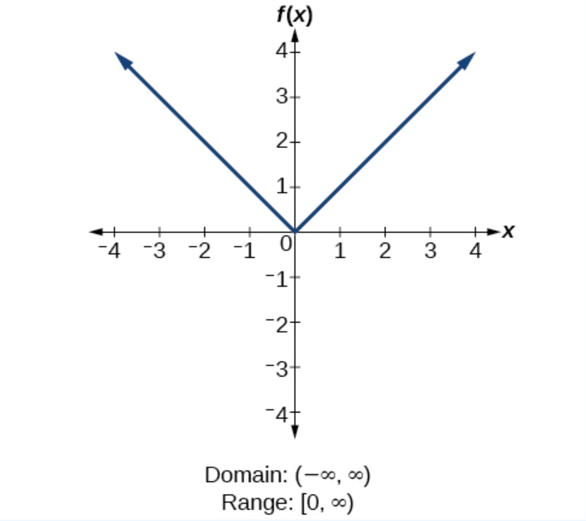


Figure 14: Graph of absolute value function.

Answer

yes

Using the Horizontal Line Test

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.

Howto: Given a graph of a function, use the horizontal line test to determine if the graph represents a one-to-one function

1. Inspect the graph to see if any horizontal line drawn would intersect the curve more than once.
2. If there is any such line, determine that the function is not one-to-one.

Example 13: Applying the Horizontal Line Test

Consider the functions shown in Figure $12a$ and Figure $12b$. Are either of the functions one-to-one?

Solution

The function in Figure $12a$ is not one-to-one. The horizontal line shown in Figure 15 intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)

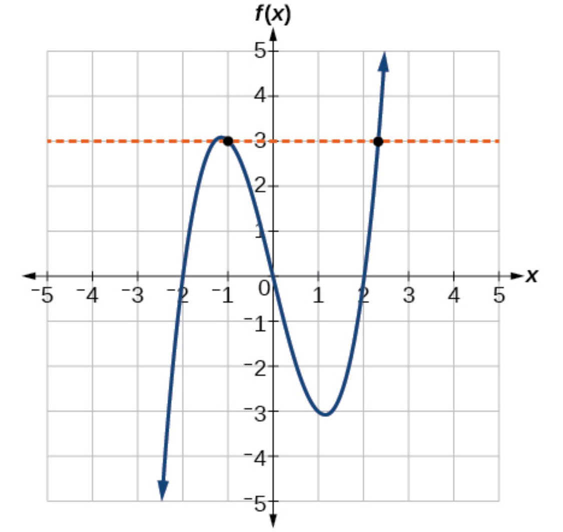


Figure 15: Graph of a polynomial with a horizontal line crossing through 2 points

The function in Figure $12b$ is one-to-one. Any horizontal line will intersect a diagonal line at most once.

Exercise 13

Is the graph shown in Figure 13 one-to-one?

Answer

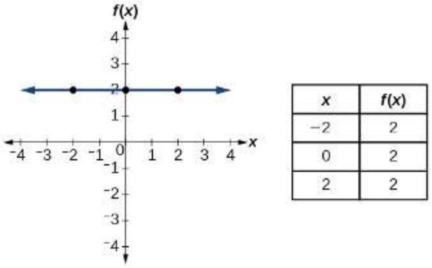
No, because it does not pass the horizontal line test.

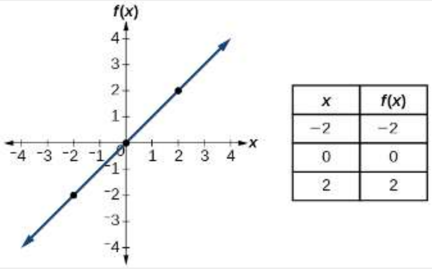
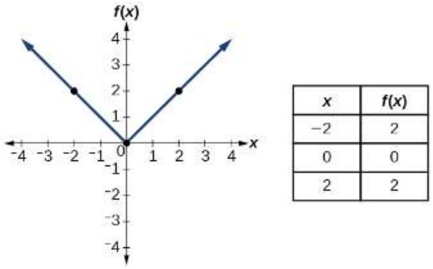
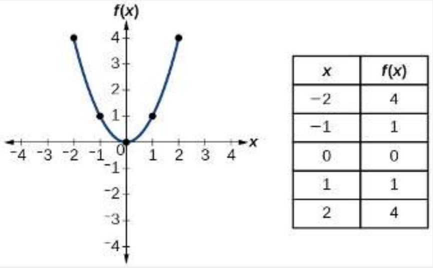
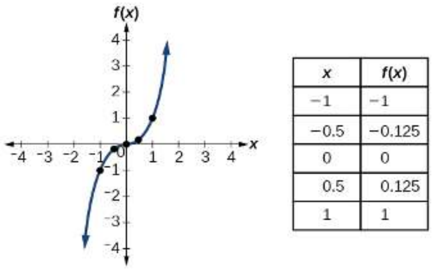
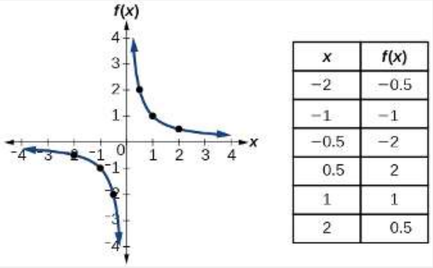
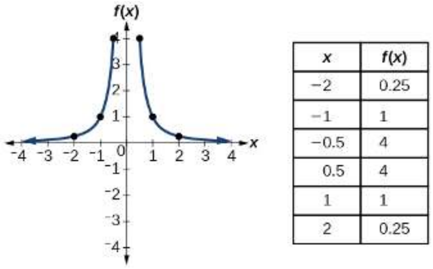
Identifying Basic Toolkit Functions

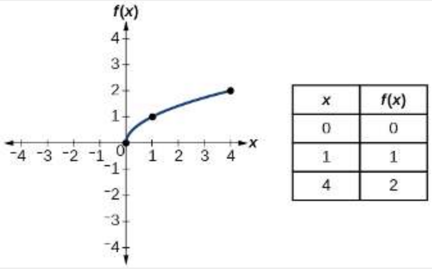
In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our “toolkit functions,” which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f(x)$ as the output variable.

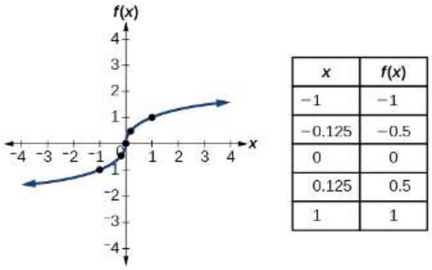
We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown in Table 14.

Table 14: Toolkit Functions

Name	Function	Graph
Constant	$f(x) = c$ where c is a constant	

Name	Function	Graph
Identity	$f(x)=x$	
Absolute Value	$f(x)=\|x\|$	
Quadratic	$f(x)=x^2$	
Cubic	$f(x)=x^3$	
reciprocal	$f(x)=\frac{1}{x}$	
Reciprocal squared	$f(x)=\frac{1}{x^2}$	

<table>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Square root</td><td>$f(x)=\sqrt{x}$</td><td>

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>4</td><td>2</td></tr>
</table></td></tr>
<tr><td>Cube root</td><td>$f(x)=\sqrt[3]{x}$</td><td>

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−1</td><td>−1</td></tr>
<tr><td>−0.125</td><td>−0.5</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>0.125</td><td>0.5</td></tr>
<tr><td>1</td><td>1</td></tr>
</table></td></tr>
</table>

Key Equations

- Constant function $f(x)=c$, where c is a constant
- Identity function $f(x)=x$
- Absolute value function $f(x)=|x|$
- Quadratic function $f(x)=x^2$
- Cubic function $f(x)=x^3$
- Reciprocal function $f(x)=\dfrac{1}{x}$
- Reciprocal squared function $f(x)=\frac{1}{x^2}$
- Square root function $f(x)=\sqrt{x}$
- Cube root function $f(x)=3\sqrt{x}$

Key Concepts

- A relation is a set of ordered pairs. A function is a specific type of relation in which each domain value, or input, leads to exactly one range value, or output.
- Function notation is a shorthand method for relating the input to the output in the form $y=f(x)$.
- In tabular form, a function can be represented by rows or columns that relate to input and output values.
- To evaluate a function, we determine an output value for a corresponding input value. Algebraic forms of a function can be evaluated by replacing the input variable with a given value.
- To solve for a specific function value, we determine the input values that yield the specific output value.
- An algebraic form of a function can be written from an equation.
- Input and output values of a function can be identified from a table.
- Relating input values to output values on a graph is another way to evaluate a function.
- A function is one-to-one if each output value corresponds to only one input value.
- A graph represents a function if any vertical line drawn on the graph intersects the graph at no more than one point.
- The graph of a one-to-one function passes the horizontal line test.

Footnotes

1 http://www.baseball-almanac.com/lege.../lisn100.shtml. Accessed 3/24/2014.

2 www.kgbanswers.com/how-long-i...y-span/4221590. Accessed 3/24/2014.

Glossary

dependent variable
an output variable

domain
the set of all possible input values for a relation

function
a relation in which each input value yields a unique output value

horizontal line test
a method of testing whether a function is one-to-one by determining whether any horizontal line intersects the graph more than once

independent variable
an input variable

input
each object or value in a domain that relates to another object or value by a relationship known as a function

one-to-one function
a function for which each value of the output is associated with a unique input value

output
each object or value in the range that is produced when an input value is entered into a function

range
the set of output values that result from the input values in a relation

relation
a set of ordered pairs

vertical line test
a method of testing whether a graph represents a function by determining whether a vertical line intersects the graph no more than once

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Inverse Functions

Learning Objectives

- Verify inverse functions.
- Determine the domain and range of an inverse function, and restrict the domain of a function to make it one-to-one.
- Find or evaluate the inverse of a function.
- Use the graph of a one-to-one function to graph its inverse function on the same axes.

A reversible heat pump is a climate-control system that is an air conditioner and a heater in a single device. Operated in one direction, it pumps heat out of a house to provide cooling. Operating in reverse, it pumps heat into the building from the outside, even in cool weather, to provide heating. As a heater, a heat pump is several times more efficient than conventional electrical resistance heating.

If some physical machines can run in two directions, we might ask whether some of the function "machines" we have been studying can also run backwards. Figure 1 provides a visual representation of this question. In this section, we will consider the reverse nature of functions.

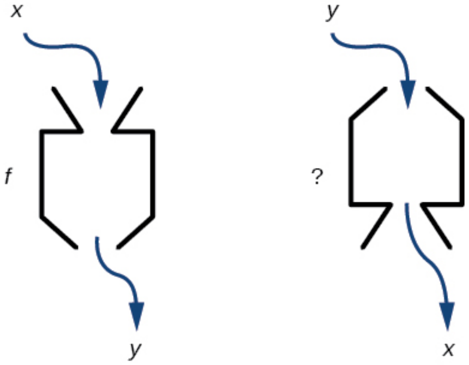

Figure 1***:*** *Can a function "machine" operate in reverse?*

Verifying That Two Functions Are Inverse Functions

Suppose a fashion designer traveling to Milan for a fashion show wants to know what the temperature will be. He is not familiar with the **Celsius** scale. To get an idea of how temperature measurements are related, he asks his assistant, Betty, to convert 75 degrees **Fahrenheit** to degrees Celsius. She finds the formula

$$C=\frac{5}{9}(F-32) \tag{1}$$

and substitutes 75 for F to calculate

$$\frac{5}{9}(75-32)\approx 24^{\circ} \tag{2}$$

Knowing that a comfortable 75 degrees Fahrenheit is about 24 degrees Celsius, he sends his assistant the week's weather forecast from Figure 2 for Milan, and asks her to convert all of the temperatures to degrees Fahrenheit.

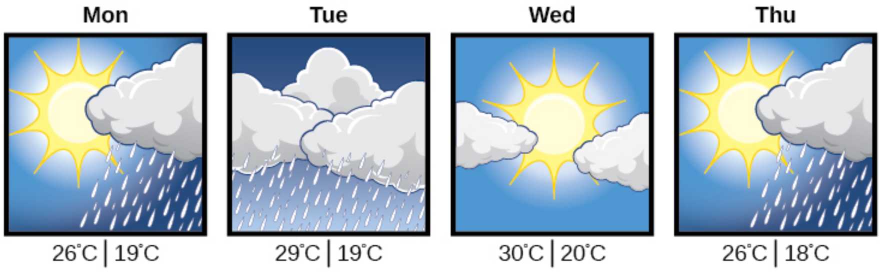

Figure 2: A forecast of Monday's through Thursday's weather.

At first, Betty considers using the formula she has already found to complete the conversions. After all, she knows her algebra, and can easily solve the equation for F after substituting a value for C. For example, to convert 26 degrees Celsius, she could write

$$26=\frac{5}{9}(F-32) \tag{3}$$

$$26\cdot\frac{9}{5}=F-32 \tag{4}$$

$$F=26\cdot\frac{9}{5}+32\approx 79 \tag{5}$$

After considering this option for a moment, however, she realizes that solving the equation for each of the temperatures will be awfully tedious. She realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one that takes the Celsius temperature and outputs the Fahrenheit temperature.

The formula for which Betty is searching corresponds to the idea of an **inverse function**, which is a function for which the input of the original function becomes the output of the inverse function and the output of the original function becomes the input of the inverse function.

Given a function $f(x)$, we represent its inverse as $f^{-1}(x)$, read as "f inverse of x." The raised -1 is part of the notation. It is not an exponent; it does not imply a power of -1 . In other words, $f^{-1}(x)$ does not mean $\frac{1}{f(x)}$ because $\frac{1}{f(x)}$ is the reciprocal of f and not the inverse.

The "exponent-like" notation comes from an analogy between function composition and multiplication: just as $a^{-1}a=1$ (1 is the identity element for multiplication) for any nonzero number a, so $f^{-1}\circ f$ equals the identity function, that is,

$$(f^{-1}\circ f)(x)=f^{-1}(f(x))=f^{-1}(y)=x \tag{6}$$

This holds for all x in the domain of f. Informally, this means that inverse functions "undo" each other. However, just as zero does not have a **reciprocal**, some functions do not have inverses.

Given a function $f(x)$, we can verify whether some other function $g(x)$ is the inverse of $f(x)$ by checking whether either $g(f(x))=x$ or $f(g(x))=x$ is true. We can test whichever equation is more convenient to work with because they are logically equivalent (that is, if one is true, then so is the other.)

For example, $y=4x$ and $y=\frac{1}{4}x$ are inverse functions.

$$(f^{-1}\circ f)(x)=f^{-1}(4x)=\frac{1}{4}(4x)=x \tag{7}$$

and

$$(f\circ f^{-1})(x)=f\left(\frac{1}{4}x\right)=4\left(\frac{1}{4}x\right)=x \tag{8}$$

A few coordinate pairs from the graph of the function $y=4x$ are $(-2,-8)$, $(0,0)$, and $(2,8)$. A few coordinate pairs from the graph of the function $y=\frac{1}{4}x$ are $(-8,-2)$, $(0,0)$, and $(8,2)$. If we interchange the input and output of each coordinate pair of a function, the interchanged coordinate pairs would appear on the graph of the inverse function.

Definition: Inverse Function

For any **one-to-one function** $f(x)=y$, a function $f^{-1}(x)$ is an **inverse function** of f if $f^{-1}(y)=x$. This can also be written as $f^{-1}(f(x))=x$ for all x in the domain of f. It also follows that $f(f^{-1}(x))=x$ for all x in the domain of f^{-1} if f^{-1} is the inverse of f.

The notation f^{-1} is read "f inverse." Like any other function, we can use any variable name as the input for f^{-1}, so we will often write $f^{-1}(x)$, which we read as "f inverse of x." Keep in mind that

$$f^{-1}(x)\neq\frac{1}{f(x)} \tag{9}$$

and not all functions have inverses.

Example 1: Identifying an Inverse Function for a Given Input-Output Pair

If for a particular one-to-one function $f(2) = 4$ and $f(5) = 12$, what are the corresponding input and output values for the inverse function?

Solution

The inverse function reverses the input and output quantities, so if

$$\begin{aligned} f(2)=4,\ \text{then}\ f^{-1}(4)=2; \\ f(5)=12,\ \text{then}\ f^{-1}(12)=5 \end{aligned} \tag{10}$$

.

Alternatively, if we want to name the inverse function g, then $g(4) = 2$ and $g(12) = 5$.

Analysis

Notice that if we show the coordinate pairs in a table form, the input and output are clearly reversed. See Table 1.

Table 1

$(x, f(x))$	$(x, g(x))$
$(2, 4)$	$(4, 2)$
$(5, 12)$	$(12, 5)$

Exercise 1

Given that $h^{-1}(6) = 2$, what are the corresponding input and output values of the original function h?

Answer

$h(2) = 6$

How To: Given two functions $f(x)$ and $g(x)$, test whether the functions are inverses of each other.

1. Determine whether $f(g(x)) = x$ or $g(f(x)) = x$.
2. If both statements are true, then $g = f^{-1}$ and $f = g^{-1}$. If either statement is false, then both are false, and $g \neq f^{-1}$ and $f \neq g^{-1}$.

Example 2: Testing Inverse Relationships Algebraically

If $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{1}{x} - 2$, is $g = f^{-1}$?

Solution

$$\begin{aligned} g(f(x)) &= \frac{1}{\left(\frac{1}{x+2}\right)-2} \qquad (11) \\ &= x+2-2 \qquad (12) \\ &= x \qquad (13) \end{aligned}$$

so

$$g = f^{-1} \text{ and } f = g^{-1} \tag{14}$$

This is enough to answer yes to the question, but we can also verify the other formula.

$$\begin{aligned} f(g(x)) &= \frac{1}{\frac{1}{x}-2+2} \qquad (15) \\ &= \frac{1}{\frac{1}{x}} \qquad (16) \\ &= x \qquad (17) \end{aligned}$$

Analysis

Notice the inverse operations are in reverse order of the operations from the original function.

Exercise 2

If $f(x) = x^3 - 4$ and $g(x) = \sqrt[3]{x+4}$, is $g = f^{-1}$?

Answer

Yes

Example 3: Determining Inverse Relationships for Power Functions

If $f(x) = x^3$ (the cube function) and $g(x) = \frac{1}{3}x$, is $g = f^{-1}$?

Solution

$$f(g(x)) = \frac{x^3}{27} \neq x \tag{18}$$

No, the functions are not inverses.

Analysis

The correct inverse to the cube is, of course, the cube root $\sqrt[3]{x} = x^{\frac{1}{3}}$, that is, the one-third is an exponent, not a multiplier.

Exercise 3

If $f(x) = (x-1)^3$ and $g(x) = \sqrt[3]{x} + 1$, is $g = f^{-1}$?

Answer

Yes

Finding Domain and Range of Inverse Functions

The outputs of the function f are the inputs to f^{-1}, so the range of f is also the domain of f^{-1}. Likewise, because the inputs to f are the outputs of f^{-1}, the domain of f is the range of f^{-1}. We can visualize the situation as in Figure 3.

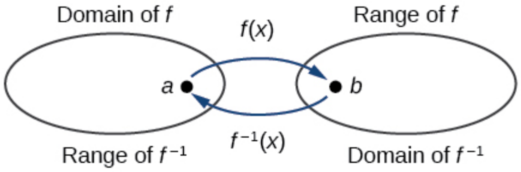

***Figure** 3: Domain and range of a function and its inverse.*

When a function has no inverse function, it is possible to create a new function where that new function on a limited domain does have an inverse function. For example, the inverse of $f(x) = \sqrt{x}$ is $f^{-1}(x) = x^2$, because a square “undoes” a square root; but the square is only the inverse of the square root on the domain $[0, \infty)$, since that is the range of $f(x) = \sqrt{x}$.

We can look at this problem from the other side, starting with the square (toolkit quadratic) function $f(x) = x^2$. If we want to construct an inverse to this function, we run into a problem, because for every given output of the quadratic function, there are two corresponding inputs (except when the input is 0). For example, the output 9 from the quadratic function corresponds to the inputs 3 and –3. But an output from a function is an input to its inverse; if this inverse input corresponds to more than one inverse output (input of the original function), then the “inverse” is not a function at all! To put it differently, the quadratic function is not a one-to-one function; it fails the horizontal line test, so it does not have an inverse function. In order for a function to have an inverse, it must be a one-to-one function.

In many cases, if a function is not one-to-one, we can still restrict the function to a part of its domain on which it is one-to-one. For example, we can make a restricted version of the square function $f(x) = x^2$ with its range limited to $[0, \infty)$, which is a one-to-one function (it passes the horizontal line test) and which has an inverse (the square-root function).

If $f(x) = (x-1)^2$ on $[1, \infty)$, then the inverse function is $f^{-1}(x) = \sqrt{x} + 1$.

- The domain of f = range of $f^{-1} = [1, \infty)$.
- The domain of f^{-1} = range of $f = [0, \infty)$.

Is it possible for a function to have more than one inverse?

> *No. If two supposedly different functions, say, g and h, both meet the definition of being inverses of another function f, then you can prove that $g = h$. We have just seen that some functions only have inverses if we restrict the domain of the original function. In these cases, there may be more than one way to restrict the domain, leading to different inverses. However, on any one domain, the original function still has only one unique inverse.*

Note: Domain and Range of Inverse Functions

The range of a function $f(x)$ is the domain of the inverse function $f^{-1}(x)$.

The domain of $f(x)$ is the range of $f^{-1}(x)$.

How To: Given a function, find the domain and range of its inverse.

1. If the function is one-to-one, write the range of the original function as the domain of the inverse, and write the domain of the original function as the range of the inverse.
2. If the domain of the original function needs to be restricted to make it one-to-one, then this restricted domain becomes the range of the inverse function.

Example 4: Finding the Inverses of Toolkit Functions

Identify which of the toolkit functions besides the quadratic function are not one-to-one, and find a restricted domain on which each function is one-to-one, if any. The toolkit functions are reviewed in Table 2. We restrict the domain in such a fashion that the function assumes all y-values exactly once.

Table 2

Constant	Identity	Quadratic	Cubic	Reciprocal
$f(x) = c$	$f(x) = x$	$f(x) = x^2$	$f(x) = x^3$	$f(x) = \frac{1}{x}$
Reciprocal squared	Cube Root	Square Root	Absolute Value	
$f(x) = \frac{1}{x^2}$	$f(x) = \sqrt[3]{x}$	$f(x) = \sqrt{x}$	$f(x) = \lvert x \rvert$	

Solution

The constant function is not one-to-one, and there is no domain (except a single point) on which it could be one-to-one, so the constant function has no meaningful inverse.

The absolute value function can be restricted to the domain $[0, \infty)$,where it is equal to the identity function.

The reciprocal-squared function can be restricted to the domain $(0, \infty)$.

Analysis

We can see that these functions (if unrestricted) are not one-to-one by looking at their graphs, shown in Figure 4. They both would fail the horizontal line test. However, if a function is restricted to a certain domain so that it passes the horizontal line test, then in that restricted domain, it can have an inverse.

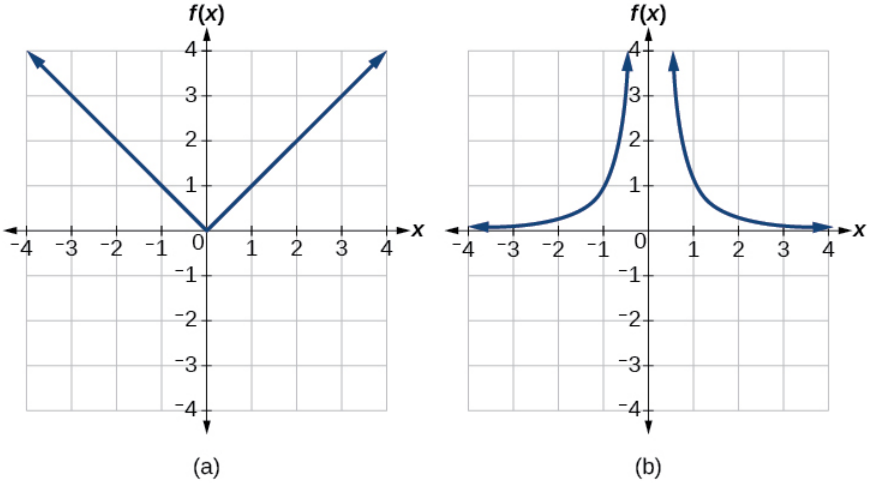

***Figure** 4: (a) Absolute value (b) Reciprocal squared*

4: The domain of function f is $(1, \infty)$ and the range of function f is $(-\infty, -2)$. Find the domain and range of the inverse function.

Solution

The domain of function f^{-1} is $(-\infty, -2)$ and the range of function f^{-1} is $(1, \infty)$.

Finding and Evaluating Inverse Functions

Once we have a one-to-one function, we can evaluate its inverse at specific inverse function inputs or construct a complete representation of the inverse function in many cases.

Inverting Tabular Functions

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

Each row (or column) of inputs becomes the row (or column) of outputs for the inverse function. Similarly, each row (or column) of outputs becomes the row (or column) of inputs for the inverse function.

Example 5: Interpreting the Inverse of a Tabular Function

A function $f(t)$ is given in Table 3, showing distance in miles that a car has traveled in t minutes. Find and interpret $f^{-1}(70)$

Table 3

t (minutes)	30	50	70	90
$f(t)$ (miles)	20	40	60	70

The inverse function takes an output of f and returns an input for f. So in the expression $f^{-1}(70)$, 70 is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function f, 90 minutes, so $f^{-1}(70) = 90$. The interpretation of this is that, to drive 70 miles, it took 90 minutes.

Alternatively, recall that the definition of the inverse was that if $f(a) = b$, then $f^{-1}(b) = a$. By this definition, if we are given $f^{-1}(70) = a$, then we are looking for a value a so that $f(a) = 70$. In this case, we are looking for a t so that $f(t) = 70$, which is when $t = 90$.

Exercise 5

Using Table 4, find and interpret (a) $f(60)$,and (b) $f^{-1}(60)$.

Table 4

t (minutes)	30	50	60	70	90
$f(t)$ (miles)	20	40	50	60	70

Answer

$f(60)=50$. In 60 minutes, 50 miles are traveled.
$f^{-1}(60)=70$. To travel 60 miles, it will take 70 minutes.

Evaluating the Inverse of a Function, Given a Graph of the Original Function

We saw in **Functions and Function Notation** that the domain of a function can be read by observing the horizontal extent of its graph. We find the domain of the inverse function by observing the *vertical* extent of the graph of the original function, because this corresponds to the horizontal extent of the inverse function. Similarly, we find the range of the inverse function by observing the *horizontal* extent of the graph of the original function, as this is the vertical extent of the inverse function. If we want to evaluate an inverse function, we find its input within its domain, which is all or part of the vertical axis of the original function's graph.

Given the graph of a function, evaluate its inverse at specific points.

1. Find the desired input on the y-axis of the given graph.
2. Read the inverse function's output from the x-axis of the given graph.

Example 6: Evaluating a Function and Its Inverse from a Graph at Specific Points

A function $g(x)$ is given in Figure 5. Find $g(3)$ and $g^{-1}(3)$.

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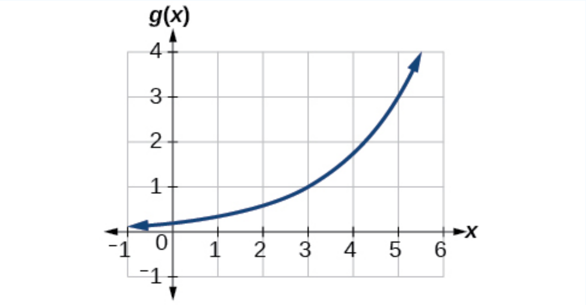

Figure 5: Graph of $g(x)$

Solution

To evaluate $g(3)$, we find 3 on the x-axis and find the corresponding output value on the y-axis. The point $(3,1)$ tells us that $g(3)=1$.

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means the value of x for which $g(x)=3$. By looking for the output value 3 on the vertical axis, we find the point $(5,3)$ on the graph, which means $g(5)=3$, so by definition, $g^{-1}(3)=5$. See Figure 6.

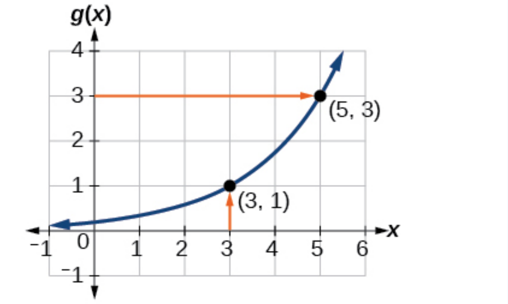

Figure 6: Graph of $g(x)$.

Exercise 6

Using the graph in Figure 6, (a) find $g^{-1}(1)$,and (b) estimate $g^{-1}(4)$.

Answer a

3

Answer b

5.6

Finding Inverses of Functions Represented by Formulas

Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula— for example, y as a function of x— we can often find the inverse function by solving to obtain x as a function of y.

How To: Given a function represented by a formula, find the inverse.

1. Make sure f is a one-to-one function.
2. Solve for x
3. Interchange x and y.

Example 7: Inverting the Fahrenheit-to-Celsius Function

Find a formula for the inverse function that gives Fahrenheit temperature as a function of Celsius temperature.

$$C=\frac{5}{9}(F-32) \tag{19}$$

Solution

$$C=\frac{5}{9}(F-32) \tag{20}$$

$$C\cdot\frac{9}{5}=F-32 \tag{21}$$

$$F=\frac{9}{5}C+32 \tag{22}$$

By solving in general, we have uncovered the inverse function. If

$$C=h(F)=\frac{5}{9}(F-32) \tag{23}$$

,

then

$$F=h^{-1}(C)=\frac{9}{5}C+32. \tag{24}$$

In this case, we introduced a function h to represent the conversion because the input and output variables are descriptive, and writing C^{-1} could get confusing.

Exercise 7

Solve for x in terms of y given $y=\frac{1}{3}(x-5)$

Answer

$x=3y+5$

Example 8: Solving to Find an Inverse Function

Find the inverse of the function $f(x) = \frac{2}{x-3} + 4$.

Solution

$$\begin{aligned}
y &= \frac{2}{x-3+4} && \text{Set up an equation.} \qquad (25)\\
y-4 &= \frac{2}{x-3} && \text{Subtract 4 from both sides.} \qquad (26)\\
x-3 &= \frac{2}{y-4} && \text{Multiply both sides by x−3 and divide by y−4.} \qquad (27)\\
x &= \frac{2}{y-4}+3 && \text{Add 3 to both sides.} \qquad (28)
\end{aligned}$$

So $f^{-1}(y) = \frac{2}{y-4} + 3$ or $f^{-1}(x) = \frac{2}{x-4} + 3$.

Analysis

The domain and range of f exclude the values 3 and 4, respectively. f and f^{-1} are equal at two points but are not the same function, as we can see by creating Table 5.

Table 5

x	1	2	5	$f^{-1}(y)$
$f(x)$	3	2	5	y

Example 9: Solving to Find an Inverse with Radicals

Find the inverse of the function $f(x) = 2 + \sqrt{x-4}$.

Solution

$$\begin{aligned}
y &= 2+\sqrt{x-4} \qquad (29)\\
(y-2)^2 &= x-4 \qquad (30)\\
x &= (y-2)^2+4 \qquad (31)
\end{aligned}$$

So $f^{-1}(x) = (x-2)^2 + 4$.

The domain of f is $[4, \infty)$. Notice that the range of f is $[2, \infty)$, so this means that the domain of the inverse function f^{-1} is also $[2, \infty)$

Analysis

The formula we found for $f^{-1}(x)$ looks like it would be valid for all real x. However, f^{-1} itself must have an inverse (namely, f) so we have to restrict the domain of f^{-1} to $[2, \infty)$ in order to make f^{-1} a one-to-one function. This domain of f^{-1} is exactly the range of f.

Exercise 8

What is the inverse of the function $f(x) = 2 - \sqrt{x}$? State the domains of both the function and the inverse function.

Answer

$f^{-1}(x) = (2-x)^2$; domain of f: $[0, \infty)$; domain of f^{-1}: $(-\infty, 2]$

Finding Inverse Functions and Their Graphs

Now that we can find the inverse of a function, we will explore the graphs of functions and their inverses. Let us return to the quadratic function $f(x) = x^2$ restricted to the domain $[0, \infty)$, on which this function is one-to-one, and graph it as in Figure 7.

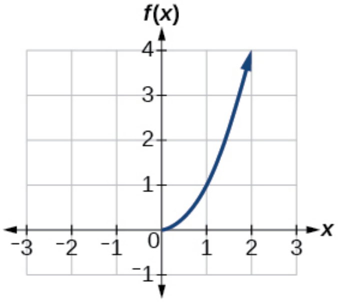

***Figure** 7: Quadratic function with domain restricted to* $[0, \infty)$.

Restricting the domain to $[0, \infty)$ makes the function one-to-one (it will obviously pass the horizontal line test), so it has an inverse on this restricted domain.

We already know that the inverse of the toolkit quadratic function is the square root function, that is, $f^{-1}(x) = \sqrt{x}$. What happens if we graph both f and f^{-1} on the same set of axes, using the x-axis for the input to both f and f^{-1}?

We notice a distinct relationship: The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected about the diagonal line $y = x$, which we will call the identity line, shown in Figure 8**.**

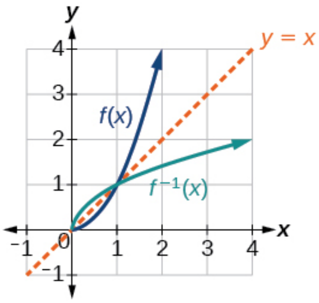

***Figure** 8: Square and square-root functions on the non-negative domain*

This relationship will be observed for all one-to-one functions, because it is a result of the function and its inverse swapping inputs and outputs. This is equivalent to interchanging the roles of the vertical and horizontal axes.

Example 10: Finding the Inverse of a Function Using Reflection about the Identity Line

Given the graph of $f(x)$ in Figure 9, sketch a graph of $f^{-1}(x)$.

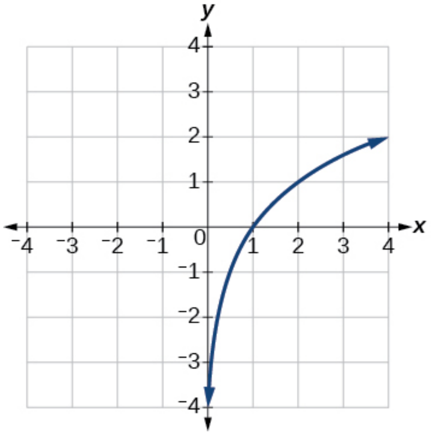

Figure 9: Graph of $f^(-1)(x)$.

This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of $(0, \infty)$ and range of $(-\infty, \infty)$, so the inverse will have a domain of $(-\infty, \infty)$ and range of $(0, \infty)$.

If we reflect this graph over the line $y = x$, the point $(1, 0)$ reflects to $(0, 1)$ and the point $(4, 2)$ reflects to $(2, 4)$. Sketching the inverse on the same axes as the original graph gives Figure 10.

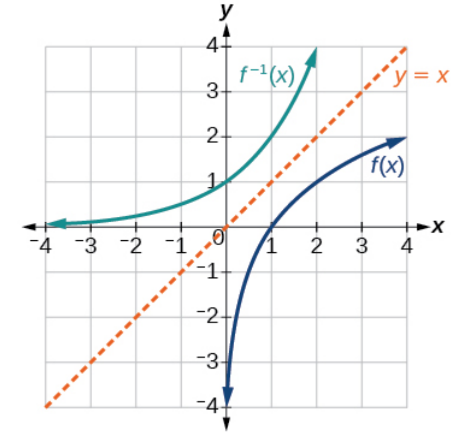

Figure 10: The function and its inverse, showing reflection about the identity line

Exercise 1

Draw graphs of the functions f and f^{-1} from Example 8.

Answer

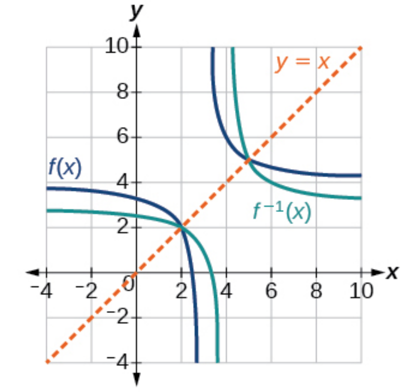

Figure 11: Graph of $f(x)$ and $f^{(}-1)(x)$.

Is there any function that is equal to its own inverse?

Yes. If $f=f^{-1}$, then $f(f(x))=x$, and we can think of several functions that have this property. The identity function does, and so does the reciprocal function, because

$$\frac{1}{\frac{1}{x}}=x \tag{32}$$

Any function $f(x)=c-x$, where c is a constant, is also equal to its own inverse.

Key Concepts

- If $g(x)$ is the inverse of $f(x)$, then $g(f(x))=f(g(x))=x$.
- Each of the toolkit functions has an inverse.
- For a function to have an inverse, it must be one-to-one (pass the horizontal line test).
- A function that is not one-to-one over its entire domain may be one-to-one on part of its domain.
- For a tabular function, exchange the input and output rows to obtain the inverse.
- The inverse of a function can be determined at specific points on its graph.
- To find the inverse of a formula, solve the equation $y=f(x)$ for x as a function of y. Then exchange the labels x and y.
- The graph of an inverse function is the reflection of the graph of the original function across the line $y=x$.

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Prelude to Functions

Toward the end of the twentieth century, the values of stocks of internet and technology companies rose dramatically. As a result, the Standard and Poor’s stock market average rose as well. The *Standard and Poor’s Index* tracks the value of that initial investment of just under \$100 over the 40 years. It shows that an investment that was worth less than \$500 until about 1995 skyrocketed up to about \$1100 by the beginning of 2000. That five-year period became known as the “dot-com bubble” because so many internet startups were formed. As bubbles tend to do, though, the dot-com bubble eventually burst. Many companies grew too fast and then suddenly went out of business. The result caused the sharp decline represented on the graph beginning at the end of 2000.

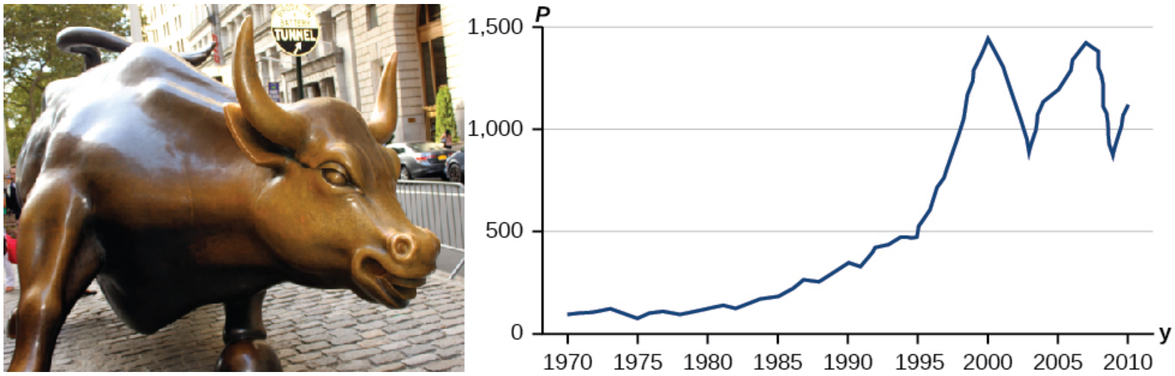

Figure 1: Figure of a Bull and a graph of market prices. Standard and Poor’s Index with dividends reinvested (credit "bull": modification of work by Prayitno Hadinata; credit "graph": modification of work by MeasuringWorth)

Notice, as we consider this example, that there is a definite relationship between the year and stock market average. For any year we choose, we can determine the corresponding value of the stock market average. In this chapter, we will explore these kinds of relationships and their properties.

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Rates of Change and Behavior of Graphs

Learning Objectives

- Find the average rate of change of a function.
- Use a graph to determine where a function is increasing, decreasing, or constant.
- Use a graph to locate local maxima and local minima.
- Use a graph to locate the absolute maximum and absolute minimum.

Gasoline costs have experienced some wild fluctuations over the last several decades. Table 1 lists the average cost, in dollars, of a gallon of gasoline for the years 2005–2012. The cost of gasoline can be considered as a function of year.

Table 1

y	2005	2006	2007	2008	2009	2010	2011	2012
$C(y)$	2.31	2.62	2.84	3.30	2.41	2.84	3.58	3.68

If we were interested only in how the gasoline prices changed between 2005 and 2012, we could compute that the cost per gallon had increased from \$2.31 to \$3.68, an increase of \$1.37. While this is interesting, it might be more useful to look at how much the price changed per year. In this section, we will investigate changes such as these.

Finding the Average Rate of Change of a Function

The price change per year is a **rate of change** because it describes how an output quantity changes relative to the change in the input quantity. We can see that the price of gasoline in Table 1 did not change by the same amount each year, so the rate of change was not constant. If we use only the beginning and ending data, we would be finding the **average rate of change** over the specified period of time. To find the average rate of change, we divide the change in the output value by the change in the input value.

$$\begin{aligned}\text{Average rate of change} &= \frac{\text{Change in output}}{\text{Change in input}} \\ &= \frac{\Delta y}{\Delta x} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{f(x_2) - f(x_1)}{x_2 - x_1}\end{aligned}$$

The Greek letter Δ (delta) signifies the change in a quantity; we read the ratio as “delta-y over delta-x” or “the change in y divided by the change in x.” Occasionally we write Δf instead of Δy, which still represents the change in the function’s output value resulting from a change to its input value. It does not mean we are changing the function into some other function.

In our example, the gasoline price increased by \$1.37 from 2005 to 2012. Over 7 years, the average rate of change was

$$\frac{\Delta y}{\Delta x} = \frac{\$1.37}{7\text{years}} \approx 0.196 \text{ dollars per year.} \tag{1}$$

On average, the price of gas increased by about 19.6¢ each year. Other examples of rates of change include:

- A population of rats increasing by 40 rats per week
- A car traveling 68 miles per hour (distance traveled changes by 68 miles each hour as time passes)
- A car driving 27 miles per gallon (distance traveled changes by 27 miles for each gallon)
- The current through an electrical circuit increasing by 0.125 amperes for every volt of increased voltage
- The amount of money in a college account decreasing by \$4,000 per quarter

Definition: Rate of Change

A **rate of change** describes how an output quantity changes relative to the change in the input quantity. The units on a rate of change are “output units per input units.”

The average rate of change between two input values is the total change of the function values (output values) divided by the change in the input values.

$$\frac{\Delta y}{\Delta x}=\frac{f(x_2)-f(x_1)}{x_2-x_1} \tag{2}$$

How To...

Given the value of a function at different points, calculate the average rate of change of a function for the interval between two values x_1 and x_2.

1. Calculate the difference $y_2-y_1=\Delta y$.
2. Calculate the difference $x_2-x_1=\Delta x$.
3. Find the ratio $\frac{\Delta y}{\Delta x}$.

Example 1: Computing an Average Rate of Change

Using the data in Table 1, find the average rate of change of the price of gasoline between 2007 and 2009.

Solution

In 2007, the price of gasoline was \$2.84. In 2009, the cost was \$2.41. The average rate of change is

$$\begin{aligned}\frac{\Delta y}{\Delta x}&=\frac{y_2-y_1}{x_2-x_1}\\&=\frac{\$2.41-\$2.84}{2009-2007}\\&=\frac{-\$0.43}{2\text{ years}}\\&=-\$0.22\text{ per year}\end{aligned}$$

Analysis

Note that a decrease is expressed by a negative change or “negative increase.” A rate of change is negative when the output decreases as the input increases or when the output increases as the input decreases.

Exercise 1

Using the data in Table 1, find the average rate of change between 2005 and 2010.

Solution

$$\frac{\$2.84-\$2.315}{5\text{ years}}=\frac{\$0.535}{5\text{ years}}=\$0.106\text{per year}.$$

Example 2: Computing Average Rate of Change from a Graph

Given the function $g(t)$ shown in Figure 1, find the average rate of change on the interval $[-1,2]$.

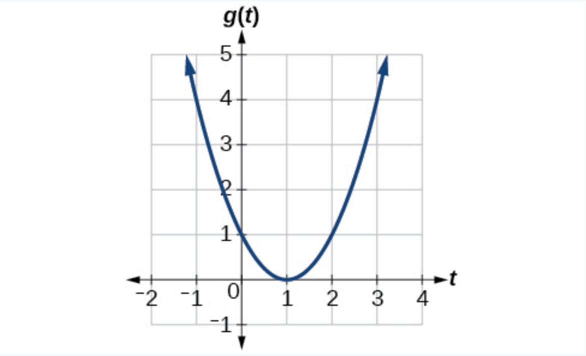


Figure 1: Graph of a parabola.

Solution

At $t=-1$, Figure 2 shows $g(-1)=4$. At $t=2$,the graph shows $g(2)=1$.

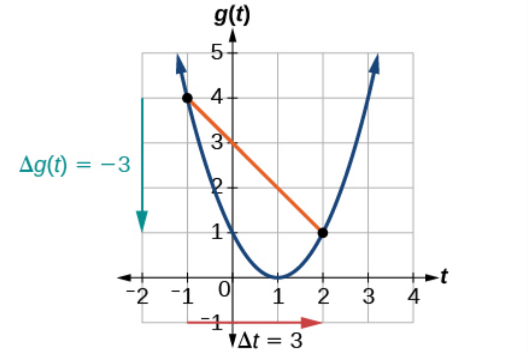


Figure 2: Graph of a parabola with a line from points (-1, 4) and (2, 1) to show the changes for g(t) and t.

The horizontal change $\Delta t=3$ is shown by the red arrow, and the vertical change $\Delta g(t)=-3$ is shown by the turquoise arrow. The output changes by –3 while the input changes by 3, giving an average rate of change of

$$\frac{1-4}{2-(-1)}=\frac{-3}{3}=-1 \tag{3}$$

Analysis

Note that the order we choose is very important. If, for example, we use $\frac{y_2-y_1}{x_1-x_2}$, we will not get the correct answer. Decide which point will be 1 and which point will be 2, and keep the coordinates fixed as (x_1,y_1) and (x_2,y_2).

Example 3: Computing Average Rate of Change from a Table

After picking up a friend who lives 10 miles away, Anna records her distance from home over time. The values are shown in Table 2. Find her average speed over the first 6 hours.

Table 2

t (hours)	0	1	2	3	4	5	6	7
$D(t)$ (miles)	10	55	90	153	214	240	292	300

Solution

Here, the average speed is the average rate of change. She traveled 292 miles in 6 hours, for an average speed of

$$\begin{aligned}\frac{292-10}{6-0}&=\frac{282}{6}\\&=47\end{aligned}$$

The average speed is about 47 miles per hour.

Analysis

Because the speed is not constant, the average speed depends on the interval chosen. For the interval $[2,3]$, the average speed is 63 miles per hour.

Example 4: Computing Average Rate of Change for a Function Expressed as a Formula

Compute the average rate of change of $f(x)=x^2-\frac{1}{x}$ on the interval $[2,4]$.

Solution

We can start by computing the function values at each endpoint of the interval.

$$\begin{aligned} f(2) &= 2^2 - \frac{1}{2} f(4) && = 4^2 - \frac{1}{4} \\ &= 4 - \frac{1}{2} && = 16 - \frac{1}{4} \\ &= 72 && = \frac{63}{4} \end{aligned}$$

Now we compute the average rate of change.

$$\begin{aligned} \text{Average rate of change} &= \frac{f(4) - f(2)}{4 - 2} \\ &= \frac{\frac{63}{4} - \frac{7}{2}}{4 - 2} \\ &= \frac{\frac{49}{4}}{2} \\ &= \frac{49}{8} \end{aligned}$$

Exercise 2

Find the average rate of change of $f(x) = x - 2\sqrt{x}$ on the interval $[1, 9]$.

Solution

$\frac{1}{2}$

Example 5: Finding the Average Rate of Change of a Force

The electrostatic force F, measured in newtons, between two charged particles can be related to the distance between the particles d,in centimeters, by the formula $F(d) = \frac{2}{d^2}$. Find the average rate of change of force if the distance between the particles is increased from 2 cm to 6 cm.

Solution

We are computing the average rate of change of $F(d) = \dfrac{2}{d^2}$ on the interval $[2, 6]$.

$$\begin{aligned} \text{Average rate of change } &= \frac{F(6) - F(2)}{6 - 2} \\ &= \frac{\frac{2}{6^2} - \frac{2}{2^2}}{6 - 2} && \text{Simplify} \\ &= \frac{\frac{2}{36} - \frac{2}{4}}{4} \\ &= \frac{-\frac{16}{36}}{4} && \text{Combine numerator terms.} \\ &= -\frac{1}{9} && \text{Simplify} \end{aligned}$$

The average rate of change is $-\frac{1}{9}$ newton per centimeter.

Example 6: Finding an Average Rate of Change as an Expression

Find the average rate of change of $g(t) = t^2 + 3t + 1$ on the interval $[0, a]$. The answer will be an expression involving a.

Solution

We use the average rate of change formula.

$$\begin{aligned}
\text{Average rate of change} &= \frac{g(a)-g(0)}{a-0} && \text{Evaluate.}\\
&= \frac{(a^2+3a+1)-(0^2+3(0)+1)}{a-0} && \text{Simplify.}\\
&= \frac{a^2+3a+1-1}{a} && \text{Simplify and factor.}\\
&= \frac{a(a+3)}{a} && \text{Divide by the common factor a.}\\
&= a+3
\end{aligned}$$

This result tells us the average rate of change in terms of a between $t=0$ and any other point $t=a$. For example, on the interval $[0,5]$, the average rate of change would be $5+3=8$.

Exercise 3

Find the average rate of change of $f(x)=x^2+2x-8$ on the interval $[5,a]$.

Solution

$a+7$

Using a Graph to Determine Where a Function is Increasing, Decreasing, or Constant

As part of exploring how functions change, we can identify intervals over which the function is changing in specific ways. We say that a function is increasing on an interval if the function values increase as the input values increase within that interval. Similarly, a function is decreasing on an interval if the function values decrease as the input values increase over that interval. The average rate of change of an increasing function is positive, and the average rate of change of a decreasing function is negative. Figure 3 shows examples of increasing and decreasing intervals on a function.

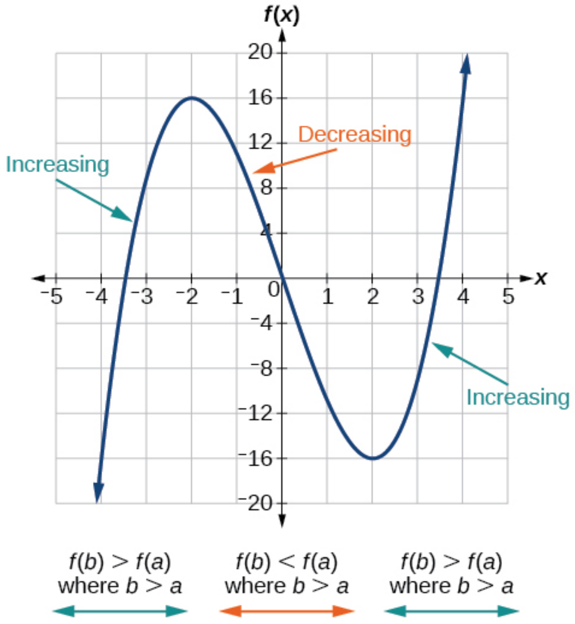


Figure 3: The function $f(x)=x^3-12x$ is increasing on $(-\infty,-2)\cup(2,\infty)$ and is decreasing on $(-2,2)$.

While some functions are increasing (or decreasing) over their entire domain, many others are not. A value of the input where a function changes from increasing to decreasing (as we go from left to right, that is, as the input variable increases) is called a **local maximum**. If a function has more than one, we say it has local maxima. Similarly, a value of the input where a function changes from decreasing to increasing as the input variable increases is called a **local minimum**. The plural form is “local minima.” Together, local maxima and minima are called **local extrema**, or local extreme values, of the function. (The singular form is “extremum.”) Often, the term local is replaced by the term relative. In this text, we will use the term local.

Clearly, a function is neither increasing nor decreasing on an interval where it is constant. A function is also neither increasing nor decreasing at extrema. Note that we have to speak of local extrema, because any given local extremum as defined here is not necessarily the highest maximum or lowest minimum in the function's entire domain.

For the function whose graph is shown in Figure 4, the local maximum is 16, and it occurs at $x = -2$. The local minimum is −16 and it occurs at $x = 2$.

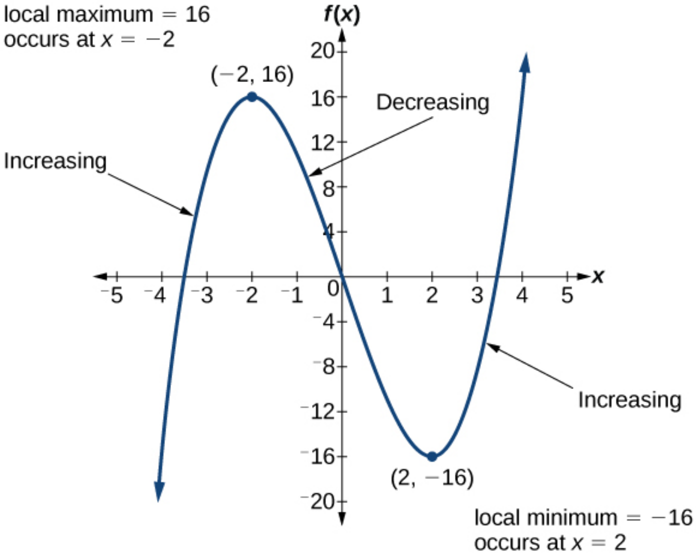


Figure 4: Graph of a polynomial that shows the increasing and decreasing intervals and local maximum.maximum

To locate the local maxima and minima from a graph, we need to observe the graph to determine where the graph attains its highest and lowest points, respectively, within an open interval. Like the summit of a roller coaster, the graph of a function is higher at a local maximum than at nearby points on both sides. The graph will also be lower at a local minimum than at neighboring points. Figure 5 illustrates these ideas for a local maximum.

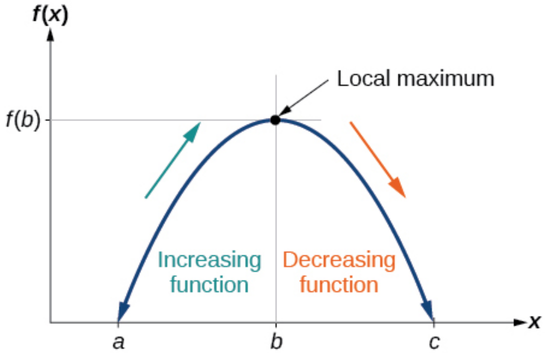


Figure 5: Definition of a local maximum

These observations lead us to a formal definition of local extrema.

Local Minima and Local Maxima

- A function f is an **increasing function** on an open interval if $f(b) > f(a)$ for every a, b interval where $b > a$.
- A function f is a **decreasing function** on an open interval if $f(b) < f(a)$ for every a, b interval where $b > a$.

A function f has a local maximum at a point b in an open interval (a, c) if $f(b)$ is greater than or equal to $f(x)$ for every point x (x does not equal b) in the interval. Likewise, f has a local minimum at a point b in (a, c) if $f(b)$ is less than or equal to $f(x)$ for every x (x does not equal b) in the interval.

Example 7 Finding Increasing and Decreasing Intervals on a Graph

Given the function $p(t)$ in Figure 6, identify the intervals on which the function appears to be increasing.

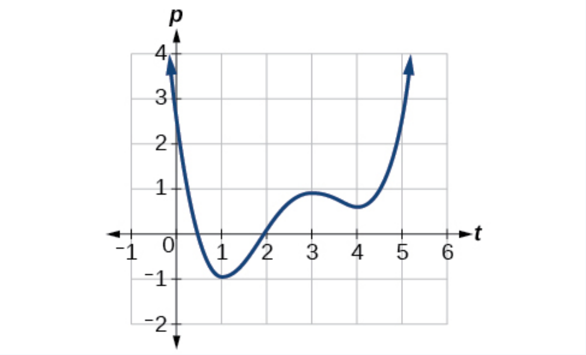


Figure 6: Graph of a polynomial.

Solution

We see that the function is not constant on any interval. The function is increasing where it slants upward as we move to the right and decreasing where it slants downward as we move to the right. The function appears to be increasing from $t = 1$ to $t = 3$ and from $t = 4$ on.

In interval notation, we would say the function appears to be increasing on the interval $(1, 3)$ and the interval $(4, \infty)$.

Analysis

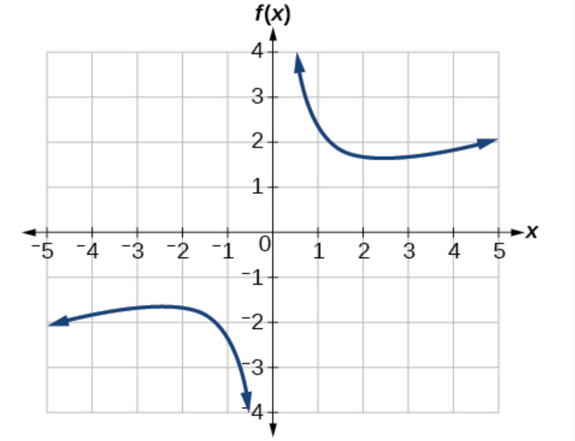
Notice in this example that we used open intervals (intervals that do not include the endpoints), because the function is neither increasing nor decreasing at $t = 1$, $t = 3$, and $t = 4$. These points are the local extrema (two minima and a maximum).

Example 8: Finding Local Extrema from a Graph

Graph the function $f(x) = \frac{2}{x} + \frac{x}{3}$. Then use the graph to estimate the local extrema of the function and to determine the intervals on which the function is increasing.

Solution

Using technology, we find that the graph of the function looks like that in Figure 7. It appears there is a low point, or local minimum, between $x = 2$ and $x = 3$, and a mirror-image high point, or local maximum, somewhere between $x = -3$ and $x = -2$



. Figure 7: Graph of a reciprocal function.

Analysis

Most graphing calculators and graphing utilities can estimate the location of maxima and minima. Figure 8 provides screen images from two different technologies, showing the estimate for the local maximum and minimum.

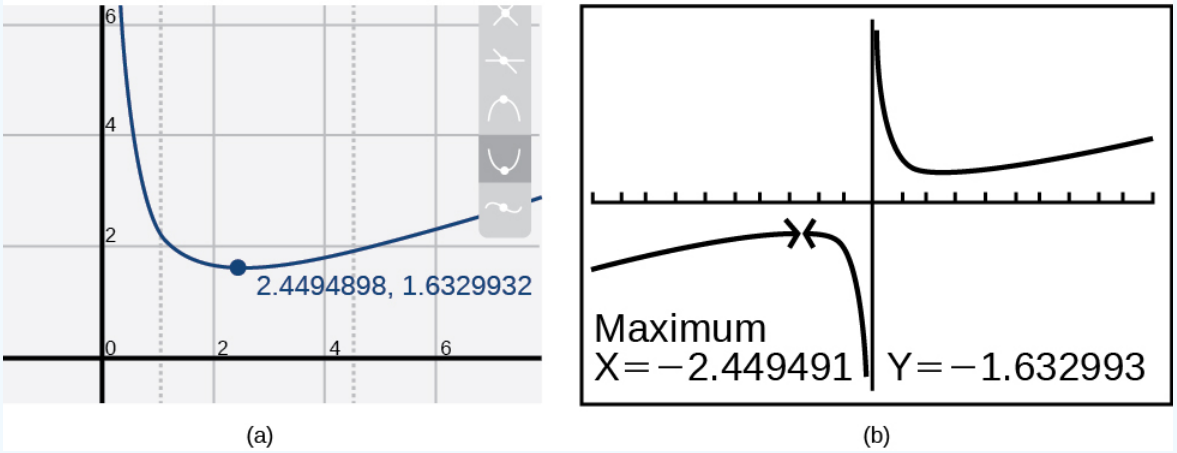


Figure 8: Graph of the reciprocal function on a graphing calculator.

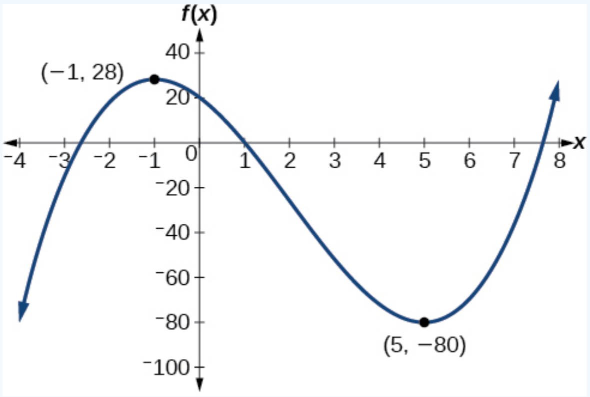
Based on these estimates, the function is increasing on the interval $(-\infty, -2.449)$ and $(2.449, \infty)$. Notice that, while we expect the extrema to be symmetric, the two different technologies agree only up to four decimals due to the differing approximation algorithms used by each. (The exact location of the extrema is at $\pm\sqrt{6}$, but determining this requires calculus.)

Exercise 8

Graph the function $f(x) = x^3 - 6x^2 - 15x + 20$ to estimate the local extrema of the function. Use these to determine the intervals on which the function is increasing and decreasing.

Solution

The local maximum appears to occur at $(-1, 28)$, and the local minimum occurs at $(5, -80)$. The function is increasing on $(-\infty, -1) \cup (5, \infty)$ and decreasing on $(-1, 5)$.



Graph of a polynomial with a local maximum at (-1, 28) and local minimum at (5, -80).

Example 9: Finding Local Maxima and Minima from a Graph

For the function f whose graph is shown in Figure 9, find all local maxima and minima.

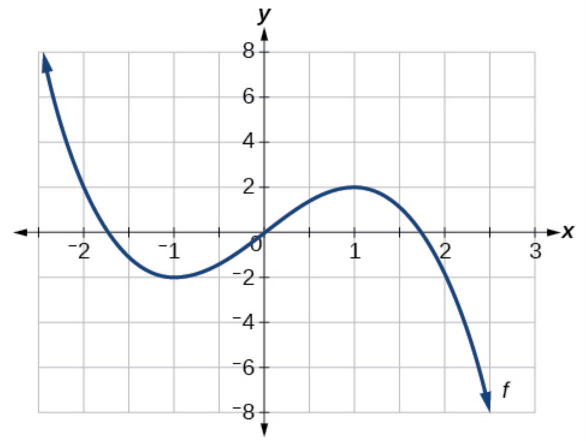


Figure 9: Graph of a polynomial.

Solution

Observe the graph of f. The graph attains a local maximum at $x = 1$ because it is the highest point in an open interval around $x = 1$.The local maximum is the y-coordinate at $x = 1$, which is 2.

The graph attains a local minimum at $x = -1$ because it is the lowest point in an open interval around $x = -1$. The local minimum is the y-coordinate at $x = -1$, which is −2.

Analyzing the Toolkit Functions for Increasing or Decreasing Intervals

We will now return to our toolkit functions and discuss their graphical behavior in Figure 10, Figure 11, and Figure 12.

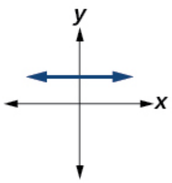
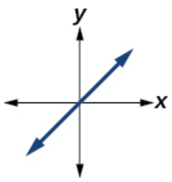
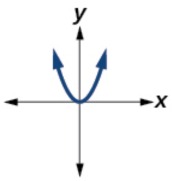
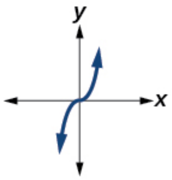
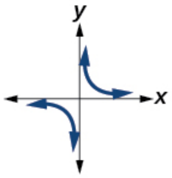
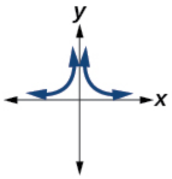
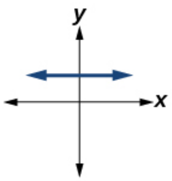
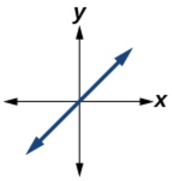
Function	Increasing/Decreasing	Example
Constant Function $f(x) = c$	Neither increasing nor decreasing	
Identity Function $f(x) = x$	Increasing	
Quadratic Function $f(x) = x^2$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ Minimum at $x = 0$	

Figure 10

Function	Increasing/Decreasing	Example
Cubic Function $f(x) = x^3$	Increasing	
Reciprocal $f(x) = \frac{1}{x}$	Decreasing $(-\infty, 0) \cup (0, \infty)$	
Reciprocal Squared $f(x) = \frac{1}{x^2}$	Increasing on $(-\infty, 0)$ Decreasing on $(0, \infty)$	

. Figure 11

Function	Increasing/Decreasing	Example
Constant Function $f(x) = c$	Neither increasing nor decreasing	
Identity Function $f(x) = x$	Increasing	
Quadratic Function	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$	

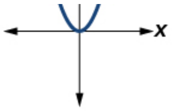
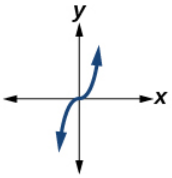
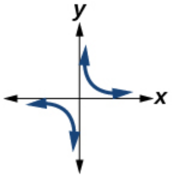
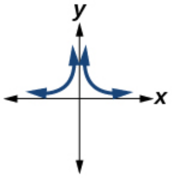
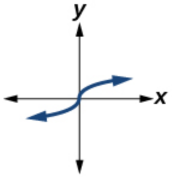
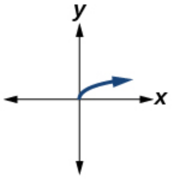
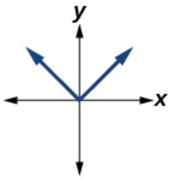
$f(x)=x^2$	Decreasing on $(-\infty, 0)$ Minimum at $x=0$	
Cubic Function $f(x)=x^3$	Increasing	
Reciprocal $f(x)=\frac{1}{x}$	Decreasing $(-\infty, 0)\cup(0, \infty)$	
Reciprocal Squared $f(x)=\frac{1}{x^2}$	Increasing on $(-\infty, 0)$ Decreasing on $(0, \infty)$	
Cube Root $f(x)=\sqrt[3]{x}$	Increasing	
Square Root $f(x)=\sqrt{x}$	Increasing on $(0, \infty)$	
Absolute Value $f(x)=\|x\|$	Increasing on $(0, \infty)$ Decreasing on $(\infty, 0)$	

Figure 12

Use A Graph to Locate the Absolute Maximum and Absolute Minimum

There is a difference between locating the highest and lowest points on a graph in a region around an open interval (locally) and locating the highest and lowest points on the graph for the entire domain. The y-coordinates (output) at the highest and lowest points are called the **absolute maximum** and **absolute minimum**, respectively. To locate absolute maxima and minima from a graph, we need to observe the graph to determine where the graph attains it highest and lowest points on the domain of the function (Figure 13).

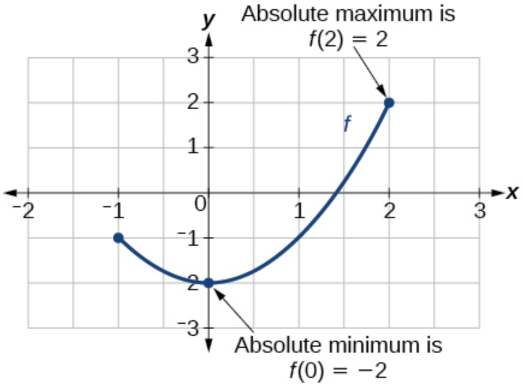


Figure 13: Graph of a segment of a parabola with an absolute minimum at (0, -2) and absolute maximum at (2, 2).

Not every function has an absolute maximum or minimum value. The toolkit function $f(x)=x^3$ is one such function.

Absolute Maxima and Minima

- The **absolute maximum** of f at $x=c$ is $f(c)$ where $f(c)\geq f(x)$ for all x in the domain of f.
- The **absolute minimum** of f at $x=d$ is $f(d)$ where $f(d)\leq f(x)$ for all x in the domain of f.

Example 10: Finding Absolute Maxima and Minima from a Graph

For the function f shown in Figure 14, find all absolute maxima and minima.

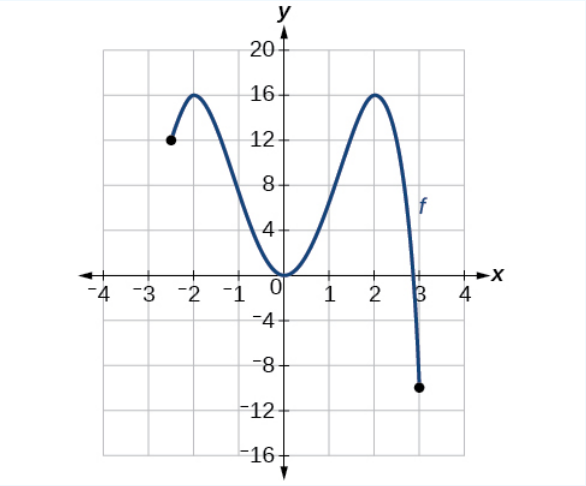


Figure 14: Graph of a polynomial.

Solution

Observe the graph of f. The graph attains an absolute maximum in two locations, $x=-2$ and $x=2$, because at these locations, the graph attains its highest point on the domain of the function. The absolute maximum is the y-coordinate at $x=-2$ and $x=2$, which is 16.

The graph attains an absolute minimum at x=3, because it is the lowest point on the domain of the function’s graph. The absolute minimum is the y-coordinate at x=3,which is−10.

Key Equations

- Average rate of change: $\dfrac{\Delta y}{\Delta x}=\dfrac{f(x_2)-f(x_1)}{x_2-x_1}$

Key Concepts

- A rate of change relates a change in an output quantity to a change in an input quantity. The average rate of change is determined using only the beginning and ending data. See Example.
- Identifying points that mark the interval on a graph can be used to find the average rate of change. See Example.

- Comparing pairs of input and output values in a table can also be used to find the average rate of change. See Example.
- An average rate of change can also be computed by determining the function values at the endpoints of an interval described by a formula. See Example and Example.
- The average rate of change can sometimes be determined as an expression. See Example.
- A function is increasing where its rate of change is positive and decreasing where its rate of change is negative. See Example.
- A local maximum is where a function changes from increasing to decreasing and has an output value larger (more positive or less negative) than output values at neighboring input values.
- A local minimum is where the function changes from decreasing to increasing (as the input increases) and has an output value smaller (more negative or less positive) than output values at neighboring input values.
- Minima and maxima are also called extrema.
- We can find local extrema from a graph. See Example and Example.
- The highest and lowest points on a graph indicate the maxima and minima. See Example.

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Transformation of Functions

Learning Objectives

- Graph functions using vertical and horizontal shifts.
- Graph functions using reflections about the x-axis and the y-axis.
- Determine whether a function is even, odd, or neither from its graph.
- Graph functions using compressions and stretches.
- Combine transformations.

We all know that a flat mirror enables us to see an accurate image of ourselves and whatever is behind us. When we tilt the mirror, the images we see may shift horizontally or vertically. But what happens when we bend a flexible mirror? Like a carnival funhouse mirror, it presents us with a distorted image of ourselves, stretched or compressed horizontally or vertically. In a similar way, we can distort or transform mathematical functions to better adapt them to describing objects or processes in the real world. In this section, we will take a look at several kinds of transformations.

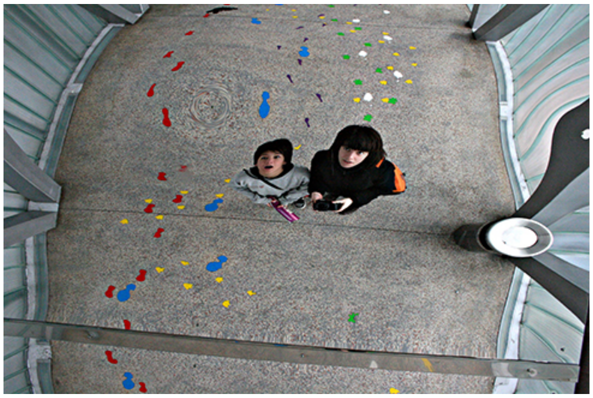


Figure 1: (credit: "Misko"/Flickr)

Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs, and equations. One method we can employ is to adapt the basic graphs of the toolkit functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

Identifying Vertical Shifts

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest shift is a vertical shift, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x) = f(x)+k$, the function $f(x)$ is shifted vertically k units. See Figure 2 for an example.

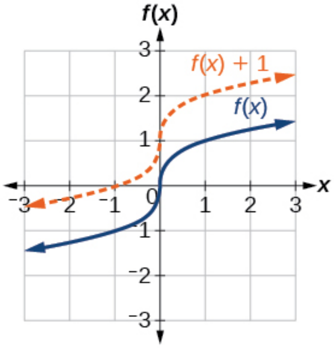


Figure 2: Vertical shift by $k=1$ of the cube root function $f(x)=\sqrt[3]{x}$.

To help you visualize the concept of a vertical shift, consider that $y=f(x)$. Therefore, $f(x)+k$ is equivalent to $y+k$. Every unit of y is replaced by $y+k$, so the y-value increases or decreases depending on the value of k. The result is a shift upward or downward.

Definition: Vertical Shift

Given a function $f(x)$, a new function $g(x) = f(x) + k$, where k is a constant, is a **vertical shift** of the function $f(x)$. All the output values change by k units. If k is positive, the graph will shift up. If k is negative, the graph will shift down.

Example 1: Adding a Constant to a Function

To regulate temperature in a green building, airflow vents near the roof open and close throughout the day. Figure 3 shows the area of open vents V (in square feet) throughout the day in hours after midnight, t. During the summer, the facilities manager decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day and night. Sketch a graph of this new function.

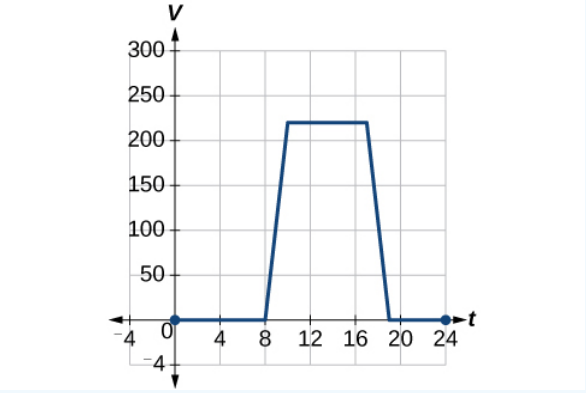


Figure 3

Solution

We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph vertically up, as shown in Figure 4.

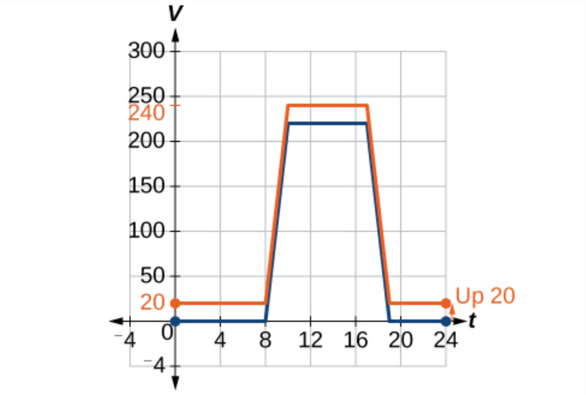


Figure 4

Notice that in Figure 4, for each input value, the output value has increased by 20, so if we call the new function $S(t)$,we could write

$$S(t) = V(t) + 20 \tag{1}$$

This notation tells us that, for any value of t,$S(t)$ can be found by evaluating the function V at the same input and then adding 20 to the result. This defines S as a transformation of the function V, in this case a vertical shift up 20 units. Notice that, with a vertical shift, the input values stay the same and only the output values change. See Table 1.

Table 1

t	0	8	10	17	19	24
$V(t)$	0	0	220	220	0	0
$S(t)$	20	20	240	240	20	20

How To...

Given a tabular function, create a new row to represent a vertical shift.

1. Identify the output row or column.
2. Determine the **magnitude** of the shift.
3. Add the shift to the value in each output cell. Add a positive value for up or a negative value for down.

Example 2: Shifting a Tabular Function Vertically

A function $f(x)$ is given in Table 2. Create a table for the function $g(x)=f(x)-3$.

Table 2

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x)=f(x)-3$ tells us that we can find the output values of g by subtracting 3 from the output values of f. For example:

$$\begin{aligned} f(x)&=1 && \text{Given} \\ g(x)&=f(x)-3 && \text{Given Transformation} \\ g(2)&=f(2)-3 \\ &=1-3 \\ &=-2 \end{aligned}$$

Subtracting 3 from each $f(x)$ value, we can complete a table of values for $g(x)$ as shown in Table 3.

Table 3

x	2	4	6	8
$f(x)$	1	3	7	11
$g(x)$	-2	0	4	8

Analysis

As with the earlier vertical shift, notice the input values stay the same and only the output values change.

Exercise 1

The function $h(t)=-4.9t^2+30t$ gives the height h of a ball (in meters) thrown upward from the ground after t seconds. Suppose the ball was instead thrown from the top of a 10-m building. Relate this new height function $b(t)$ to $h(t)$, and then find a formula for $b(t)$.

Answer

$b(t)=h(t)+10=-4.9t^2+30t+10$

Identifying Horizontal Shifts

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function left or right in what is known as a **horizontal shift**, shown in Figure 4.

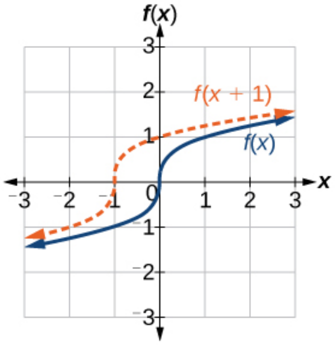


Figure 4: Horizontal shift of the function $f(x)=\sqrt[3]{x}$. Note that $h=-1$ shifts the graph to the left, that is, towards negative values of x.

For example, if $f(x)=x^2$, then $g(x)=(x-2)^2$ is a new function. Each input is reduced by 2 prior to squaring the function. The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in f.

Definition: Horizontal Shift

Given a function f, a new function $g(x)=f(x-h)$, where h is a constant, is a **horizontal shift** of the function f. If h is positive, the graph will shift right. If h is negative, the graph will shift left.

Example 4: Adding a Constant to an Input

Returning to our building airflow example from Figure 2, suppose that in autumn the facilities manager decides that the original venting plan starts too late, and wants to begin the entire venting program 2 hours earlier. Sketch a graph of the new function.

Solution

We can set $V(t)$ to be the original program and $F(t)$ to be the revised program.

$$V(t)=\text{ the original venting plan}$$

$$F(t)=\text{ starting 2 hrs sooner}$$

In the new graph, at each time, the airflow is the same as the original function V was 2 hours later. For example, in the original function V, the airflow starts to change at 8 a.m., whereas for the function F, the airflow starts to change at 6 a.m. The comparable function values are $V(8)=F(6)$. See Figure 5. Notice also that the vents first opened to 220ft^2 at 10 a.m. under the original plan, while under the new plan the vents reach 220ft^2 at 8 a.m., so $V(10)=F(8)$.

In both cases, we see that, because $F(t)$ starts 2 hours sooner, $h=-2$. That means that the same output values are reached when $F(t)=V(t-(-2))=V(t+2)$.

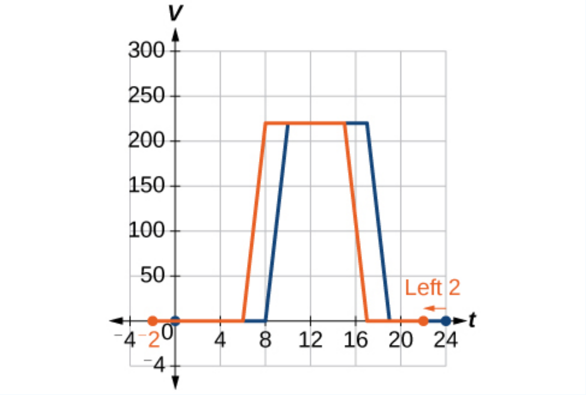


Figure 5

Analysis

Note that $V(t+2)$ has the effect of shifting the graph to the left.

Horizontal changes or "inside changes" affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function $F(t)$ uses the same outputs as $V(t)$, but matches those outputs to inputs 2 hours earlier than those of $V(t)$. Said another way, we must add 2 hours to the input of V to find the corresponding output for $F: F(t) = V(t+2)$.

How To...

Given a tabular function, create a new row to represent a horizontal shift.

1. Identify the input row or column.
2. Determine the magnitude of the shift.
3. Add the shift to the value in each input cell.

Example 5: Shifting a Tabular Function Horizontally

A function $f(x)$ is given in Table 4. Create a table for the function $g(x) = f(x-3)$.

Table 4

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = f(x-3)$ tells us that the output values of g are the same as the output value of f when the input value is 3 less than the original value. For example, we know that $f(2) = 1$. To get the same output from the function g, we will need an input value that is 3 larger. We input a value that is 3 larger for $g(x)$ because the function takes 3 away before evaluating the function f.

$$\begin{aligned} g(5) &= f(5-3) \\ &= f(2) \\ &= 1 \end{aligned}$$

We continue with the other values to create Table 5.

Table 5

x	5	7	9	11
$x-3$	2	4	6	8
$f(x)$	1	3	7	11
$g(x)$	1	3	7	11

The result is that the function $g(x)$ has been shifted to the right by 3. Notice the output values for $g(x)$ remain the same as the output values for $f(x)$, but the corresponding input values, x, have shifted to the right by 3. Specifically, 2 shifted to 5, 4 shifted to 7, 6 shifted to 9, and 8 shifted to 11.

Analysis

Figure 6 represents both of the functions. We can see the horizontal shift in each point.

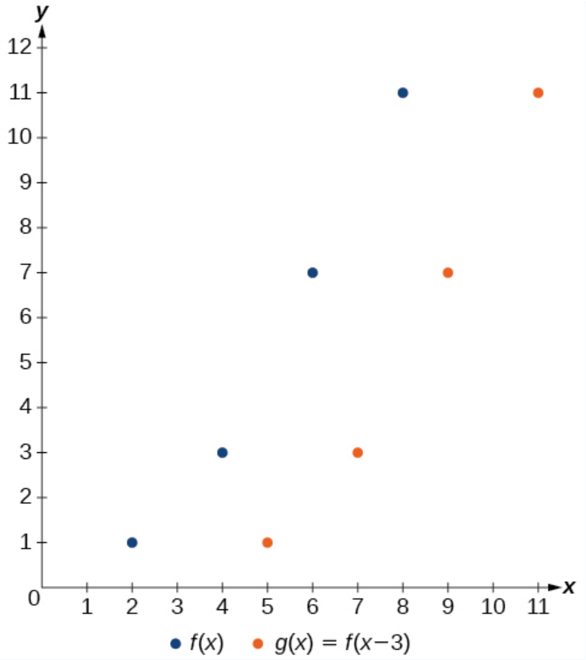


Figure 6: Graph of the points from the previous table for $f(x)$ and $g(x) = f(x-3)$

Example 6: Identifying a Horizontal Shift of a Toolkit Function

Figure 7 represents a transformation of the toolkit function $f(x)=x^2$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

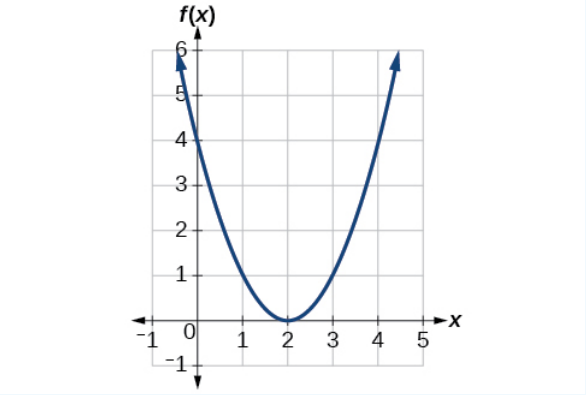


Figure 7: Graph of a parabola

Solution

Notice that the graph is identical in shape to the $f(x)=x^2$ function, but the x-values are shifted to the right 2 units. The vertex used to be at $(0,0)$, but now the vertex is at $(2,0)$. The graph is the basic quadratic function shifted 2 units to the right, so

$$g(x)=f(x-2)$$

Notice how we must input the value $x=2$ to get the output value $y=0$; the x-values must be 2 units larger because of the shift to the right by 2 units. We can then use the definition of the $f(x)$ function to write a formula for $g(x)$ by evaluating $f(x-2)$.

$$\begin{aligned}
&f(x)=x^2\\
&g(x)=f(x-2)\\
&g(x)=f(x-2)=(x-2)^2
\end{aligned}$$

Analysis

To determine whether the shift is $+2$ or -2, consider a single reference point on the graph. For a quadratic, looking at the vertex point is convenient. In the original function, $f(0)=0$. In our shifted function, $g(2)=0$. To obtain the output value of 0 from the function f, we need to decide whether a plus or a minus sign will work to satisfy $g(2)=f(x-2)=f(0)=0$. For this to work, we will need to subtract 2 units from our input values.

Example 7: Interpreting Horizontal versus Vertical Shifts

The function $G(m)$ gives the number of gallons of gas required to drive m miles. Interpret $G(m)+10$ and $G(m+10)$

Solution

$G(m)+10$ can be interpreted as adding 10 to the output, gallons. This is the gas required to drive m miles, plus another 10 gallons of gas. The graph would indicate a vertical shift.

$G(m+10)$ can be interpreted as adding 10 to the input, miles. So this is the number of gallons of gas required to drive 10 miles more than m miles. The graph would indicate a horizontal shift.

Exercise 7

Given the function $f(x)=\sqrt{x}$, graph the original function $f(x)$ and the transformation $g(x)=f(x+2)$ on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?

Answer

The graphs of $f(x)$ and $g(x)$ are shown below. The transformation is a horizontal shift. The function is shifted to the left by 2 units.

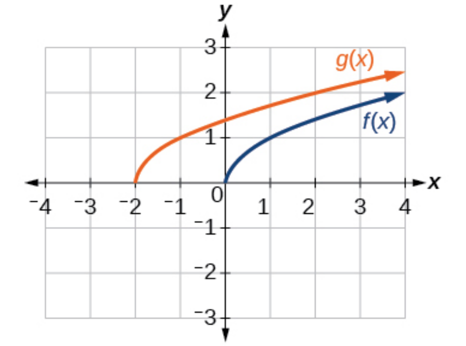


Figure 8

Combining Vertical and Horizontal Shifts

Now that we have two transformations, we can combine them together. Vertical shifts are outside changes that affect the output $(y-)$ axis values and shift the function up or down. Horizontal shifts are inside changes that affect the input $(x-)$ axis values and shift the function left or right. Combining the two types of shifts will cause the graph of a function to shift up or down and right or left.

How To...

Given a function and both a vertical and a horizontal shift, sketch the graph.

1. Identify the vertical and horizontal shifts from the formula.
2. The vertical shift results from a constant added to the output. Move the graph up for a positive constant and down for a negative constant.
3. The horizontal shift results from a constant added to the input. Move the graph left for a positive constant and right for a negative constant.
4. Apply the shifts to the graph in either order.

Example 8: Graphing Combined Vertical and Horizontal Shifts

Given $f(x)=|x|$, sketch a graph of $h(x)=f(x+1)-3$.

Solution

The function f is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of h has transformed f in two ways: $f(x+1)$ is a change on the inside of the function, giving a horizontal shift left by 1, and the subtraction by 3 in $f(x+1)-3$ is a change to the outside of the function, giving a vertical shift down by 3. The transformation of the graph is illustrated in Figure 9.

Let us follow one point of the graph of $f(x)=|x|$.

- The point $(0,0)$ is transformed first by shifting left 1 unit:$(0,0)\rightarrow(-1,0)$
- The point $(-1,0)$ is transformed next by shifting down 3 units:$(-1,0)\rightarrow(-1,-3)$

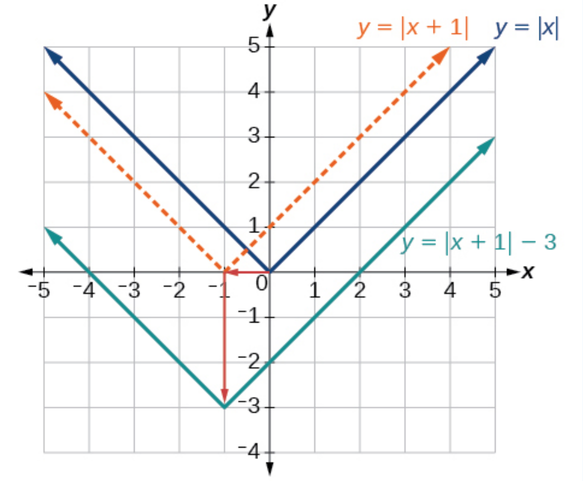


Figure 9: Graph of an absolute function, $y=|x|$, and how it was transformed to $y=|x+1|-3$

Figure 10 shows the graph of h.

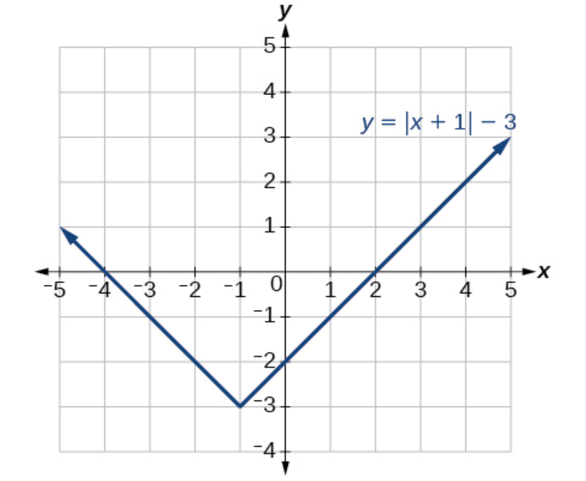


Figure 10: The final function $y=|x+1|-3$.

Exercise 8

Given $f(x)=|x|$, sketch a graph of $h(x)=f(x-2)+4$.

Answer

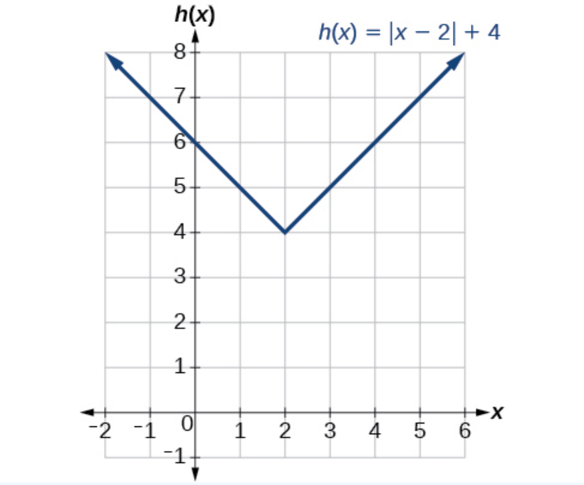


Figure 11

Example 9: Identifying Combined Vertical and Horizontal Shifts

Write a formula for the graph shown in Figure 12, which is a transformation of the toolkit square root function.

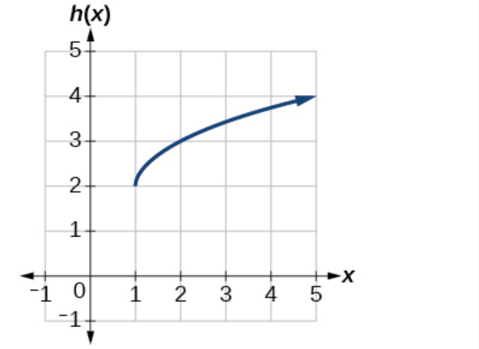


Figure 12: Graph of a square root function transposed right one unit and up 2.

Solution

The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right and up 2. In function notation, we could write that as

$$h(x) = f(x-1) + 2$$

Using the formula for the square root function, we can write

$$h(x) = \sqrt{x-1} + 2$$

Analysis

Note that this transformation has changed the domain and range of the function. This new graph has domain $[1, \infty)$ and range $[2, \infty)$.

Exercise 9

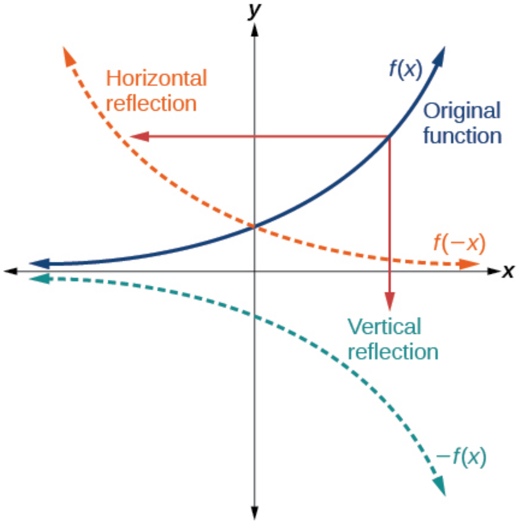
Write a formula for a transformation of the toolkit reciprocal function $f(x) = \frac{1}{x}$ that shifts the function's graph one unit to the right and one unit up.

Answer

$$g(x) = \frac{1}{x-1} + 1$$

Graphing Functions Using Reflections about the Axes

Another transformation that can be applied to a function is a reflection over the x- or y-axis. A **vertical reflection** reflects a graph vertically across the x-axis, while a **horizontal reflection** reflects a graph horizontally across the y-axis. The reflections are shown in Figure 13.



. Figure 13: Graph of the vertical and horizontal reflection of a function.

Notice that the vertical reflection produces a new graph that is a mirror image of the base or original graph about the x-axis. The horizontal reflection produces a new graph that is a mirror image of the base or original graph about the y-axis.

Definitions: Reflections

Given a function $f(x)$, a new function $g(x)=-f(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a reflection about (or over, or through) the x-axis.

Given a function $f(x)$, a new function $g(x)=f(-x)$ is a **horizontal reflection** of the function $f(x)$, sometimes called a reflection about the y-axis.

How To...

Given a function, reflect the graph both vertically and horizontally.

1. Multiply all outputs by –1 for a vertical reflection. The new graph is a reflection of the original graph about the x-axis.
2. Multiply all inputs by –1 for a horizontal reflection. The new graph is a reflection of the original graph about the y-axis.

Example 10: Reflecting a Graph Horizontally and Vertically

Reflect the graph of $s(t)=\sqrt{t}$ (a) vertically and (b) horizontally.

Solution

a. Reflecting the graph vertically means that each output value will be reflected over the horizontal t-axis as shown in Figure 14.

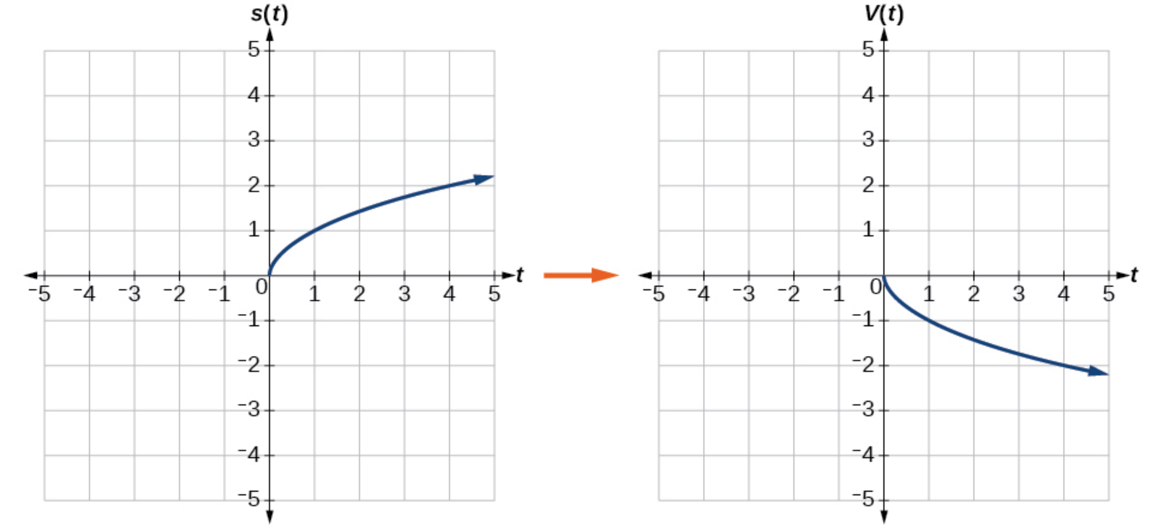


Figure 14: Graph of the vertical reflection of the square root function.

Because each output value is the opposite of the original output value, we can write

$$V(t) = -s(t) \text{ or } V(t) = -\sqrt{t}$$

Notice that this is an outside change, or vertical shift, that affects the output $s(t)$ values, so the negative sign belongs outside of the function.

b. Reflecting horizontally means that each input value will be reflected over the vertical axis as shown in Figure 15.

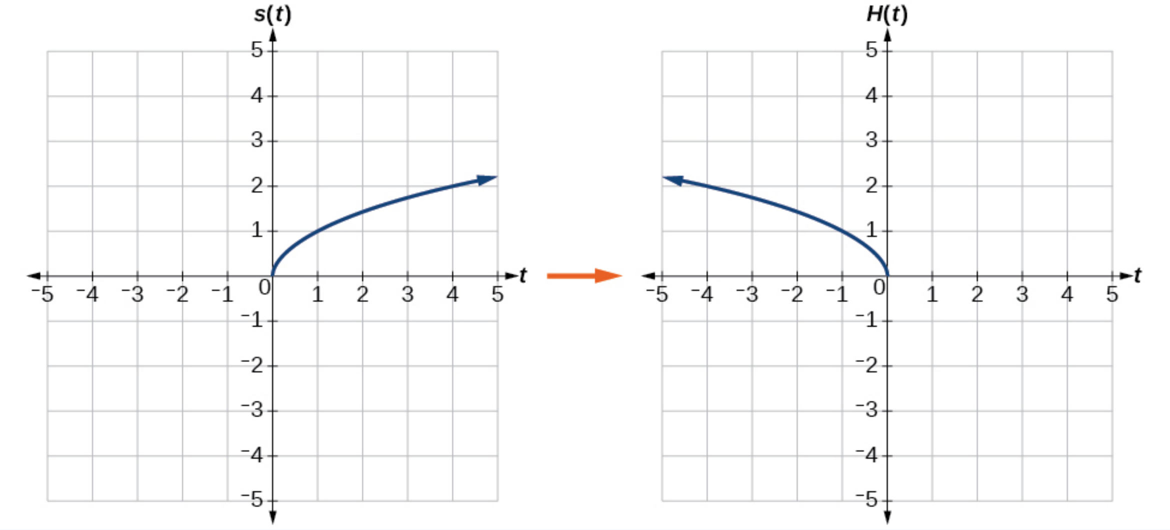


Figure 15: Horizontal reflection of the square root function

Because each input value is the opposite of the original input value, we can write

$$H(t) = s(-t) \text{ or } H(t) = \sqrt{-t}$$

Notice that this is an inside change or horizontal change that affects the input values, so the negative sign is on the inside of the function.

Note that these transformations can affect the domain and range of the functions. While the original square root function has domain $[0, \infty)$ and range $[0, \infty)$, the vertical reflection gives the $V(t)$ function the range $(-\infty, 0]$ and the horizontal reflection gives the $H(t)$ function the domain $(-\infty, 0]$.

Exercise 5

Reflect the graph of $f(x)=|x-1|$ (a) vertically and (b) horizontally.

Answer

a.

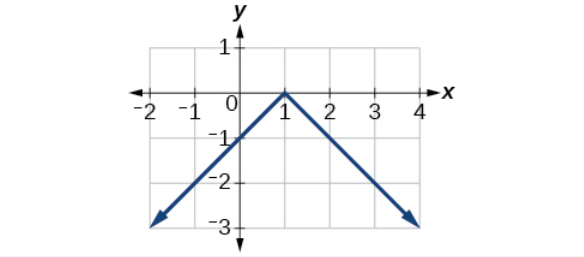


Figure 16: Graph of a vertically reflected absolute function.

b.

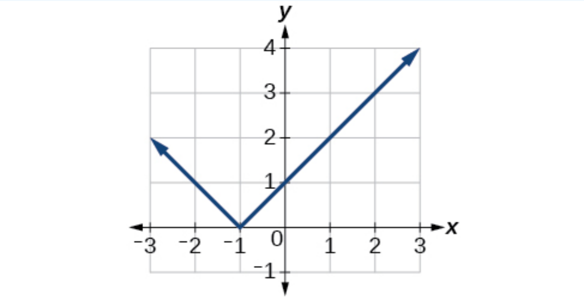


Figure 17: Graph of an absolute function translated one unit left.

Example 11: Reflecting a Tabular Function Horizontally and Vertically

A function $f(x)$ is given as Table 6. Create a table for the functions below.

a. $g(x)=-f(x)$
b. $h(x)=f(-x)$

Table 6

x	2	4	6	8
$f(x)$	1	3	7	11

a. For $g(x)$, the negative sign outside the function indicates a vertical reflection, so the x-values stay the same and each output value will be the opposite of the original output value. See Table 7.

Table 7

x	2	4	6	8
$g(x)$	-1	-3	-7	-11

b. For $h(x)$, the negative sign inside the function indicates a horizontal reflection, so each input value will be the opposite of the original input value and the $h(x)$ values stay the same as the $f(x)$ values. See Table 8.

Table 8

x	-2	-4	-6	-8
$h(x)$	1	3	7	11

Exercise 6

A function $f(x)$ is given as Table 9. Create a table for the functions below.

a. $g(x)=-f(x)$
b. $h(x)=f(-x)$

Table 9

x	-2	0	2	4
$f(x)$	5	10	15	20

Answer

a. $g(x)=-f(x)$

Table 10

x	-2	0	2	4
$g(x)$	-5	-10	-15	-20

b. $h(x)=f(-x)$

Table 11

x	-2	0	2	-4
$h(x)$	15	10	5	20

Example 12: Applying a Learning Model Equation

A common model for learning has an equation similar to $k(t)=-2^{-t}+1$, where k is the percentage of mastery that can be achieved after t practice sessions. This is a transformation of the function $f(t)=2^t$ shown inFigure 18. Sketch a graph of $k(t)$.

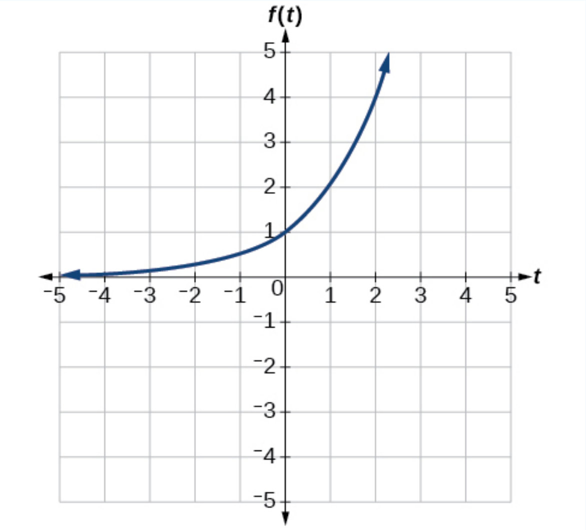


Figure 18: Graph of $k(t)$.

Solution

This equation combines three transformations into one equation.

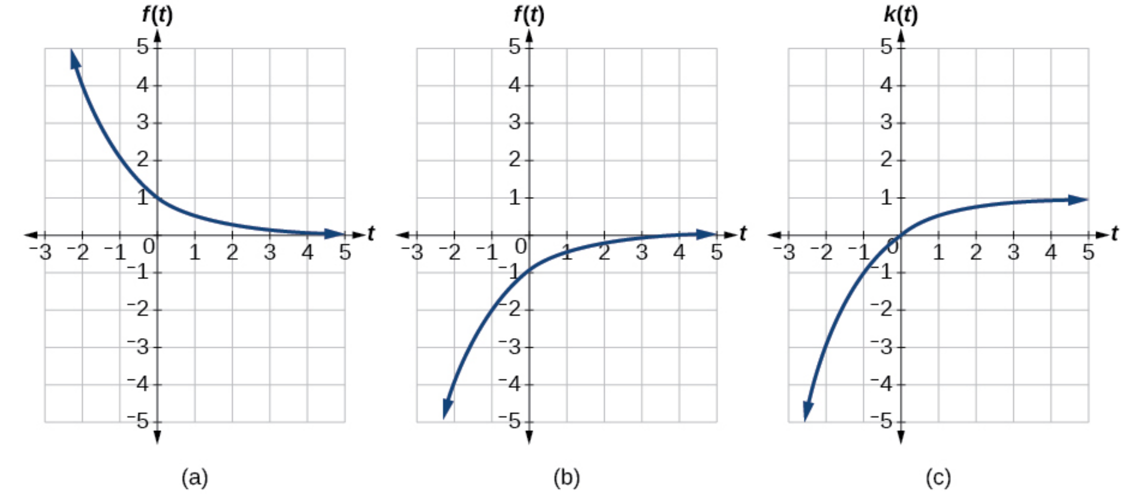
- A horizontal reflection: $f(-t)=2^{-t}$
- A vertical reflection: $-f(-t)=-2^{-t}$
- A vertical shift: $-f(-t)+1=-2^{-t}+1$

We can sketch a graph by applying these transformations one at a time to the original function. Let us follow two points through each of the three transformations. We will choose the points $(0,1)$ and $(1,2)$.

- First, we apply a horizontal reflection: $(0,1)$ $(-1,2)$.
- Then, we apply a vertical reflection: $(0,-1)$ $(-1,-2)$.
- Finally, we apply a vertical shift: $(0,0)$ $(-1,-1)$.

This means that the original points, $(0,1)$ and $(1,2)$ become $(0,0)$ and $(-1,-1)$ after we apply the transformations.

In Figure 19, the first graph results from a horizontal reflection. The second results from a vertical reflection. The third results from a vertical shift up 1 unit.



(a) (b) (c)

Figure 19: Graphs of all the transformations.

Analysis

As a model for learning, this function would be limited to a domain of $t \geq 0$, with corresponding range $[0,1)$.

Exercise 7

Given the toolkit function $f(x)=x^2$, graph $g(x)=-f(x)$ and $h(x)=f(-x)$. Take note of any surprising behavior for these functions.

Answer

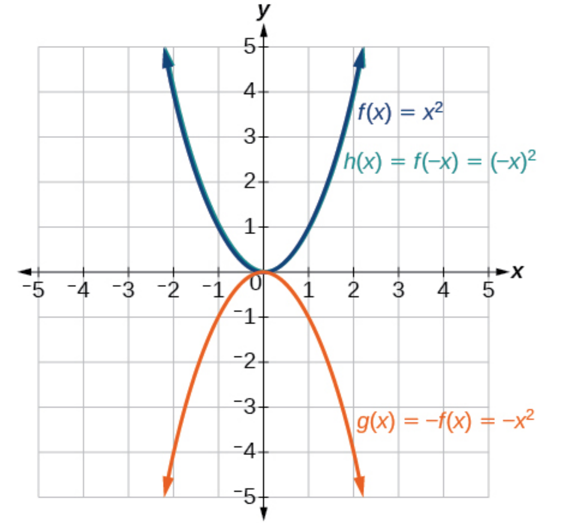


Figure 20: Graph of x^2 and its reflections.

Notice: $g(x)=f(-x)$ looks the same as $f(x)$.

Determining Even and Odd Functions

Some functions exhibit symmetry so that reflections result in the original graph. For example, horizontally reflecting the toolkit functions $f(x)=x^2$ or $f(x)=|x|$ will result in the original graph. We say that these types of graphs are symmetric about the y-axis. Functions whose graphs are symmetric about the y-axis are called **even functions**.

If the graphs of $f(x)=x^3$ or $f(x)=\frac{1}{x}$ were reflected over both axes, the result would be the original graph, as shown in Figure 21.

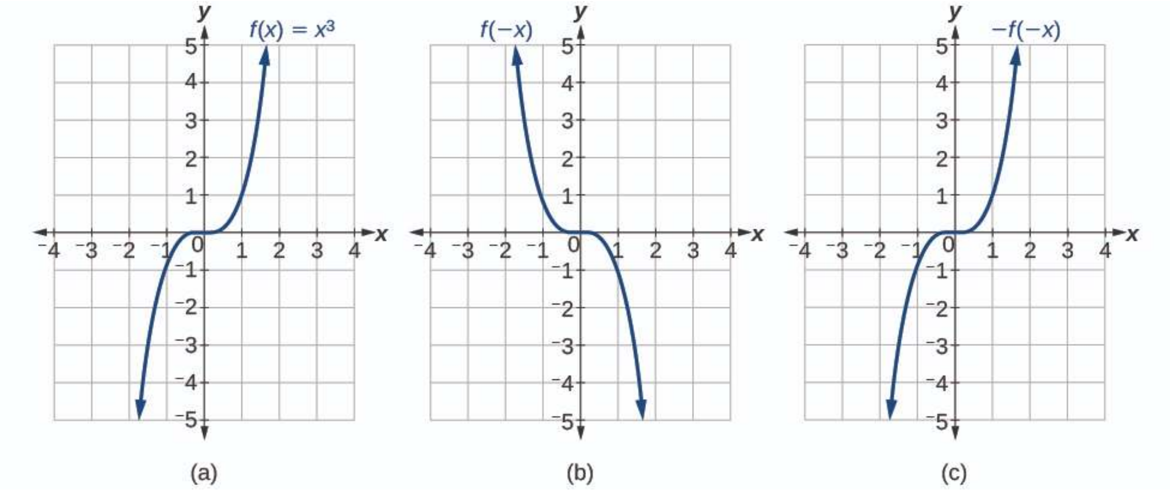


Figure 21: (a) The cubic toolkit function (b) Horizontal reflection of the cubic toolkit function (c) Horizontal and vertical reflections reproduce the original cubic function.

We say that these graphs are symmetric about the origin. A function with a graph that is symmetric about the origin is called an **odd function**.

Note: A function can be neither even nor odd if it does not exhibit either symmetry. For example, $f(x)=2^x$ is neither even nor odd. Also, the only function that is both even and odd is the constant function $f(x)=0$.

Definitions: Even and Odd Functions

A function is called an **even function** if for every input x

$f(x)=f(-x)$

The graph of an even function is symmetric about the y-axis.

A function is called an **odd function** if for every input x

$f(x)=-f(-x)$

The graph of an odd function is symmetric about the origin.

How To...

Given the formula for a function, determine if the function is even, odd, or neither.

1. Determine whether the function satisfies $f(x)=f(-x)$. If it does, it is even.
2. Determine whether the function satisfies $f(x)=-f(-x)$. If it does, it is odd.
3. If the function does not satisfy either rule, it is neither even nor odd.

Example 13: Determining whether a Function Is Even, Odd, or Neither

Is the function $f(x)=x^3+2x$ even, odd, or neither?

Solution

Without looking at a graph, we can determine whether the function is even or odd by finding formulas for the reflections and determining if they return us to the original function. Let's begin with the rule for even functions.

$$f(-x)=(-x)^3+2(-x)=-x^3-2x$$

This does not return us to the original function, so this function is not even. We can now test the rule for odd functions.

$$-f(-x) = -(-x^3 - 2x) = x^3 + 2x$$

Because $-f(-x) = f(x)$, this is an odd function.

Analysis

Consider the graph of f in Figure 22. Notice that the graph is symmetric about the origin. For every point (x, y) on the graph, the corresponding point $(-x, -y)$ is also on the graph. For example, $(1, 3)$ is on the graph of f, and the corresponding point $(-1, -3)$ is also on the graph.

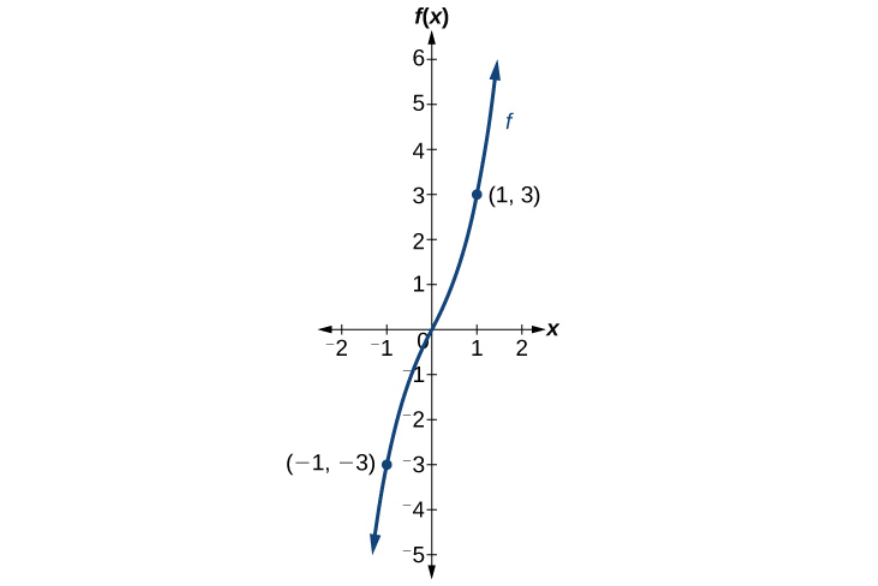


Figure 22: Graph of $f(x)$ with labeled points at $(1, 3)$ and $(-1, -3)$.

Exercise 8

Is the function $f(s) = s^4 + 3s^2 + 7$ even, odd, or neither?

Answer

even

Graphing Functions Using Stretches and Compressions

Adding a constant to the inputs or outputs of a function changed the position of a graph with respect to the axes, but it did not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. Figure 23 shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.

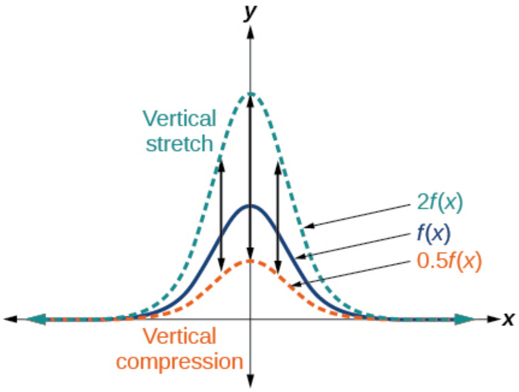


Figure 23: Vertical stretch and compression

Definitions: Vertical Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = af(x)$, where a is a constant, is a **vertical stretch** or **vertical compression** of the function $f(x)$.

- If $a > 1$, then the graph will be stretched.
- If $0 < a < 1$, then the graph will be compressed.
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

How To...

Given a function, graph its vertical stretch.

a. Identify the value of a.
b. Multiply all range values by a
c. If $a > 1$, the graph is stretched by a factor of a.
d. If $0 < a < 1$, the graph is compressed by a factor of a.
e. If $a < 0$, the graph is either stretched or compressed and also reflected about the x-axis.

Example 1.5.14: Graphing a Vertical Stretch

A function $P(t)$ models the population of fruit flies. The graph is shown in Figure 24.

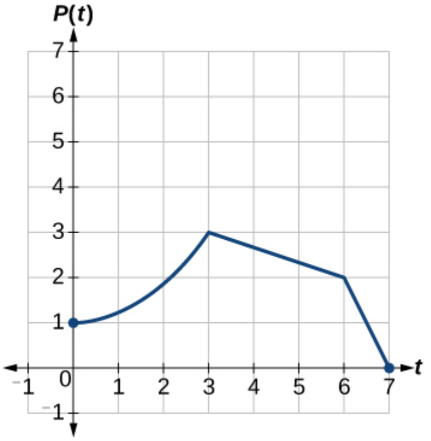


Figure 24: Graph to represent the growth of the population of fruit flies.

A scientist is comparing this population to another population, Q, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

Solution

Because the population is always twice as large, the new population's output values are always twice the original function's output values. Graphically, this is shown in Figure 25.

If we choose four reference points, $(0,1)$, $(3,3)$, $(6,2)$ and $(7,0)$ we will multiply all of the outputs by 2.

The following shows where the new points for the new graph will be located.

$$(0,1) \rightarrow (0,2) \tag{2}$$

$$(3,3) \rightarrow (3,6) \tag{3}$$

$$(6,2) \rightarrow (6,4) \tag{4}$$

$$(7,0) \rightarrow (7,0) \tag{5}$$

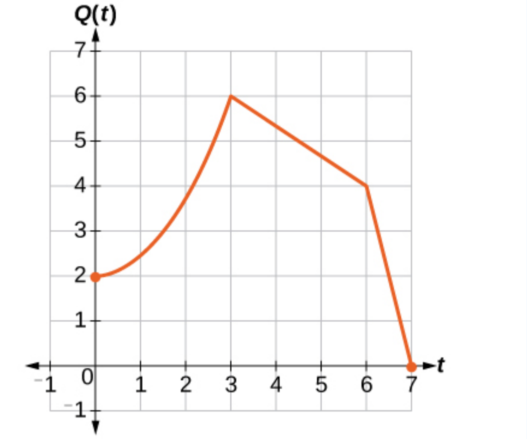


Figure 25: Graph of the population function doubled.

Symbolically, the relationship is written as

$$Q(t) = 2P(t)$$

This means that for any input t, the value of the function Q is twice the value of the function P. Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t, stay the same while the output values are twice as large as before.

How To...

Given a tabular function and assuming that the transformation is a vertical stretch or compression, create a table for a vertical compression.

1. Determine the value of a.
2. Multiply all of the output values by a.

Example 15: Finding a Vertical Compression of a Tabular Function

A function f is given as Table 12. Create a table for the function $g(x) = \frac{1}{2}f(x)$.

Table 12

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = \frac{1}{2}f(x)$ tells us that the output values of g are half of the output values of f with the same inputs. For example, we know that $f(4) = 3$. Then

$$g(4) = \frac{1}{2}f(4) = \frac{1}{2}(3) = \frac{3}{2}$$

We do the same for the other values to produce Table 13.

Table 13

x	2	4	6	8
$g(x)$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{11}{2}$

Analysis

The result is that the function $g(x)$ has been compressed vertically by $\frac{1}{2}$. Each output value is divided in half, so the graph is half the original height.

Exercise 9

A function f is given as Table 14. Create a table for the function $g(x) = \frac{3}{4}f(x)$.

Table 14

x	2	4	6	8
$f(x)$	12	16	20	0

Answer

Table 15

x	2	4	6	8
$g(x)$	9	12	15	0

Example 16: Recognizing a Vertical Stretch

The graph in Figure 26 is a transformation of the toolkit function $f(x) = x^3$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

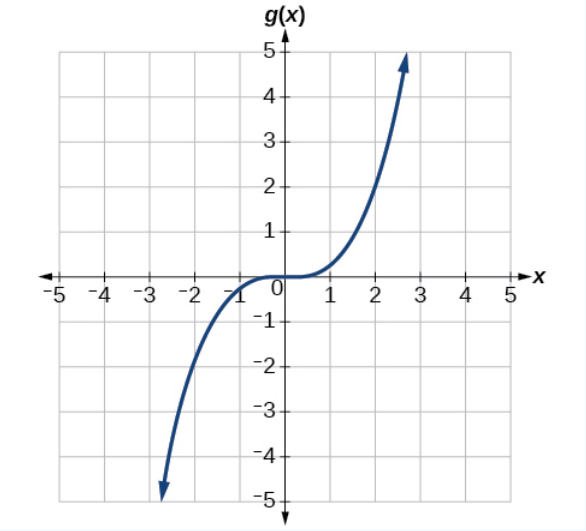


Figure 26: Graph of a transformation of $f(x) = x^3$

When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$. Based on that, it appears that the outputs of g are $\frac{1}{4}$ the outputs of the function f because $g(2) = \frac{1}{4}f(2)$. From this we can fairly safely conclude that $g(x) = \frac{1}{4}f(x)$.

We can write a formula for g by using the definition of the function f.

$$g(x) = \frac{1}{4}f(x) = \frac{1}{4}x^3. \tag{6}$$

Exercise 1

Write the formula for the function that we get when we stretch the identity toolkit function by a factor of 3, and then shift it down by 2 units.

Answer

$g(x) = 3x - 2$

Horizontal Stretches and Compressions

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is between 0 and 1, we get a **horizontal stretch**; if the constant is greater than 1, we get a **horizontal compression** of the function.

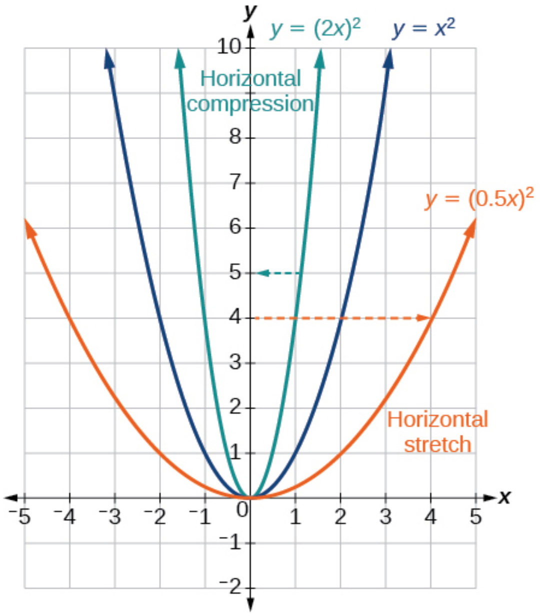


Figure 27: Graph of the vertical stretch and compression of x^2.

Given a function $y = f(x)$, the form $y = f(bx)$ results in a horizontal stretch or compression. Consider the function $y = x^2$. Observe Figure 27. The graph of $y = (0.5x)^2$ is a horizontal stretch of the graph of the function $y = x^2$ by a factor of 2. The graph of $y = (2x)^2$ is a horizontal compression of the graph of the function $y = x^2$ by a factor of 2.

Definitions: Horizontal Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = f(bx)$, where b is a constant, is a **horizontal stretch** or **horizontal compression** of the function $f(x)$.

- If $b > 1$, then the graph will be compressed by $\frac{1}{b}$.
- If $0 < b < 1$, then the graph will be stretched by $\frac{1}{b}$.
- If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

How To...

Given a description of a function, sketch a horizontal compression or stretch.

1. Write a formula to represent the function.
2. Set $g(x) = f(bx)$ where $b > 1$ for a compression or $0 < b < 1$ for a stretch.

Example 17: Graphing a Horizontal Compression

Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, R, will progress in 1 hour the same amount as the original population does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.

Solution

Symbolically, we could write

$$R(1) = P(2), \tag{7}$$
$$R(2) = P(4), \quad \text{and in general,} \tag{8}$$
$$R(t) = P(2t). \tag{9}$$

See Figure 28 for a graphical comparison of the original population and the compressed population.

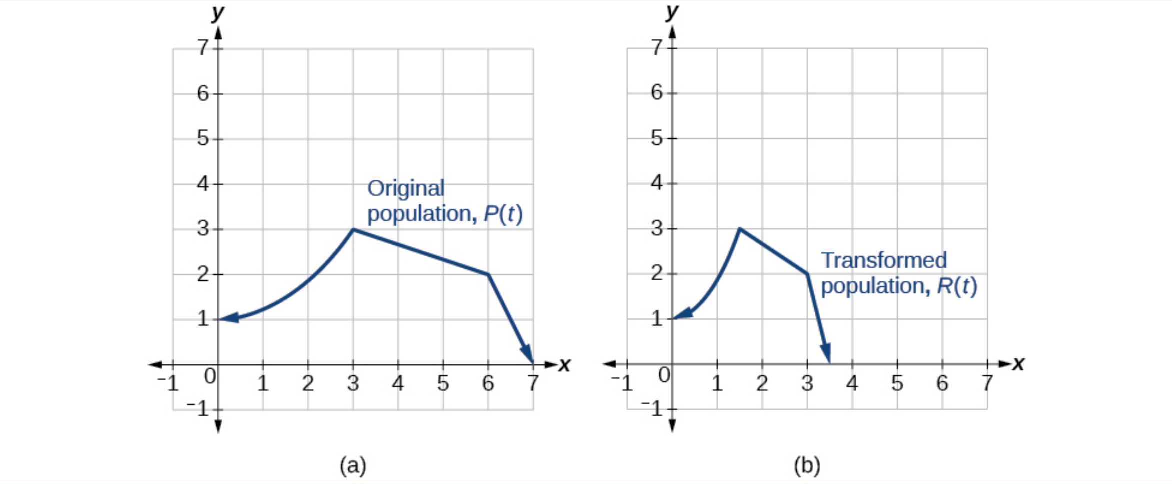


Figure 28: (a) Original population graph (b) Compressed population graph

Analysis

Note that the effect on the graph is a horizontal compression where all input values are half of their original distance from the vertical axis.

Example 18: Finding a Horizontal Stretch for a Tabular Function

A function $f(x)$ is given as Table 16. Create a table for the function $g(x) = f(\frac{1}{2}x)$.

Table 16

x	2	4	6	8
$f(x)$	1	3	7	11

The formula $g(x) = f(\frac{1}{2}x)$ tells us that the output values for g are the same as the output values for the function f at an input half the size. Notice that we do not have enough information to determine $g(2)$ because $g(2) = f(\frac{1}{2} \cdot 2) = f(1)$, and we do not have a value for $f(1)$ in our table. Our input values to g will need to be twice as large to get inputs for f that we can evaluate. For example, we can determine $g(4)$.

$$g(4) = f(\frac{1}{2} \cdot 4) = f(2) = 1 \tag{10}$$

We do the same for the other values to produce Table 17.

Table 17

x	4	8	12	16
$g(x)$	1	3	7	11

Figure 29 shows the graphs of both of these sets of points.

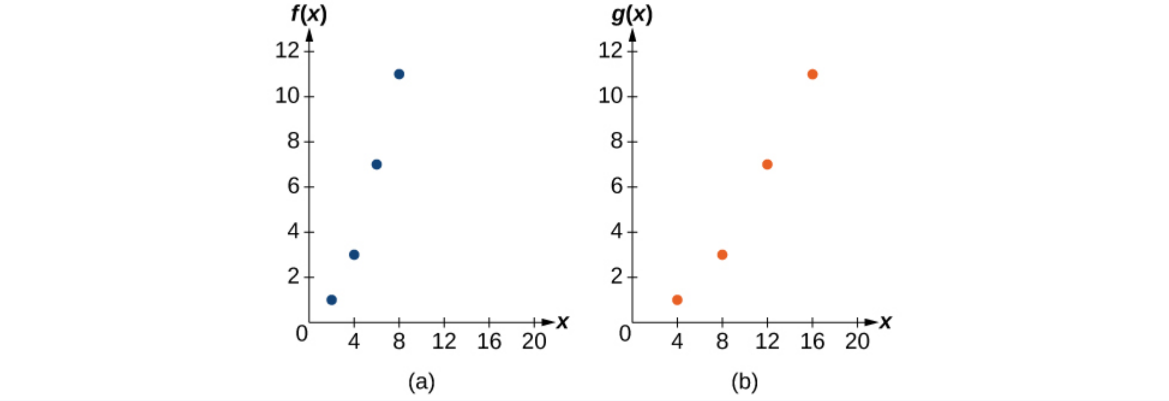


Figure 29: Graph of the previous table.

Analysis

Because each input value has been doubled, the result is that the function $g(x)$ has been stretched horizontally by a factor of 2.

Example 19: Recognizing a Horizontal Compression on a Graph

Relate the function $g(x)$ to $f(x)$ in Figure 30.

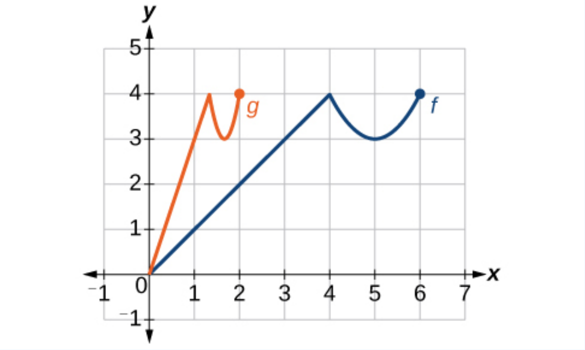


Figure 30: Graph of $f(x)$ being vertically compressed to $g(x)$.

Solution

The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Because $f(x)$ ends at (6,4) and $g(x)$ ends at (2,4), we can see that the x-values have been compressed by $\frac{1}{3}$, because $6(\frac{1}{3}) = 2$. We might also notice that $g(2) = f(6)$ and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by $\frac{1}{3}$.

Analysis

Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of 4, we need a coefficient of $\frac{1}{4}$ in our function: $f(\frac{1}{4}x)$. This means that the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

Exercise 11

Write a formula for the toolkit square root function horizontally stretched by a factor of 3.

Answer

$g(x) = f(\frac{1}{3}x)$, so using the square root function we get $g(x) = \sqrt{\frac{1}{3}x}$

Performing a Sequence of Transformations

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as $2f(x)+3$, which transformation should we start with? The answer here follows nicely from the order of operations. Given the output value of $f(x)$, we first multiply by 2, causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

Horizontal transformations are a little trickier to think about. When we write $g(x)=f(2x+3)$, for example, we have to think about how the inputs to the function g relate to the inputs to the function f. Suppose we know $f(7)=12$. What input to g would produce that output? In other words, what value of x will allow $g(x)=f(2x+3)=12$? We would need $2x+3=7$. To solve for x, we would first subtract 3, resulting in a horizontal shift, and then divide by 2, causing a horizontal compression.

This format ends up being very difficult to work with, because it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.

$$f(bx+p)=f(b(x+\frac{p}{b}))$$

Let’s work through an example.

$$f(x)=(2x+4)^2$$

We can factor out a 2.

$$f(x)=(2(x+2))^2$$

Now we can more clearly observe a horizontal shift to the left 2 units and a horizontal compression. Factoring in this way allows us to horizontally stretch first and then shift horizontally.

Combining Transformations

- When combining vertical transformations written in the form $af(x)+k$, first vertically stretch by a and then vertically shift by k.
- When combining horizontal transformations written in the form $f(bx+h)$, first horizontally shift by h and then horizontally stretch by $\frac{1}{b}$.
- When combining horizontal transformations written in the form $f(b(x+h))$, first horizontally stretch by $\frac{1}{b}$ and then horizontally shift by h.
- Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are performed first.

Example 20: Finding a Triple Transformation of a Tabular Function

Given Table 18 for the function $f(x)$, create a table of values for the function $g(x)=2f(3x)+1$.

Table 18

x	6	12	18	24
$f(x)$	10	14	15	17

Solution

There are three steps to this transformation, and we will work from the inside out. Starting with the horizontal transformations, $f(3x)$ is a horizontal compression by $\frac{1}{3}$, which means we multiply each x-value by $\frac{1}{3}$.See Table 19.

Table 19

x	2	4	6	8
$f(3x)$	10	14	15	17

Looking now to the vertical transformations, we start with the vertical stretch, which will multiply the output values by 2. We apply this to the previous transformation. See Table 20.

Table 20

x	2	4	6	8
$2f(3x)$	20	28	30	34

Finally, we can apply the vertical shift, which will add 1 to all the output values. See Table 21.

Table 21

x	2	4	6	8
$g(x)=2f(3x)+1+1$	21	29	31	35

Example 21: Finding a Triple Transformation of a Graph

Use the graph of $f(x)$ in Figure 31 to sketch a graph of $k(x)=f\left(\frac{1}{2}x+1\right)-3$.

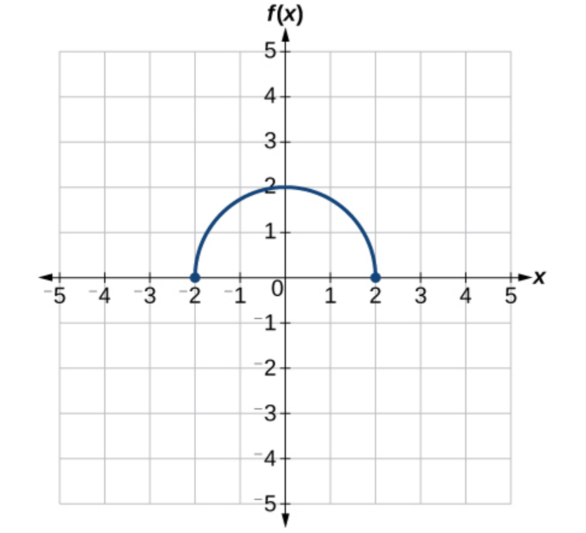


Figure 31: Graph of a half-circle.

To simplify, let’s start by factoring out the inside of the function.

$$f\left(\frac{1}{2}x+1\right)-3=f\left(\frac{1}{2}(x+2)\right)-3 \tag{11}$$

By factoring the inside, we can first horizontally stretch by 2, as indicated by the $\frac{1}{2}$ on the inside of the function. Remember that twice the size of 0 is still 0, so the point $(0,2)$ remains at $(0,2)$ while the point $(2,0)$ will stretch to $(4,0)$. See Figure 32.

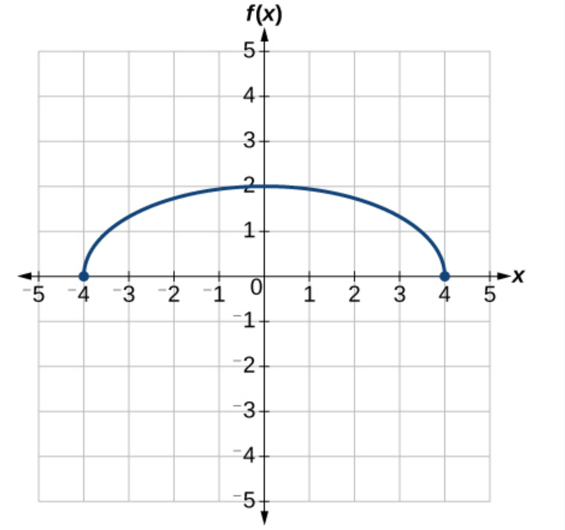


Figure 32: Graph of a vertically stretch half-circle.

Next, we horizontally shift left by 2 units, as indicated by $x + 2$. See Figure 33.

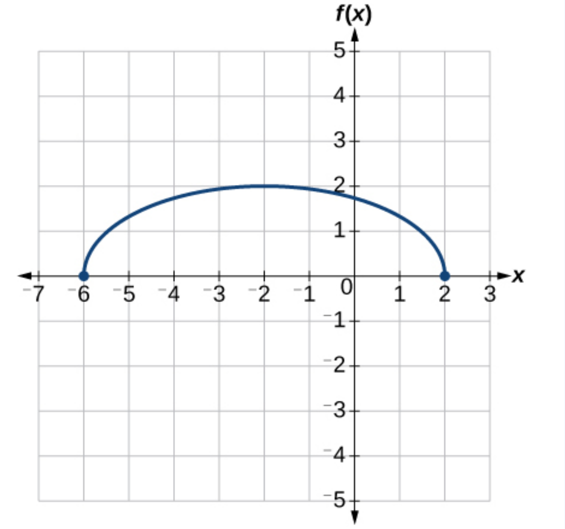


Figure 33: Graph of a vertically stretch and translated half-circle.

Last, we vertically shift down by 3 to complete our sketch, as indicated by the −3 on the outside of the function. See Figure 34.

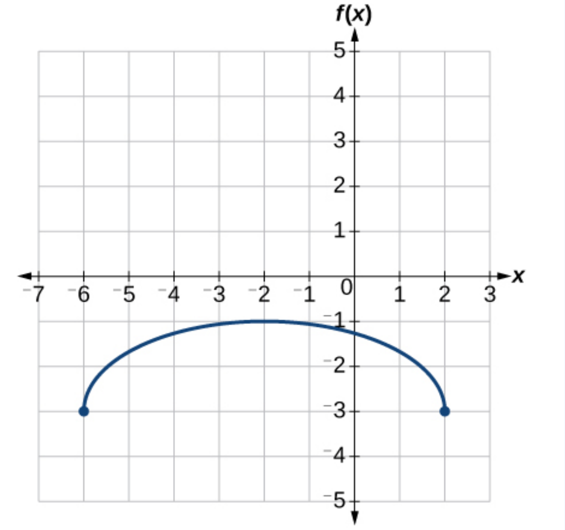


Figure 34: Graph of a vertically stretch and translated half-circle.

Key Equations

- Vertical shift $g(x)=f(x)+k$ (up for $k>0$)
- Horizontal shift $g(x)=f(x-h)$(right) for $h>0$
- Vertical reflection $g(x)=-f(x)$
- Horizontal reflection $g(x)=f(-x)$
- Vertical stretch $g(x)=af(x)$ (a>0)
- Vertical compression $g(x)=af(x)$ (0<a<1)
- Horizontal stretch $g(x)=f(bx)(0<b<1)$
- Horizontal compression $g(x)=f(bx)$ (b>1)

Key Concepts

- A function can be shifted vertically by adding a constant to the output.
- A function can be shifted horizontally by adding a constant to the input.
- Relating the shift to the context of a problem makes it possible to compare and interpret vertical and horizontal shifts.
- Vertical and horizontal shifts are often combined.
- A vertical reflection reflects a graph about the x-axis. A graph can be reflected vertically by multiplying the output by –1.
- A horizontal reflection reflects a graph about the y-axis. A graph can be reflected horizontally by multiplying the input by –1.
- A graph can be reflected both vertically and horizontally. The order in which the reflections are applied does not affect the final graph.
- A function presented in tabular form can also be reflected by multiplying the values in the input and output rows or columns accordingly.
- A function presented as an equation can be reflected by applying transformations one at a time.
- Even functions are symmetric about the y-axis, whereas odd functions are symmetric about the origin.
- Even functions satisfy the condition $f(x)=f(-x)$.
- Odd functions satisfy the condition $f(x)=-f(-x)$.
- A function can be odd, even, or neither.
- A function can be compressed or stretched vertically by multiplying the output by a constant.
- A function can be compressed or stretched horizontally by multiplying the input by a constant.
- The order in which different transformations are applied does affect the final function. Both vertical and horizontal transformations must be applied in the order given. However, a vertical transformation may be combined with a horizontal transformation in any order.

Glossary

even function

a function whose graph is unchanged by horizontal reflection, $f(x)=f(-x)$, and is symmetric about the y-axis

horizontal compression

a transformation that compresses a function’s graph horizontally, by multiplying the input by a constant b>1

horizontal reflection

a transformation that reflects a function’s graph across the y-axis by multiplying the input by −1

horizontal shift

a transformation that shifts a function’s graph left or right by adding a positive or negative constant to the input

horizontal stretch

a transformation that stretches a function’s graph horizontally by multiplying the input by a constant 0<b<1

odd function

a function whose graph is unchanged by combined horizontal and vertical reflection, $f(x)=-f(-x)$, and is symmetric about the origin

vertical compression

a function transformation that compresses the function’s graph vertically by multiplying the output by a constant 0<a<1

vertical reflection
a transformation that reflects a function’s graph across the x-axis by multiplying the output by -1

vertical shift
a transformation that shifts a function’s graph up or down by adding a positive or negative constant to the output

vertical stretch
a transformation that stretches a function’s graph vertically by multiplying the output by a constant $a>1$

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CHAPTER OVERVIEW

4: Polynomial and Rational Functions

4.1: Algebra of Functions
4.2: Factoring Polynomials
4.3: Factoring Trinomials
4.4: Solve Polynomial Equations by Factoring
4.5: Rational Functions - Multiplication and Division
4.6: Rational Functions - Addition and Subtraction
4.7: Solving Rational Equations
4.8: Applications and Variation
4.E: Polynomial and Rational Functions (Exercises)

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4.1: Algebra of Functions

Skills to develop

- Identify and evaluate polynomial functions.
- Add and subtract functions.
- Multiply and divide functions.
- Add functions graphically.

Polynomial Functions

Any polynomial with one variable is a function and can be written in the form

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0.$$

Here a_n represents any real number and n represents any whole number. The **degree of a polynomial** with one variable is the largest exponent of all the terms. Typically, we arrange terms of polynomials in descending order based on their degree and classify them as follows:

$$\begin{aligned} f(x)&=2 && \mathit{Constant\ function\ (degree\ 0)}\\ g(x)&=3x+2 && \mathit{Linear\ function\ (degree\ 1)}\\ h(x)&=4x^2+3x+2 && \mathit{Quadratic\ function\ (degree\ 2)}\\ r(x)&=5x^3+4x^2+3x+2 && \mathit{Cubic\ function\ (degree\ 3)} \end{aligned}$$

In this textbook, we call any polynomial with degree higher than 3 an nth-degree polynomial. For example, if the degree is 4, we call it a fourth-degree polynomial; if the degree is 5, we call it a fifth-degree polynomial, and so on.

Example $4.1.1$:

Given $f(x)=x^2-8x+17$, find $f(2)$ and $f(4)$.

Solution

Replace each instance of x with the value given inside the parentheses.

Table $4.1.1$

$f(2)=(2)^2-8(2)+17$ $=4-16+17$ $=4+1$ $=5$	$f(4)=(4)^2-8(4)+17$ $=16-32+17$ $=-16+17$ $=1$

We can write $f(2)=5$ and $f(4)=1$. Remember that $f(x)=y$ and so we can interpret these results on the graph as follows:

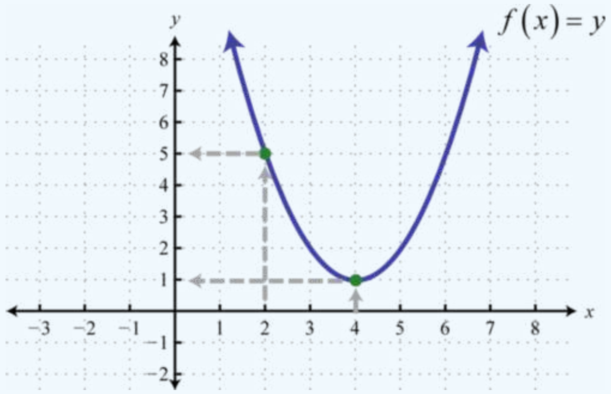

Table $4.1.1$

Answer:

$f(2)=5; f(4)=1$

Often we will be asked to evaluate polynomials for algebraic expressions.

Example 4.1.2:

Given $g(x) = x^3 - x + 5$, find $g(-2u)$ and $g(x-2)$.

Solution

Replace x with the expressions given inside the parentheses.

Table $4.1.2$

$\begin{aligned} g(-2u) &= (-2u)^3 - (-2u) + 5 \\ &= -8u^3 + 2u + 5 \end{aligned}$	$\begin{aligned} g(x-2) &= (x-2)^3 - (x-2) + 5 \\ &= (x-2)(x-2)(x-2) - (x-2) + 5 \\ &= (x-2)(x^2 - 4x + 4) - x + 7 \\ &= x^3 - 4x^2 + 4x - 2x^2 + 8x - 8 - x + 7 \\ &= x^3 - 6x^2 + 11x - 1 \end{aligned}$

Answer:

$g(-2u) = -8u^2 + 2u + 5$ and $g(x-2) = x^3 - 6x^2 + 11x - 1$

The height of an object launched upward, ignoring the effects of air resistance, can be modeled with the following quadratic function:

$$h(t) = -\tfrac{1}{2}gt^2 + v_0 t + s_0$$

With this formula, the height $h(t)$ can be calculated at any given time t after the object is launched. The letter g represents acceleration due to gravity on the surface of the Earth, which is 32 feet per second squared (or, using metric units, $g = 9.8$ meters per second squared). The variable v_0, pronounced "*v-naught*," or sometimes "*v-zero*," represents the initial velocity of the object, and s_0 represents the initial height from which the object was launched.

Example 4.1.3

An object is launched from the ground at a speed of 64 feet per second. Write a function that models the height of the object and use it to calculate the objects height at 1 second and at 3.5 seconds.

Solution

We know that the acceleration due to gravity is $g = 32$ feet per second squared and we are given the initial velocity $v_0 = 64$ feet per second. Since the object is launched from the ground, the initial height is $s_0 = 0$ feet. Create the mathematical model by substituting these coefficients into the following formula:

$$\begin{aligned} h(t) &= -\tfrac{1}{2}gt^2 + v_0 t + s_0 \\ h(t) &= -\tfrac{1}{2}(32)t^2 + (64)t + 0 \\ h(t) &= -16t^2 + 64t \end{aligned}$$

Use this model to calculate the height of the object at 1 second and 3.5 seconds.

$$\begin{aligned} h(1) &= -16(1)^2 + 64(1) = -16 + 64 = 48 \\ h(3.5) &= -16(3.5)^2 + 64(3.5) = -196 + 224 = 28 \end{aligned}$$

Answer:

$h(t) = -16t^2 + 64t$; At 1 second the object is at a height of 48 feet, and at 3.5 seconds it is at a height of 28 feet.

Exercise 4.1.1

An object is dropped from a height of 6 meters. Write a function that models the height of the object and use it to calculate the object's height 1 second after it is dropped.

Answer

$h(t) = -4.9t^2 + 6$; At 1 second the object is at a height of 1.1 meters.

www.youtube.com/v/yy8KzQSPzxI

Adding and Subtracting Functions

The notation used to indicate **addition**[1] and **subtraction**[2] of functions follows:

Addition of functions: $(f+g)(x)=f(x)+g(x)$

Subtraction of functions: $(f-g)(x)=f(x)-g(x)$

When using function notation, be careful to group the entire function and add or subtract accordingly.

Example 4.1.4

Given $f(x)=x^3-5x-7$ and $g(x)=3x^2+7x-2$, find $(f+g)(x)$ and $(f-g)(x)$.

Solution

The notation $f+g$ indicates that we should add the given expressions.

$$\begin{aligned}(f+g)(x)&=f(x)+g(x)\\&=\left(x^3-5x-7\right)+\left(3x^2+7x-2\right)\\&=x^3-5x-7+3x^2+7x-2\\&=x^3+3x^2+2x-9\end{aligned}$$

The notation $f-g$ indicates that we should subtract the given expressions. When subtracting, the parentheses become very important. Recall that we can eliminate them after applying the distributive property.

$$\begin{aligned}(f-g)(x)&=f(x)-g(x)\\&=\left(x^3-5x-7\right)-\left(3x^2+7x-2\right)\\&=x^3-5x-7-3x^2-7x+2\\&=x^3-3x^2-12x-5\end{aligned}$$

Answer:

$(f+g)(x)=x^3+3x^2+2x-9$ and $(f-g)(x)=x^3-3x^2-12x-5$

We may be asked to evaluate the sum or difference of two functions. We have the option to first find the sum or difference in general and then use the resulting function to evaluate for the given variable, or evaluate each first and then find the sum or difference.

Example 4.1.5

Evaluate $(f-g)(3)$ given $f(x)=5x^2-x+4$ and $g(x)=x^2+2x-3$.

Solution

First, find $(f-g)(x)$.

$$\begin{aligned}(f-g)(x)&=f(x)-g(x)\\&=\left(5x^2-x+4\right)-\left(x^2+2x-3\right)\\&=5x^2-x+4-x^2-2x+3\\&=4x^2-3x+7\end{aligned}$$

Therefore,

$$(f-g)(x)=4x^2-3x+7.$$

Next, substitute 3 in for the variable x.

$$\begin{aligned}(f-g)(3)&=4(3)^2-3(3)+7\\&=36-9+7\\&=34\end{aligned}$$

Hence $(f-g)(3)=34$.

Alternate Solution

Since $(f-g)(3)=f(3)-g(3)$, we can find $f(3)$ and $g(3)$ and then subtract the results.

Table 4.1.3

$f(x)=5x^2-x+4$ $f(3)=5(3)^2-(3)+4$ $=45-3+4$ $=46$	$g(x)=x^2+2x-3$ $g(3)=(3)^2+2(3)-3$ $=9+6-3$ $=12$

Therefore,

$$\begin{aligned}(f-g)(3)&=f(3)-g(3)\\&=46-12\\&=34\end{aligned}$$

Notice that we obtain the same answer.

Answer:

$(f-g)(3)=34$

Note

If multiple values are to be evaluated, it is best to find the sum or difference in general first and then use it to evaluate.

Exercise $4.1.2$

Evaluate $(f+g)(-1)$ given $f(x)=x^3+x-8$ and $g(x)=2x^2-x+9$.

Answer

2

www.youtube.com/v/aYkzZ9U6xPw

Multiplying and Dividing Functions

The notation used to indicate **multiplication**[3] and **division**[4] of functions follows:

Table $4.1.4$

Multiplication of functions:	$(f\cdot g)(x)=f(x)\cdot g(x)$
Division of functions:	$(f/g)(x)=\frac{f(x)}{g(x)},\ \text{where } g(x)\neq 0$

Example $4.1.6$:

Given $f(x)=15x^4-9x^3+6x^2$ and $g(x)=3x^2$, find $(f\cdot g)(x)$ and $(f/g)(x)$.

Solution

The notation $f\cdot g$ indicates that we should multiply. Apply the distributive property and simplify.

$$\begin{aligned}(f\cdot g)(x)&=f(x)\cdot g(x)\\&=\left(15x^4-9x^3+6x^2\right)\left(3x^2\right)\\&=15x^4\cdot 3x^2-9x^3\cdot 3x^2+6x^2\cdot 3x^2\\&=45x^6-27x^5+18x^4\end{aligned}$$

The notation f/g indicates that we should divide. For this quotient, assume $x\neq 0$.

$$\begin{aligned}(f/g)(x)&=\frac{f(x)}{g(x)}\\&=\frac{15x^4-9x^3+6x^2}{3x^2}\\&=\frac{15x^4}{3x^2}-\frac{9x^3}{3x^2}+\frac{6x^2}{3x^2}\\&=5x^2-3x+2\end{aligned}$$

Answer:

$(f \cdot g)(x)=45x^6-27x^5+18x^4$ and $(f/g)(x)=5x^2-3x+2$ where $x \neq 0$.

Example 4.1.7:

Given $f(x)=6x-5$ and $g(x)=3x^2-2x-1$, evaluate $(f \cdot g)(0)$ and $(f \cdot g)(-1)$

Solution

Begin by finding $(f \cdot g)(x)$.

$$\begin{aligned}(f \cdot g)(x) &= f(x) \cdot g(x) \\ &= (6x-5)\left(3x^2-2x-1\right) \\ &= 18x^3-12x^2-6x-15x^2+10x+5 \\ &= 18x^3-27x^2+4x+5\end{aligned}$$

Therefore $(f \cdot g)(x)=18x^3-27x^2+4x+5$, and we have,

Table 4.1.5

$\begin{aligned}(f \cdot g)(0) &= 18(0)^3-27(0)^2+4(0)+5 \\ &= 5\end{aligned}$	$\begin{aligned}(f \cdot g)(-1) &= 18(-1)^3-27(-1)^2+4(-1)+5 \\ &= -18-27-4+5 \\ &= -44\end{aligned}$

Answer:

$(f \cdot g)(0)=5$ and $(f \cdot g)(-1)=-44$

Exercise 4.1.3

Evaluate $(f \cdot g)(-1)$ given $f(x)=x^3+x-8$ and $g(x)=2x^2-x+9$.

Answer

-120

www.youtube.com/v/vyZFGCa9cXg

Adding Functions Graphically

Here we explore the geometry of adding functions. One way to do this is to use the fact that $(f+g)(x)=f(x)+g(x)$. Add the functions together using *x*-values for which both f and g are defined.

Example 4.1.8

Use the graphs of f and g to graph $f+g$. Also, give the domain of $f+g$.

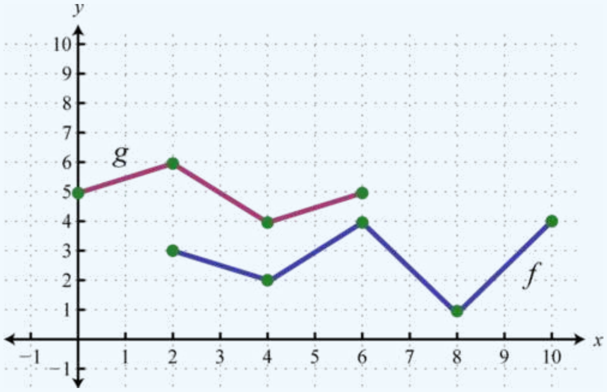

Figure 4.1.2

Solution

In this case, both functions are defined for x-values between 2 and 6. We will use 2, 4, and 6 as representative values in the domain of $f+g$ to sketch its graph.

$$(f+g)(2)=f(2)+g(2)=3+6=9$$
$$(f+g)(4)=f(4)+g(4)=2+4=6$$
$$(f+g)(6)=f(6)+g(6)=4+5=9$$

Sketch the graph of f+g using the three ordered pair solutions $(2,9)$, $(4,6)$, and $(6,9)$.

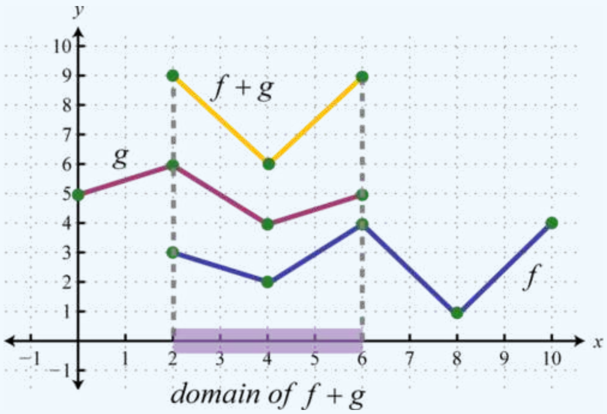

Figure 4.1.3

Answer:

$f+g$ graphed above has domain $[2,6]$.

Example 4.1.9

Use the graphs of f and g to graph $f+g$. Also, give the domain of $f+g$.

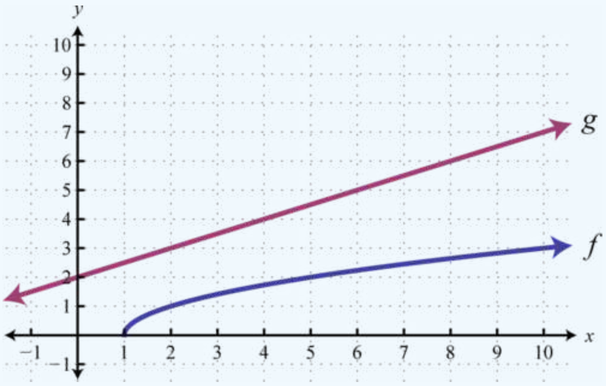

Figure 4.1.4

Solution

Another way to add nonnegative functions graphically is to copy the line segment formed from the x-axis to one of the functions onto the other as illustrated below.

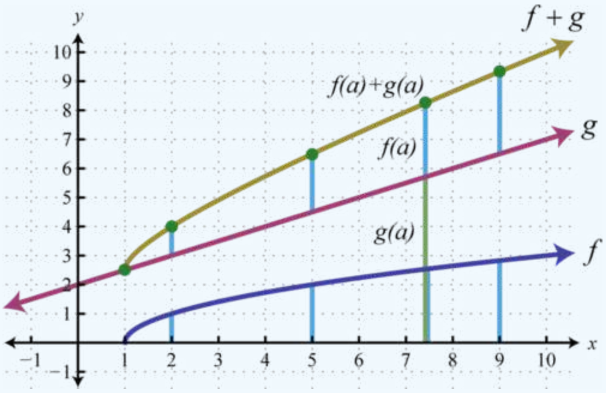

Figure 4.1.5

The line segment from the x-axis to the function f represents $f(a)$. Copy this line segment onto the other function over the same point; the endpoint represents $f(a)+g(a)$. Doing this for a number of points allows us to obtain a quick sketch of the combined graph. In this example, the domain of $f+g$ is limited to the x-values for which f is defined.

Answer:

Domain: $[1, \infty)$

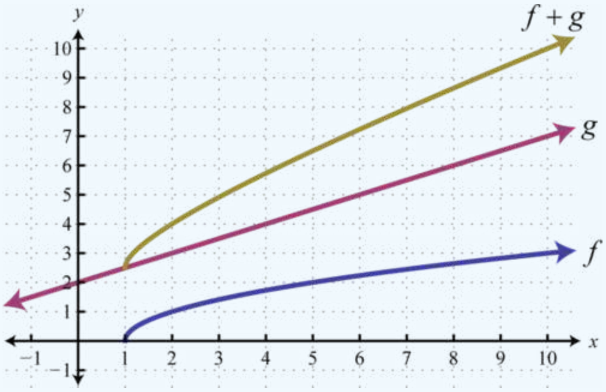

Figure 4.1.6

In general, the domain of $f+g$ is the intersection of the domain of f with the domain of g. In fact, this is the case for all of the arithmetic operations with an extra consideration for division. When dividing functions, we take extra care to remove any values that make the denominator zero. This will be discussed in more detail as we progress in algebra.

Key Takeaways

- Any polynomial with one variable is a function and can be written in the form $f(x)=a_n x^n+a_{n-1} x^{n-1}+\ldots+a_1 x+a_0$. The degree of the polynomial is the largest exponent of all the terms.
- Use function notation to streamline the evaluating process. Substitute the value or expression inside the parentheses for each instance of the variable.
- The notation $(f+g)(x)$ indicates that we should add $f(x)+g(x)$.
- The notation $(f-g)(x)$ indicates that we should subtract $f(x)-g(x)$.
- The notation $(f \cdot g)(x)$ indicates that we should multiply $f(x) g(x)$.
- The notation $(f / g)(x)$ indicates that we should divide $\frac{f(x)}{g(x)}$, where $g(x) \neq 0$.
- The domain of the function that results from these arithmetic operations is the intersection of the domain of each function. The domain of a quotient is further restricted to values that do not evaluate to zero in the denominator.

Exercise 4.1.4

Evaluate.

1. Given $f(x)=x^2-10x+3$, find $f(-3), f(0)$, and $f(5)$.
2. Given $f(x)=2x^2-x+9$, find $f(-1), f(0)$, and $f(3)$.
3. Given $g(x)=x^3-x^2+x+7$, find $g(-2), g(0)$, and $g(3)$.
4. Given $g(x)=x^3-2x+5$, find $g(-5), g(0)$, and $g(3)$.
5. Given $s(t)=5t^4-t^2+t-3$, find $s(-1), s(0)$, and $s(2)$.
6. Given $p(n)=n^4-10n^2+9$, find $p(-3), p(-1)$, and $p(2)$.
7. Given $f(x)=x^6-64$, find $f(-2), f(-1)$, and $f(0)$.
8. Given $f(x)=x^6-x^3+3$, find $f(-2), f(-1)$, and $f(0)$.
9. Given $f(x)=x^2-2x-1$, find $f(2t)$ and $f(2t-1)$.
10. Given $f(x)=x^2-2x+4$, find $f(-3t)$ and $f(2-3t)$.
11. Given $g(x)=2x^2+3x-1$, find $g(-5a)$ and $g(5-2x)$.
12. Given $g(x)=3x^2-5x+4$, find $g(-4u)$ and $g(3x-1)$.
13. Given $f(x)=x^3-1$, find $f(2a)$ and $f(x-2)$.
14. Given $f(x)=x^3-x+1$, find $f(-3x)$ and $f(2x+1)$.
15. Given $g(x)=x^3+x^2-1$, find $g(x^2)$ and $g(x-4)$.

16. Given $g(x) = 2x^3 - x + 1$, find $g(-2x^3)$ and $g(3x - 1)$.

Answer

1. $f(-3) = 42; f(0) = 3; f(5) = -22$

3. $g(-2) = -7; g(0) = 7; g(3) = 28$

5. $s(-1) = 0; s(0) = -3; s(2) = 75$

7. $f(-2) = 0; f(-1) = -63; f(0) = -64$

9. $f(2t) = 4t^2 - 4t - 1 : f(2t - 1) = 4t^2 - 8t + 2$

11. $g(-5a) = 50a^2 - 15a - 1; g(5 - 2x) = 8x^2 - 46x + 64$

13. $f(2a) = 8a^3 - 1; f(x - 2) = x^3 - 6x^2 + 12x - 9$

15. $g\left(x^2\right) = x^6 + x^4 - 1; g(x - 4) = x^3 - 11x^2 + 40x - 49$

Exercise $4.1.5$

Given the function calculate $f(x + h)$.

1. $f(x) = 5x - 3$
2. $f(x) = x^2 - 1$
3. $f(x) = x^3 - 8$
4. $f(x) = x^4$

Answer

1. $f(x + h) = 5x + 5h - 3$

3. $f(x + h) = x^3 + 3hx^2 + 3h^2x + h^3 - 8$

Exercise $4.1.6$

Given the graph of the polynomial function f find the function values.

1. Find $f(0)$, $f(1)$, and $f(2)$.

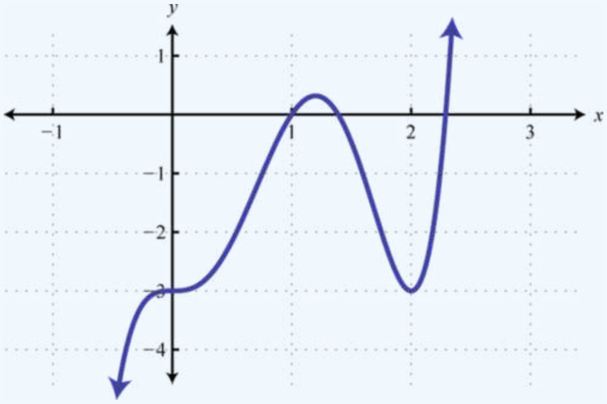

Figure $4.1.7$

2. Find $f(-1)$, $f(0)$, and $f(1)$.

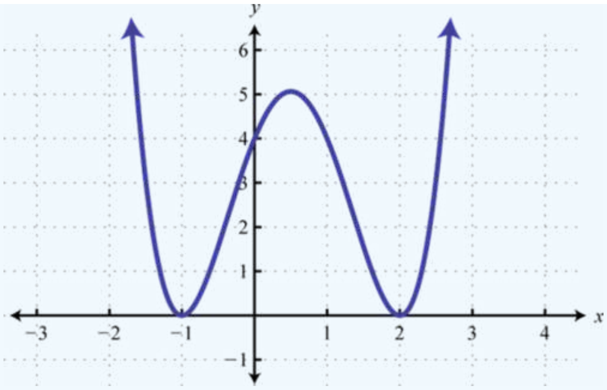

Figure 4.1.8

3. Find $f(-2)$, $f(-1)$, and $f(0)$.

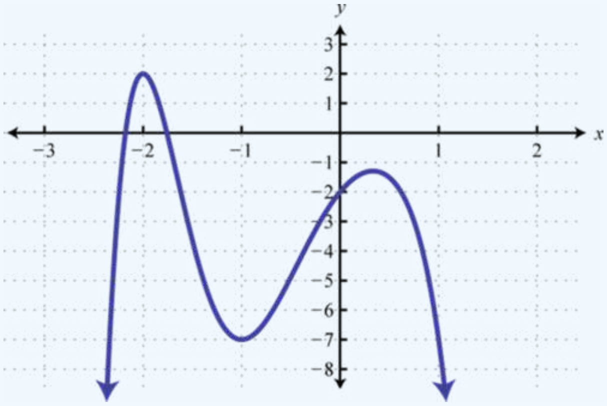

Figure 4.1.9

4. Find $f(-3)$, $f(-2)$, and $f(0)$.

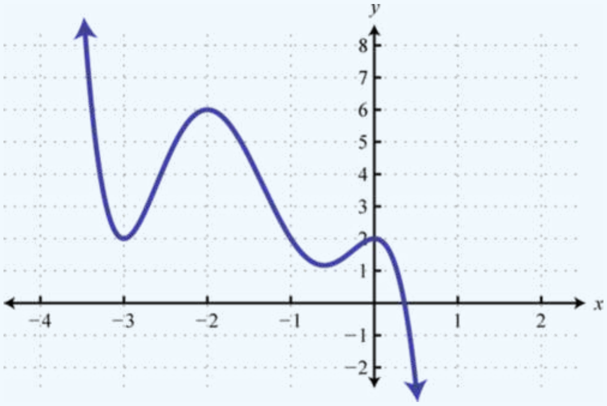

Figure 4.1.10

5. A projectile is launched upward from the ground at a speed of 48 feet per second. Write a function that models the height of the projectile and use it to calculate the height every $1/2$ second after launch. Sketch a graph that shows the height of the projectile with respect to time.

6. An object is tossed upward from a 48-foot platform at a speed of 32 feet per second. Write a function that models the height of the object and use it to calculate the height every $1/2$ second after the object is tossed. Sketch a graph that shows the height of the object with respect to time.

7. An object is dropped from a 128-foot bridge. Write a function that models the height of the object, and use it to calculate the height at 1 second and 2 seconds after it has been dropped.

8. An object is dropped from a 500-foot building. Write a function that models the height of the object, and use it to calculate the distance the object falls in the 1^{st} second, 2^{nd} second, and the 3^{rd} second.

9. A bullet is fired straight up into the air at 320 meters per second. Ignoring the effects of air friction, write a function that models the height of the bullet, and use it to calculate the bullet's height 1 minute after it was fired into the air.

10. A book is dropped from a height of 10 meters. Write a function that gives the height of the book, and use it to determine how far it will fall in $1\frac{1}{4}$ seconds.

Answer

1. $f(0)=-3; f(1)=0; f(2)=-3$

3. $f(-2)=2; f(-1)=-7; f(0)=-2$

5. $h(t)=-16t^2+48t$

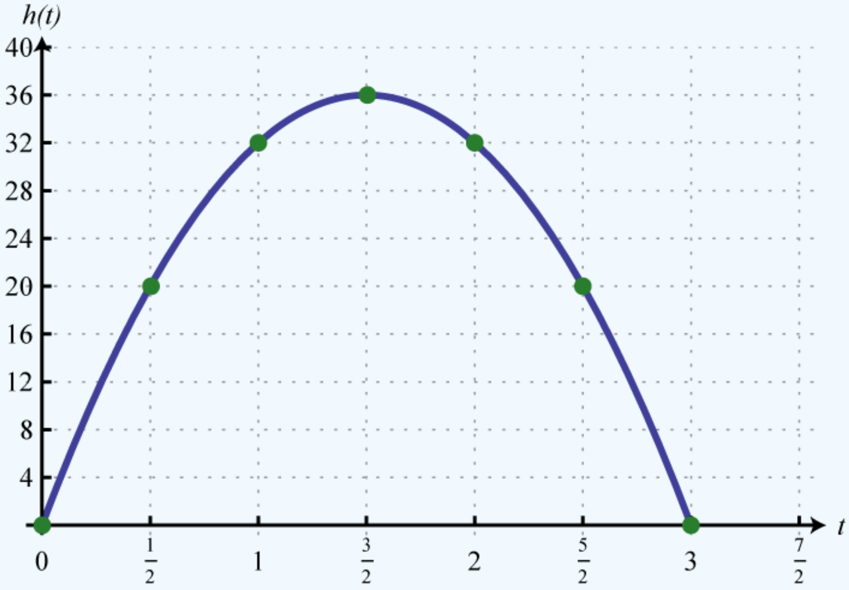

Figure $4.1.11$

7. $h(t)=-16t^2+128$; At 1 second the object's height is 112 feet and at 2 seconds its height is 64 feet.

9. $h(t)=-4.9t^2+320t; 1,560$ meters

Exercise $4.1.7$

Given functions f and g, find $(f+g)$ and $(f-g)$.

1. $f(x)=5x-3, g(x)=4x-1$
2. $f(x)=3x+2, g(x)=7x-5$
3. $f(x)=2-3x, g(x)=1-x$
4. $f(x)=8x-5, g(x)=-7x+4$
5. $f(x)=x^2-3x+2, g(x)=x^2+4x-7$
6. $f(x)=2x^2+x-3, g(x)=x^2-x+4$
7. $f(x)=x^2+5x-3, g(x)=6x+11$
8. $f(x)=9x+5, g(x)=2x^2-5x+4$
9. $f(x)=9x^2-1, g(x)=x^2+5x$
10. $f(x)=10x^2, g(x)=5x^2-8$
11. $f(x)=8x^3+x-4, g(x)=4x^3+x^2-1$
12. $f(x)=x^3-x^2+x+1, g(x)=x^3-x^2-x-1$

Answer

1. $(f+g)(x)=9x-4; (f-g)(x)=x-2$

3. $(f+g)(x)=-4x+3; (f-g)(x)=-2x+1$

5. $(f+g)(x)=2x^2+x-5; (f-g)(x)=-7x+9$
7. $(f+g)(x)=x^2+11x+8; (f-g)(x)=x^2-x-14$
9. $(f+g)(x)=10x^2+5x-1; (f-g)(x)=8x^2-5x-1$
11. $\begin{array}{l}(f+g)(x)=12x^3+x^2+x-5\\(f-g)(x)=4x^3-x^2+x-3\end{array}$

Exercise 4.1.8

Given $f(x)=x^3+2x^2-8$ and $g(x)=2x^2-3x+5$, evaluate the following.

1. $(f+g)(-2)$
2. $(f+g)(3)$
3. $(f-g)(-2)$
4. $(f-g)(3)$
5. $(g-f)(-2)$
6. $(g-f)(3)$
7. $(f+f)(1)$
8. $(g+g)(-1)$

Answer

1. 11
3. -27
5. 27
7. -10

Exercise 4.1.9

Given the graphs of f and g, evaluate the following.

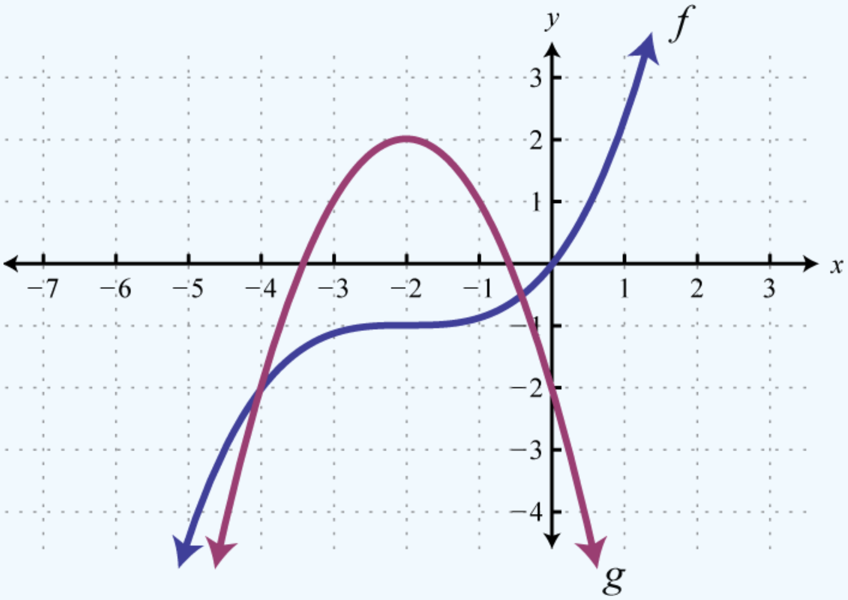

Figure 4.1.12

1. $(f+g)(-4)$
2. $(f-g)(-4)$
3. $(f+g)(-2)$
4. $(f-g)(-2)$

5. $(f+g)(0)$
6. $(f-g)(0)$

Answer

1. -4

3. 1

5. -2

Exercise 4.1.10

Given f and g, find $f \cdot g$.

1. $f(x)=5x, g(x)=x-3$
2. $f(x)=x-4, g(x)=6x$
3. $f(x)=2x-3, g(x)=3x+4$
4. $f(x)=5x-1, g(x)=2x+1$
5. $f(x)=3x+4, g(x)=3x-4$
6. $f(x)=x+5, g(x)=x-5$
7. $f(x)=x-2, g(x)=x^2-3x+2$
8. $f(x)=2x-3, g(x)=x^2+2x-1$
9. $f(x)=2x^2, g(x)=x^2-7x+5$
10. $f(x)=5x^3, g(x)=x^2-3x-1$
11. $f(x)=x^2-3x-2, g(x)=2x^2-x+3$
12. $f(x)=x^2+x-1, g(x)=x^2-x+1$

Answer

1. $(f \cdot g)(x)=5x^2-15x$

3. $(f \cdot g)(x)=6x^2-x-12$

5. $(f \cdot g)(x)=9x^2-16$

7. $(f \cdot g)(x)=x^3-5x^2+8x-4$

9. $(f \cdot g)(x)=2x^4-14x^3+10x^2$

11. $(f \cdot g)(x)=2x^4-7x^3+2x^2-7x-6$

Exercise 4.1.11

Given f and g, find f/g. (Assume all expressions in the denominator are nonzero.)

1. $f(x)=36x^3-16x^2-8x, g(x)=4x$
2. $f(x)=2x^3-6x^2+10x, g(x)=2x$
3. $f(x)=20x^7-15x^5+5x^3, g(x)=5x^3$
4. $f(x)=9x^6+12x^4-3x^2, g(x)=3x^2$
5. $f(x)=x^3+4x^2+3x-2, g(x)=x+2$
6. $f(x)=x^3-x^2-10x+12, g(x)=x-3$
7. $f(x)=6x^3-13x^2+36x-45, g(x)=2x-3$
8. $f(x)=6x^3-11x^2+15x-4, g(x)=3x-1$
9. $f(x)=3x^3-13x^2-x+8, g(x)=3x+2$
10. $f(x)=5x^3-16x^2+13x-6, g(x)=5x-1$

Answer

1. $(f/g)(x)=9x^2-4x-2$

3. $(f/g)(x)=4x^4-3x^2+1$

5. $(f/g)(x) = x^2 + 2x - 1$

7. $(f/g)(x) = 3x^2 - 2x + 15$

9. $(f/g)(x) = x^2 - 5x + 3 + \frac{2}{3x+2}$

Exercise 4.1.12

Given $f(x) = 25x^4 + 10x^3 - 5x^2$ and $g(x) = 5x^2$ evaluate the following.

1. $(f \cdot g)(-1)$
2. $(f \cdot g)(1)$
3. $(f/g)(-2)$
4. $(f/g)(-3)$
5. $(g \cdot f)(0)$
6. $(g/f)(1)$
7. $(g \cdot g)(-1)$
8. $(f \cdot f)(-1)$

Answer

1. 50

3. 15

5. 0

7. 25

Exercise 4.1.13

Given the graphs of f and g evaluate the following.

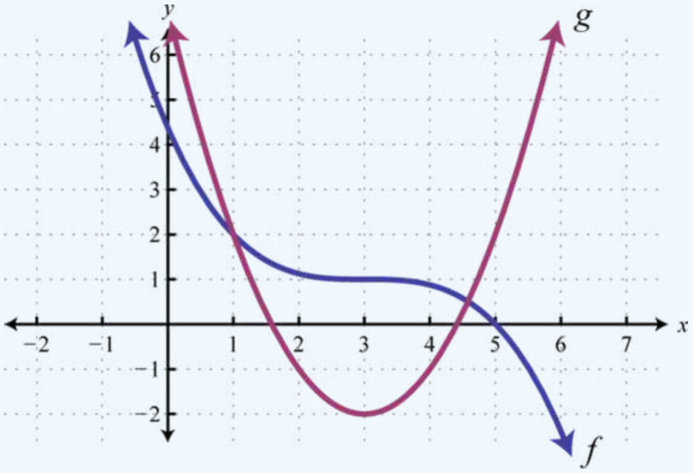

Figure 4.1.13

1. $(f \cdot g)(3)$

2. $(f \cdot g)(5)$

3. $(f/g)(5)$

4. $(f/g)(3)$

5. $(f \cdot g)(1)$

6. $(f/g)(1)$

Answer

1. -2
3. 0
5. 4

Exercise 4.1.14

Given $f(x)=5x^3-15x^2+10x, g(x)=x^2-x+3$, and $h(x)=-5x$, find the following. (Assume all expressions in the denominator are nonzero.)

1. $(f-g)(x)$
2. $(g-f)(x)$
3. $(g\cdot h)(x)$
4. $(f/h)(x)$
5. $(h+g)(x)$
6. $(h\cdot f)(x)$
7. $(g/h)(2)$
8. $(g-h)(-3)$
9. The revenue in dollars from selling MP3 players is given by the function $R(n)=125n-0.15n^2$, where n represents the number of units sold $(0\le n<833)$. The cost in dollars of producing the MP3 players is given by the formula $C(n)=1200+42n$ where n represents the number of units produced. Write a function that models the profit of producing and selling n MP3 players. Use the function to determine the profit generated from producing and selling 225 MP3 players. Recall that profit equals revenues less costs.
10. The inner radius of a washer is $\frac{1}{2}$ that of the outer radius.

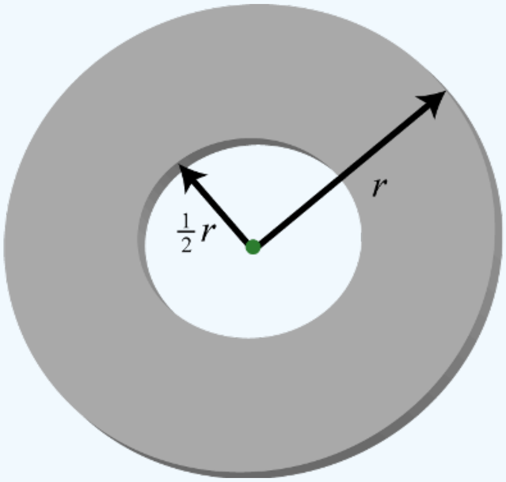

Figure $4.1.14$

Answer

1. $(f-g)(x)=5x^3-16x^2+11x-3$
3. $(g\cdot h)(x)=-5x^3+5x^2-15x$
5. $(h+g)(x)=x^2-6x+3$
7. $(g/h)(2)=-\frac{1}{2}$
9. $P(n)=-0.15n^2+83n-1200; \$9,881.25$

Exercise $4.1.15$

Use the graphs of f and g to graph $f+g$. Also, give the domain of $f+g$.

1.

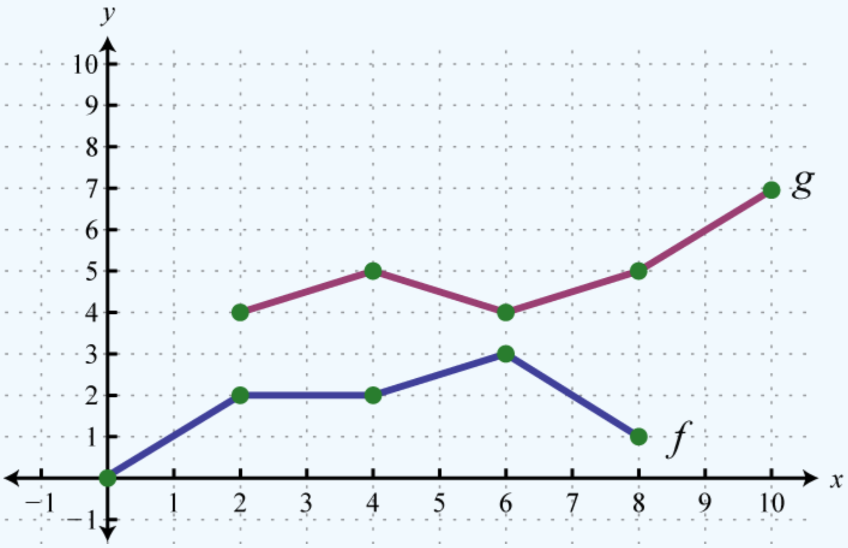

Figure $4.1.15$

2.

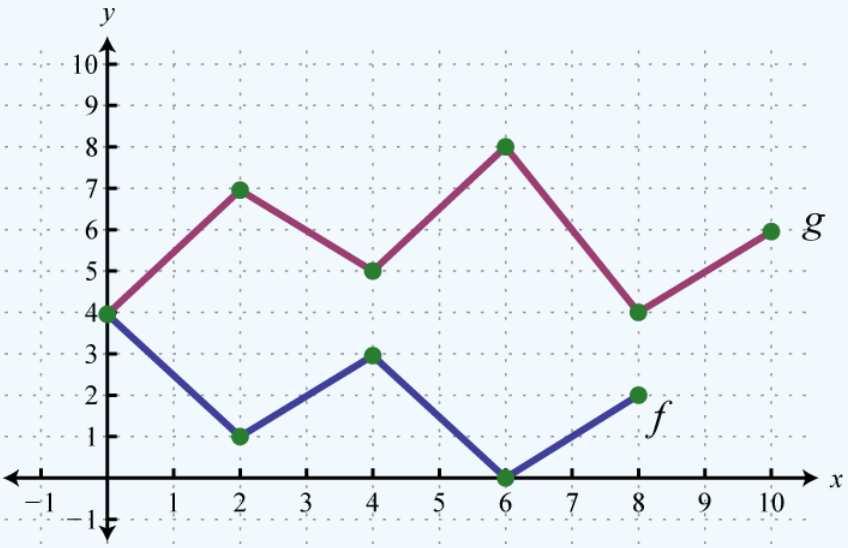

Figure $4.1.16$

3.

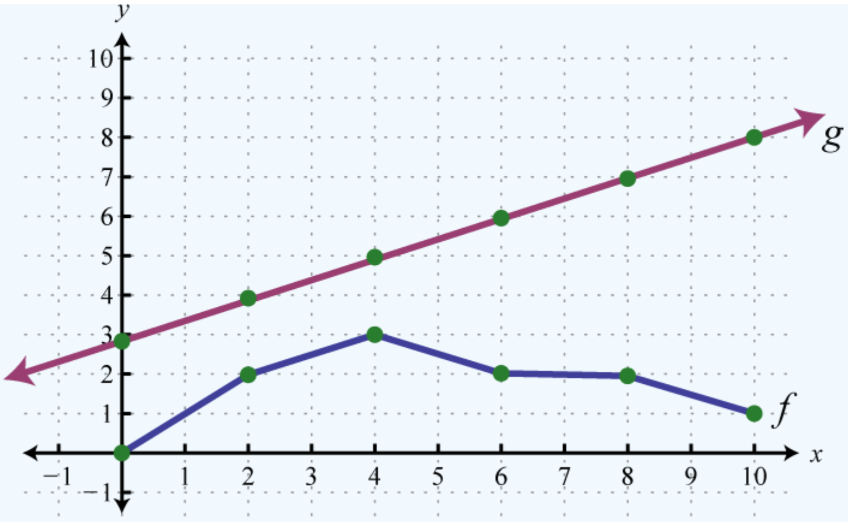

Figure 4.1.17

4.

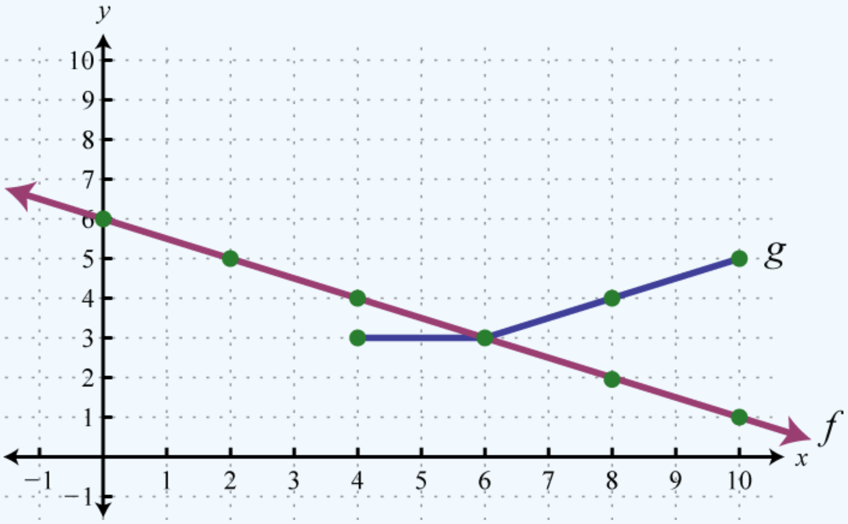

Figure 4.1.18

5.

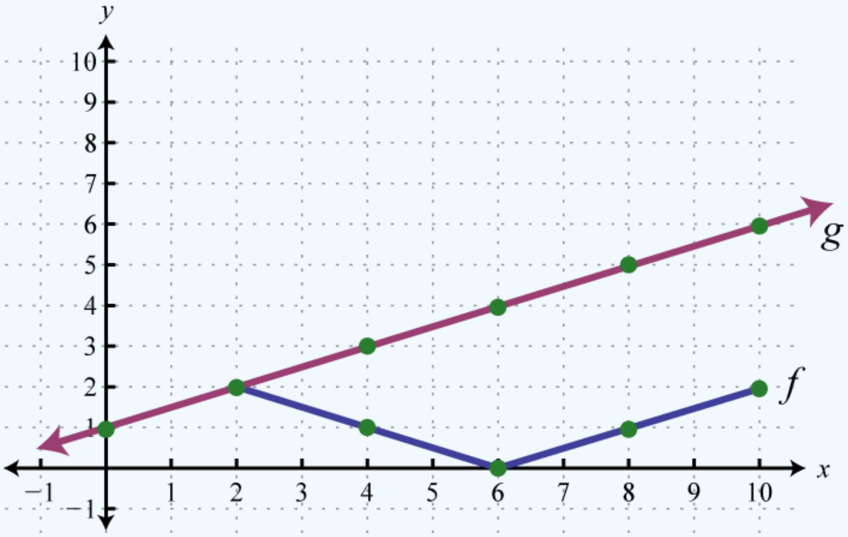

Figure 4.1.19

6.

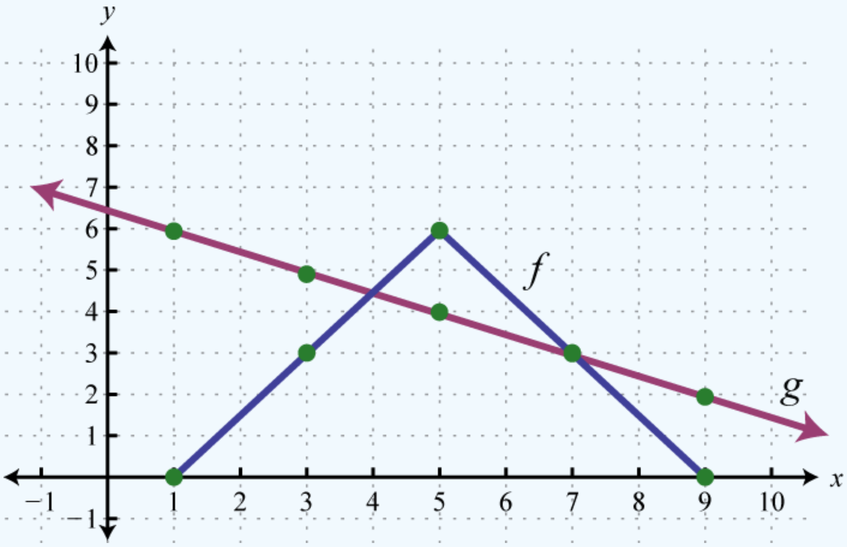

Figure 4.1.20

7.

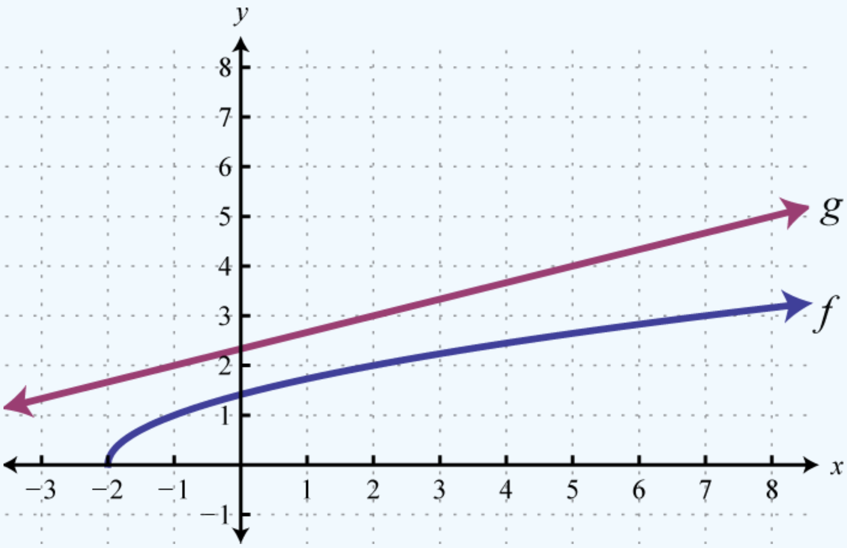

Figure 4.1.21

8.

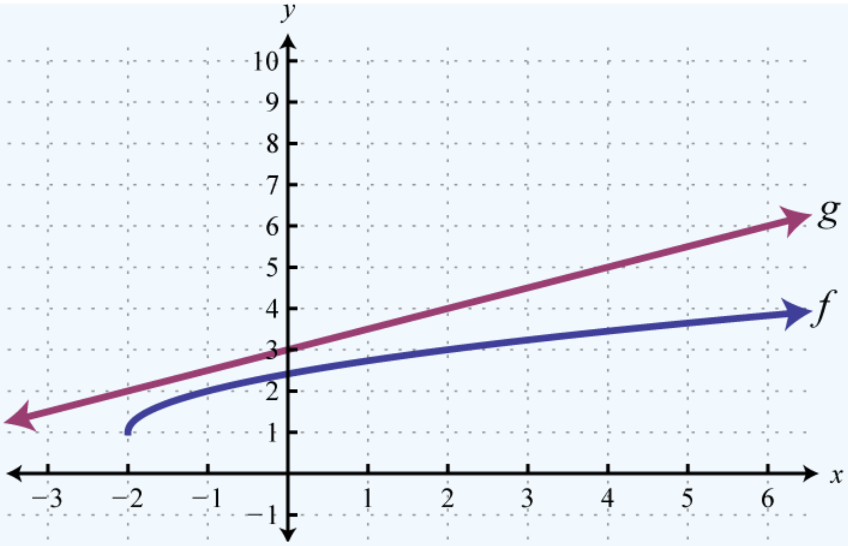

Figure 4.1.22

9.

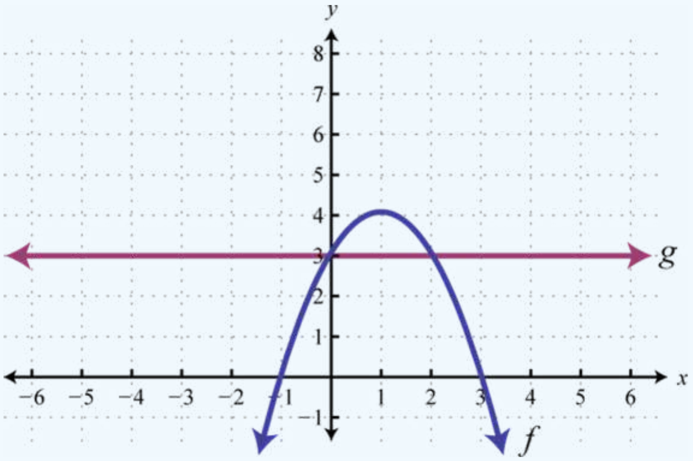

Figure 4.1.23

10.

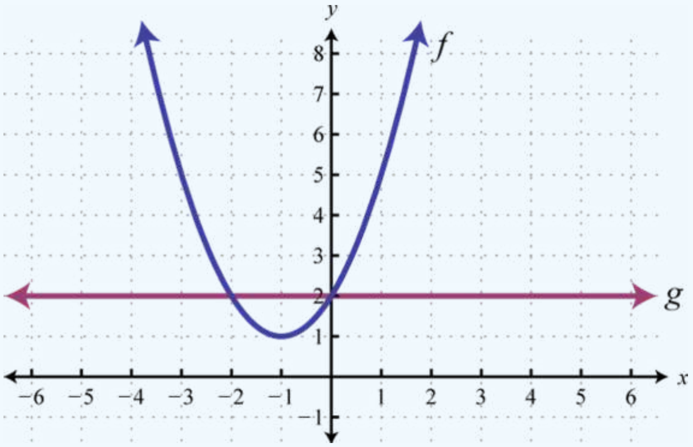

Figure 4.1.24

Answer

1. $[2, 8]$

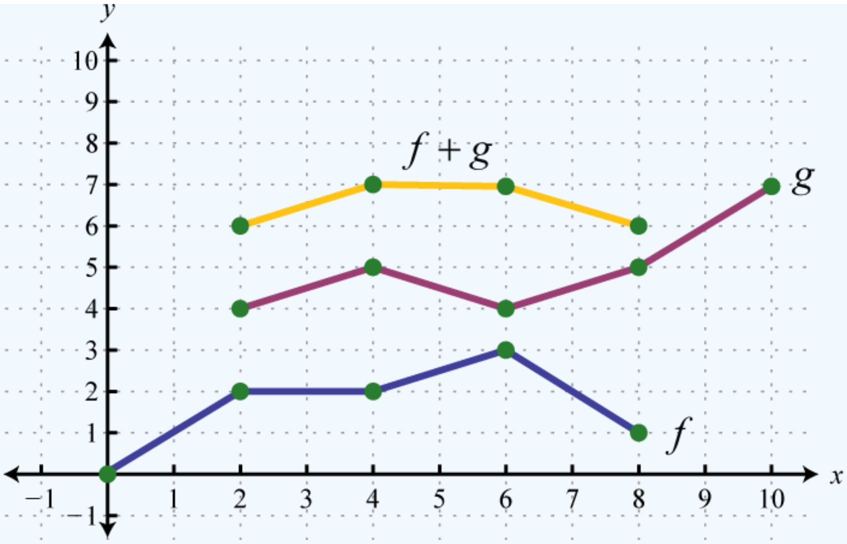

Figure 4.1.25

3. $[0,10]$

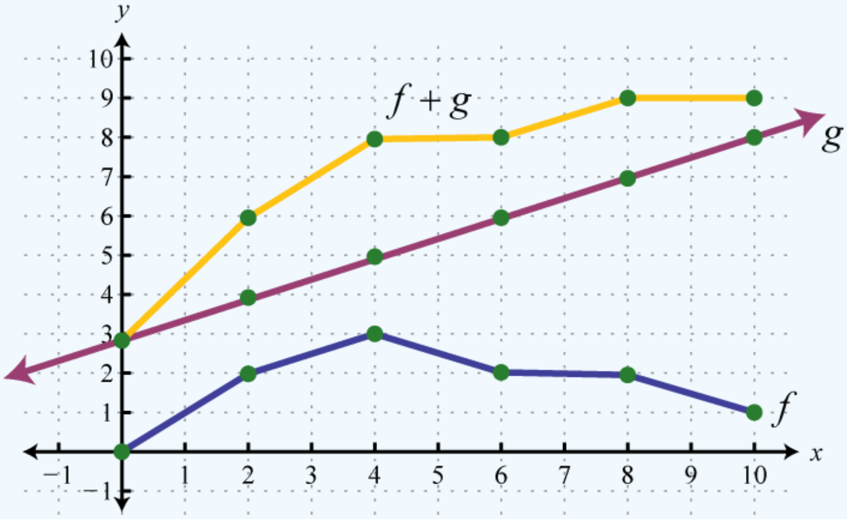

Figure 4.1.26

5. $[2,10]$

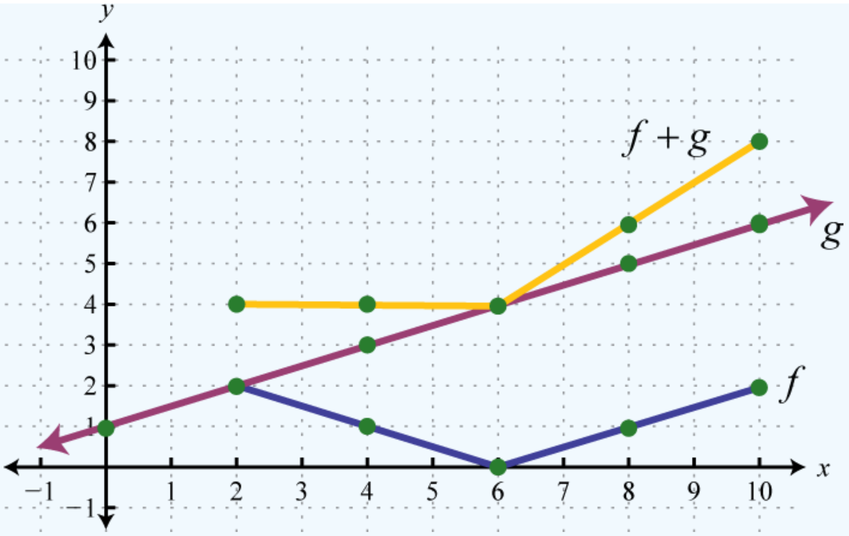

Figure 4.1.27

7. $[-2, \infty)$

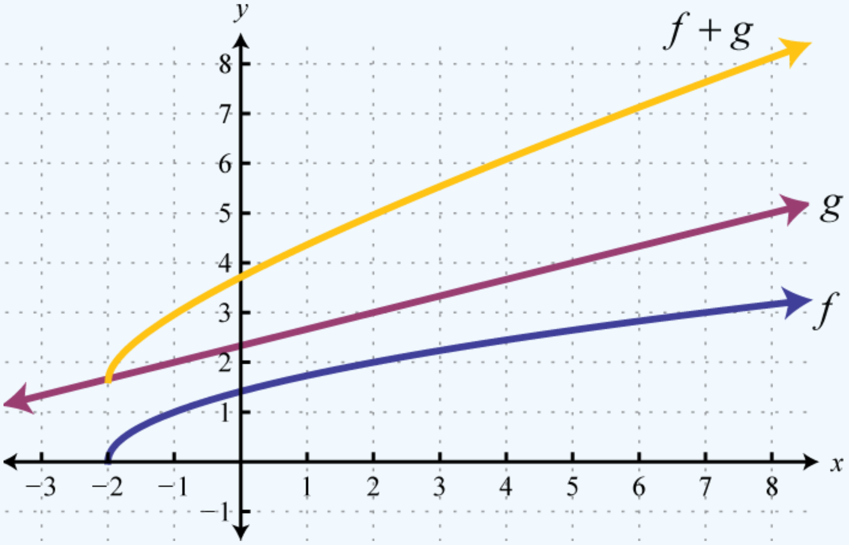

Figure 4.1.28

9. $(-\infty, \infty)$

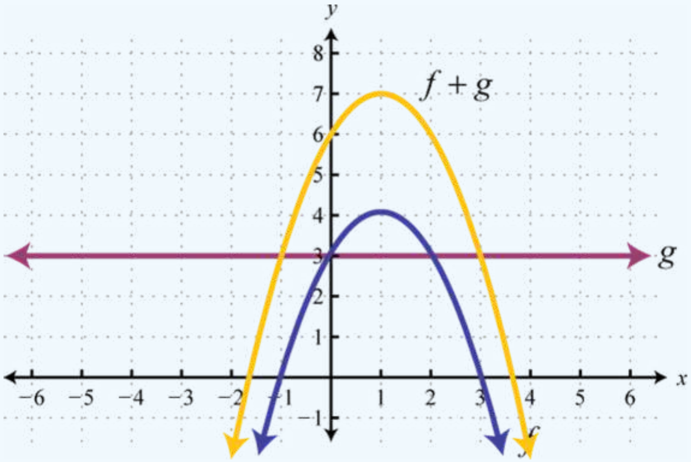

Figure 4.1.29

Exercise $4.1.16$

1. Which arithmetic operations on functions are commutative? Explain.
2. Explore ways we can add functions graphically if they happen to be negative.

Answer

1. Answer may vary

Footnotes

[1] Add functions as indicated by the notation: $(f+g)(x)=f(x)+g(x)$.

[2] Subtract functions as indicated by the notation: $(f-g)(x)=f(x)-g(x)$.

[3] Multiply functions as indicated by the notation: $(f\cdot g)(x)=f(x)\cdot g(x)$.

[4] Divide functions as indicated by the notation: $(f/g)(x)=\frac{f(x)}{g(x)}$, where $g(x)\neq 0$.

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4.2: Factoring Polynomials

Learning Objectives

- Determine the greatest common factor (GCF) of monomials.
- Factor out the GCF of a polynomial.
- Factor a four-term polynomial by grouping.
- Factor special binomials.

Determining the GCF of Monomials

The process of writing a number or expression as a product is called *factoring*[5]. If we write the monomial $8x^7 = 2x^5 \cdot 4x^2$, we say that the product $2x^5 \cdot 4x^2$ is a *factorization*[6] of $8x^7$ and that $2x^5$ and $4x^2$ are *factors*[7]. Typically, there are many ways to factor a monomial. Some factorizations of $8x^7$ follow:

$$\left.\begin{aligned} 8x^7 &= 2x^5 \cdot 4x^2 \\ 8x^7 &= 8x^6 \cdot x \\ 8x^7 &= 2x \cdot 2x^2 \cdot 2x^4 \end{aligned}\right\} \quad Factorizations\ of\ 8x^7$$

Given two or more monomials, it will be useful to find the *greatest common monomial factor (GCF)*[8] of each. The GCF of the monomials is the product of the common variable factors and the GCF of the coefficients.

Example 4.2.1

Find the GCF of $25x^7y^2z$ and $15x^3y^4z^2$.

Solution

Begin by finding the GCF of the coefficients. In this case, $25 = 5 \cdot 5$ and $15 = 3 \cdot 5$. It should be clear that

$$\mathrm{GCF}(25, 15) = 5$$

Next determine the common variable factors with the smallest exponents.

$$25x^7y^2z \quad \text{and} \quad 15x^3y^4z^2$$

The common variable factors are x^3, y^2, and z. Therefore, given the two monomials,

$$\mathrm{GCF} = 5x^3y^2z$$

Answer:

$5x^3y^2z$

It is worth pointing out that the GCF divides both expressions evenly.

$$\frac{25x^7y^2z}{5x^3y^2z} = 5x^4 \quad \text{and} \quad \frac{15x^3y^4z^2}{5x^3y^2z} = 3y^2z$$

Furthermore, we can write the following:

$$25x^7y^2z = 5x^3y^2z \cdot 5x^4 \quad \text{and} \quad 15x^3y^4z^2 = 5x^3y^2z \cdot 3y^2z$$

The factors $5x^4$ and $3y^2z$ share no common monomial factors other than 1; they are *relatively prime*[9].

Example 4.2.2

Determine the GCF of the following three expressions: $12a^5b^2(a+b)^5$, $60a^4b^3c\,(a+b)^3$, and $24a^2b^7c^3(a+b)^2$.

Solution

Begin by finding the GCF of the coefficients. To do this, determine the prime factorization of each and then multiply the common factors with the smallest exponents.

$$\begin{aligned} 12 &= 2^2 \cdot 3 \\ 60 &= 2^2 \cdot 3 \cdot 5 \\ 24 &= 2^3 \cdot 3 \end{aligned}$$

Therefore, the GCF of the coefficients of the three monomials is

$$\mathrm{GCF}(12,60,24)=2^2\cdot 3=12$$

Next, determine the common factors of the variables.

$$12a^5b^2(a+b)^5 \text{ and } 60a^4b^3c(a+b)^3 \text{ and } 24a^2b^7c^3(a+b)^2$$

The variable factors in common are a^2, b^2, and $(a+b)^2$. Therefore,

$$\mathrm{GCF}=12\cdot a^2\cdot b^2\cdot (a+b)^2$$

Note that the variable c is not common to all three expressions and thus is not included in the GCF.

Answer

$12a^2b^2(a+b)^2$

Factoring out the GCF

The application of the distributive property is the key to multiplying polynomials. For example,

$$\begin{aligned} 6xy^2(2xy+1) &= 6xy^2\cdot 2xy + 6xy^2\cdot 1 \quad \textit{Multiplying} \\ &= 12x^2y^3+6xy^2 \end{aligned}$$

The process of factoring a polynomial involves applying the distributive property in reverse to write each polynomial as a product of polynomial factors.

$$\begin{aligned} a(b+c) &= ab+ac \quad \textit{Multiplying} \\ ab+ac &= a(b+c) \quad \textit{Factoring} \end{aligned}$$

Consider factoring the result of the opening example:

$$\begin{aligned} 12x^2y^3+6xy^2 &= 6xy^2\cdot 2xy+6xy^2\cdot 1 \quad \textit{Factoring} \\ &= 6xy^2(\quad ? \quad) \\ &= 6xy^2(2xy+1) \end{aligned}$$

We see that the distributive property allows us to write the polynomial $12x^2y^3 + 6xy^2$ as a product of the two factors $6xy^2$ and $(2xy+1)$. Note that in this case, $6x^2y$ is the GCF of the terms of the polynomial.

$$\mathrm{GCF}\left(12x^2y^3, 6xy^2\right)=6xy^2$$

Factoring out the greatest common factor (GCF)[10] of a polynomial involves rewriting it as a product where a factor is the GCF of all of its terms.

$$\left.\begin{aligned} 8x^3+4x^2-16x &= 4x(2x^2+x-4) \\ 9ab^2-18a^2b-3ab &= 3ab(3b-6a-1) \end{aligned}\right\} \quad \textit{Factoring out the GCF}$$

To factor out the GCF of a polynomial, we first determine the GCF of all of its terms. Then we can divide each term of the polynomial by this factor as a means to determine the remaining factor after applying the distributive property in reverse.

Example 4.2.3

Factor out the GCF: $18x^7-30x^5+6x^3$.

Solution

In this case, the $GCF(18,30,6)=6$, and the common variable factor with the smallest exponent is x^3. The GCF of the polynomial is $6x^3$.

$$18x^7-30x^5+6x^3=6x^3(\quad ? \quad)$$

The missing factor can be found by dividing each term of the original expression by the GCF.

$$\frac{18x^7}{6x^3}=3x^4 \qquad \frac{-30x^5}{6x^3}=-5x^2 \qquad \frac{+6x^3}{6x^3}=+1$$

Apply the distributive property (in reverse) using the terms found in the previous step.

$$18x^7-30x^5+6x^3=6x^3(3x^4-5x^2+1)$$

If the GCF is the same as one of the terms, then, after the GCF is factored out, a constant term 1 will remain. The importance of remembering the constant term becomes clear when performing the check using the distributive property.

$$\begin{aligned} 6x^3(3x^4-5x^2+1) &= 6x^3 \cdot 3x^4 - 6x^3 \cdot 5x^2 + 6x^3 \cdot 1 \\ &= 18x^7 - 30x^5 + 6x^3 \quad \checkmark \end{aligned}$$

Answer

$6x^3(3x^4-5x^2+1)$

Example $4.2.4$

Factor out the GCF: $27x^5y^5z + 54x^5yz - 63x^3y^4$.

Solution

The GCF of the terms is $9x^3y$. The last term does not have a variable factor of z, and thus z cannot be a part of the greatest common factor. If we divide each term by $9x^3y$, we obtain

$$\frac{27x^5y^5z}{9x^3y} = 3x^2y^4z \qquad \frac{54x^5yz}{9x^3y} = 6x^2z \qquad \frac{-63x^3y^4}{9x^3y} = -7y^3$$

and can write

$$\begin{aligned} 27x^5y^5z + 54x^5yz - 63x^3y^4 &= 9x^3y(\quad ? \quad) \\ &= 9x^3y(3x^2y^4z + 6x^2z - 7y^3) \end{aligned}$$

Answer:

$9x^3y(3x^2y^4z + 6x^2z - 7y^3)$

Exercise $4.2.1$

Factor out the GCF: $12x^3y^4 - 6x^2y^3 - 3xy^2$

Answer

$3xy^2(4x^2y^2 - 2xy - 1)$

www.youtube.com/v/FRnIljz10XY

Of course, not every polynomial with integer coefficients can be factored as a product of polynomials with integer coefficients other than 1 and itself. If this is the case, then we say that it is a **prime polynomial**[11]. For example, a linear factor such as $10x-9$ is prime. However, it can be factored as follows:

$$10x - 9 = x\left(10 - \frac{9}{x}\right) \quad \text{or} \quad 10x - 9 = 5\left(2x - \frac{9}{5}\right)$$

If an *x* is factored out, the resulting factor is not a polynomial. If any constant is factored out, the resulting polynomial factor will not have integer coefficients. Furthermore, some linear factors are not prime. For example,

$$5x - 10 = 5(x-2)$$

In general, any linear factor of the form $ax+b$, where a and b are relatively prime integers, is prime.

Factoring by Grouping

In this section, we outline a technique for factoring polynomials with four terms. First, review a preliminary example where the terms have a common binomial factor.

Example $4.2.5$

Factor: $7x(3x-2) - (3x-2)$.

Solution

Begin by rewriting the second term $-(3x-2)$ as $-1(3x-2)$. Next, consider $(3x-2)$ as a common binomial factor and factor it out as follows:

$$\begin{aligned}7x(3x-2)-(3x-2) &= 7x(3x-2)-1(3x-2)\\ &= (3x-2)(\quad ? \quad)\\ &= (3x-2)(7x-1)\end{aligned}$$

Answer:

$(3x-2)(7x-1)$

Factoring by grouping[12] is a technique that enables us to factor polynomials with four terms into a product of binomials. This involves an intermediate step where a common binomial factor will be factored out. For example, we wish to factor

$$3x^3-12x^2+2x-8$$

Begin by grouping the first two terms and the last two terms. Then factor out the GCF of each grouping:

$$\begin{aligned}&\underbrace{3x^3-12x^2}_{group}+\underbrace{2x-8}_{group}\\ &=3x^2(x-4)+2(x-4)\end{aligned}$$

Figure 4.2.1

In this form, the polynomial is a binomial with a common binomial factor, $(x-4)$.

$$\begin{aligned}&=(x-4)(\quad ? \quad)\\ &=(x-4)\left(3x^2+2\right)\end{aligned}$$

Therefore,

$$3x^3-12x^2+2x-8=(x-4)\left(3x^2+2\right)$$

We can check by multiplying.

$$\begin{aligned}(x-4)\left(3x^2+2\right) &= 3x^3+2x-12x^2-8\\ &= 3x^3-12x^2+2x-8 \quad \checkmark\end{aligned}$$

Example 4.2.6

Factor by grouping $24a^4-18a^3-20a+15$.

Solution

The GCF for the first group is $6a^3$. We have to choose 5 or -5 to factor out of the second group.

$$\begin{aligned}&\underbrace{24a^4-18a^3}_{group}-\underbrace{20a+15}_{group}\\ &=6a^3(4a-3)+5(-4a+3) \quad \times\\ &=6a^3(4a-3)-5(4a-3) \quad \checkmark\end{aligned}$$

Figure 4.2.2

Factoring out $+5$ does not result in a common binomial factor. If we choose to factor out -5, then we obtain a common binomial factor and can proceed. Note that when factoring out a negative number, we change the signs of the factored terms.

$$\begin{aligned}24a^4-18a^3-20a+15 &= \underbrace{24a^4-18a^3}_{group}-\underbrace{20a+15}_{group}\\ &=6a^3(\quad ? \quad)-5(\quad ? \quad)\\ &=6a^3(4a-3)-5(4a-3)\\ &=(4a-3)(\quad ? \quad)\\ &=(4a-3)\left(6a^3-5\right)\end{aligned}$$

Figure 4.2.3

Answer

$(4a-3)(6a^3-5)$. Check by multiplying; this is left to the reader as an exercise.

Sometimes we must first rearrange the terms in order to obtain a common factor.

Example 4.2.7

Factor $ab-2a^2b+a^3-2b^2$.

Solution

Simply factoring the GCF out of the first group and last group does not yield a common binomial factor.

$$\begin{aligned} &\underbrace{ab-2a^2b}_{group}+\underbrace{a^3-2b^2}_{group} \\ &=ab(1-2a)+1(a^3-2b^2) \end{aligned}$$

Figure 4.2.4

We must rearrange the terms, searching for a grouping that produces a common factor. In this example, we have a workable grouping if we switch the terms a^3 and ab.

$$\begin{aligned} ab-2a^2b+a^3-2b^2 &= \underbrace{a^3-2a^2b}_{group}+\underbrace{ab-2b^2}_{group} \\ &=a^2(a-2b)+b(a-2b) \\ &=(a-2b)(a^2+b) \end{aligned}$$

Figure 4.2.5

Answer

$(a-2b)\left(a^2+b\right)$

Exercise 4.2.2

Factor $x^3-x^2y-xy+y^2$.

Answer

$(x-y)\left(x^2-y\right)$

www.youtube.com/v/ECdPaOZ0TRs

Not all factorable four-term polynomials can be factored with this technique. For example,

$$3x^3+5x^2-x+2$$

This four-term polynomial cannot be grouped in any way to produce a common binomial factor. Despite this, the polynomial is not prime and can be written as a product of polynomials. It can be factored as follows:

$$3x^3+5x^2-x+2=(x+2)(3x^2-x+1)$$

Factoring such polynomials is something that we will learn to do as we move further along in our study of algebra. For now, we will limit our attempt to factor four-term polynomials to using the factor by grouping technique.

Factoring Special Binomials

A binomial is a polynomial with two terms. We begin with the special binomial called **difference of squares**[13]:

$$a^2-b^2=(a+b)(a-b)$$

To verify the above formula, multiply.

$$\begin{aligned}(a+b)(a-b)&=a^2-ab+ba-b^2\\&=a^2-ab+ab-b^2\\&=a^2-b^2\end{aligned}$$

We use this formula to factor certain special binomials.

Example 4.2.8

Factor x^2-9y^2.

Solution

Identify the binomial as difference of squares and determine the square factors of each term.

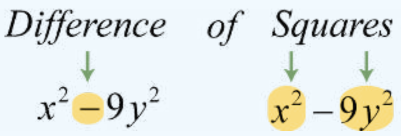

Figure 4.2.6

Here we can write

$$x^2-9y^2=(x)^2-(3y)^2$$

Substitute into the difference of squares formula where $a=x$ and $b=3y$.

$$a^2-b^2=(a+b)(a-b)$$
$$\downarrow\quad\downarrow\quad\downarrow\quad\downarrow$$
$$x^2-9y^2=(x+3y)(x-3y)$$

Multiply to check:

$$\begin{aligned}(x+3y)(x-3y)&=x^2-3xy+3yx-9y^2\\&=x^2-3xy+3xy-9y^2\\&=x^2-9y^2\ \checkmark\end{aligned}$$

Answer

$(x+3y)(x-3y)$

Example 4.2.9

Factor: $x^2-(2x-1)^2$.

Solution

First, identify this expression as a difference of squares.

$$x^2-(2x-1)^2=(x)^2-(2x-1)^2$$

Use $a=x$ and $b=2x-1$ in the formula for a difference of squares and then simplify.

$$a^2-b^2=(a+b)(a-b)$$

$$\begin{aligned}x^2-(2x-1)^2&=[x+(2x-1)][x-(2x-1)]\\&=(x+2x-1)(x-2x+1)\\&=(3x-1)(-x+1)\end{aligned}$$

Answer

$(3x-1)(-x+1)$

Given any real number b, a polynomial of the form x^2+b^2 is prime. Furthermore, the **sum of squares**[14] a^2+b^2 does not have a general factored equivalent. Care should be taken not to confuse this with a perfect square trinomial.

$$\begin{aligned}(a+b)^2&=(a+b)(a+b)\\&=a^2+ab+ba+b^2\\&=a^2+2ab+b^2\end{aligned}$$

Therefore,

$$(a+b)^2 \neq a^2+b^2$$

For example, the sum of squares binomial x^2+9 is prime. Two other special binomials of interest are the **sum**[15] and **difference of cubes**[16]:

$$\begin{aligned} a^3+b^3 &= (a+b)\left(a^2-ab+b^2\right) \\ a^3-b^3 &= (a-b)\left(a^2+ab+b^2\right) \end{aligned}$$

We can verify these formulas by multiplying.

$$\begin{aligned} (a+b)\left(a^2-ab+b^2\right) &= a^3-a^2b+ab^2+a^2b-ab^2+b^3 \\ &= a^3+b^3 \qquad \checkmark \\ (a-b)\left(a^2+ab+b^2\right) &= a^3+a^2b+ab^2-a^2b-ab^2-b^3 \\ &= a^3-b^3 \qquad \checkmark \end{aligned}$$

The process for factoring sums and differences of cubes is very similar to that of differences of squares. We first identify a and b and then substitute into the appropriate formula. The separate formulas for the sum and difference of cubes allow us to always choose a and b to be positive.

Example 4.2.10

Factor x^3-8y^3.

Solution

First, identify this binomial as a difference of cubes.

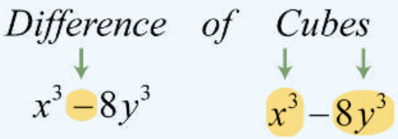

Figure 4.2.7

Next, identify what is being cubed.

$$x^3-8y^3=(x)^3-(2y)^3$$

In this case, $a=x$ and $b=2y$. Substitute into the difference of cubes formula.

$$\begin{array}{rl} a^3+b^3 = & (a \quad -b\ )\left(a^2 \quad +a\ \cdot\ b \quad +b^2\right) \\ & \ \downarrow \quad\ \downarrow \quad \downarrow \qquad \downarrow \ \ \downarrow \qquad \downarrow \\ x^3-8y^3 = & (x-2y)\left((x)^2+x\cdot 2y+(2y)^2\right) \\ = & (x-2y)\left(x^2+2xy+4y^2\right) \end{array}$$

We can check this factorization by multiplying.

$$\begin{aligned} (x-2y)\left(x^2+2xy+4y^2\right) &= x^3+2x^2y+4xy^2-2x^2y-4xy^2-8y^3 \\ &= x^3+2x^2y+4xy^2-2x^2y-4xy^2-8y^3 \qquad \checkmark \\ &= x^3-8y^3 \end{aligned}$$

Answer

$(x-2y)\left(x^2+2xy+4y^2\right)$

It may be the case that the terms of the binomial have a common factor. If so, it will be difficult to identify it as a special binomial until we first factor out the GCF.

Example 4.2.11

Factor: $81x^4y+3xy^4$.

Solution

The terms are not perfect squares or perfect cubes. However, notice that they do have a common factor. First, factor out the GCF, $3xy$.

$$81x^4y+3xy^4=3xy\left(27x^3+y^3\right)$$

The resulting binomial factor is a sum of cubes with $a = 3x$ and $b = y$.

$$\begin{aligned} 81x^4y + 3xy^4 &= 3xy\left(27x^3 + y^3\right) \\ &= 3xy(3x+y)\left(9x^2 - 3xy + y^2\right) \end{aligned}$$

Answer

$3xy(3x+y)\left(9x^2 - 3xy + y^2\right)$

When the degree of the special binomial is greater than two, we may need to apply the formulas multiple times to obtain a complete factorization. A polynomial is *completely factored*[17] when it is prime or is written as a product of prime polynomials.

Example 4.2.12

Factor completely $x^4 - 81y^4$.

Solution

First, identify what is being squared.

$$x^4 - 81y^4 = (\ \ )^2 - (\ \ )^2$$

To do this, recall the power rule for exponents, $(x^m)^n = x^{mn}$. When exponents are raised to a power, multiply them. With this in mind, we find

$$x^4 - 81y^4 = \left(x^2\right)^2 - \left(9y^2\right)^2$$

Therefore, $a = x^2$ and $b = 9y^2$. Substitute into the formula for difference of squares.

$$x^4 - 81y^4 = \left(x^2 + 9y^2\right)\left(x^2 - 9y^2\right)$$

At this point, notice that the factor $(x^2 - 9y^2)$ is itself a difference of two squares and thus can be further factored using $a = x^2$ and $b = 3y$. The factor $(x^2 + 9y^2)$ is prime and cannot be factored using real numbers.

$$\begin{aligned} x^4 - 81y^4 &= \left(x^2 + 9y^2\right)\left(x^2 - 9y^2\right) \\ &= \left(x^2 + 9y^2\right)(x+3y)(x-3y) \end{aligned}$$

Answer

$\left(x^2 + 9y^2\right)(x+3y)(x-3y)$

When factoring, always look for resulting factors to factor further.

Example 4.2.13

Factor completely $64x^6 - y^6$.

Solution

This binomial is both a difference of squares and difference of cubes.

$$\begin{aligned} 64x^6 - y^6 &= \left(4x^2\right)^3 - \left(y^2\right)^3 \quad \textit{Difference of cubes} \\ 64x^6 - y^6 &= \left(8x^3\right)^2 - \left(y^3\right)^2 \quad \textit{Difference of squares} \end{aligned}$$

When confronted with a binomial that is a difference of both squares and cubes, as this is, make it a rule to factor using difference of squares first. Therefore, $a = 8x^3$ and $b = y^3$. Substitute into the difference of squares formula.

$$64x^6 - y^6 = \left(8x^3 + y^3\right)\left(8x^3 - y^3\right)$$

The resulting two binomial factors are sum and difference of cubes. Each can be factored further. Therefore, we have

$$\begin{aligned} 64x^6 - y^6 &= \overbrace{(2x+y)\left(4x^2 - 2xy + y^2\right)}^{\left(8x^3 + y^3\right)} \cdot \overbrace{(2x-y)\left(4x^2 + 2xy + y^2\right)}^{\left(8x^3 - y^3\right)} \end{aligned}$$

Figure 4.2.8

The trinomial factors are prime and the expression is completely factored.

Answer

$(2x+y)\left(4x^{2}-2xy+y^{2}\right)(2x-y)\left(4x^{2}+2xy+y^{2}\right)$

As an exercise, factor the previous example as a difference of cubes first and then compare the results. Why do you think we make it a rule to factor using difference of squares first?

Exercise 4.2.3

Factor: $a^{6}b^{6}-1$

Answer

$(ab+1)\left(a^{2}b^{2}-ab+1\right)(ab-1)\left(a^{2}b^{2}+ab+1\right)$

www.youtube.com/v/oZZXRfb7APo

Key Takeaways

- The GCF of two or more monomials is the product of the GCF of the coefficients and the common variable factors with the smallest power.
- If the terms of a polynomial have a greatest common factor, then factor out that GCF using the distributive property. Divide each term of the polynomial by the GCF to determine the terms of the remaining factor.
- Some four-term polynomials can be factored by grouping the first two terms and the last two terms. Factor out the GCF of each group and then factor out the common binomial factor.
- When factoring by grouping, you sometimes have to rearrange the terms to find a common binomial factor. After factoring out the GCF, the remaining binomial factors must be the same for the technique to work.
- When factoring special binomials, the first step is to identify it as a sum or difference. Once we identify the binomial, we then determine the values of a and b and then substitute into the appropriate formula.
- If a binomial is both a difference of squares and cubes, then first factor it as a difference of squares.

Exercise 4.2.4

Determine the GCF of the given expressions.

1. $9x^{5}, 27x^{2}, 15x^{7}$
2. $20y^{4}, 12y^{7}, 16y^{3}$
3. $50x^{2}y^{3}, 35xy^{3}, 10x^{3}y^{2}$
4. $12x^{7}y^{2}, 36x^{4}y^{2}, 18x^{3}y$
5. $15a^{7}b^{2}c^{5}, 75a^{7}b^{3}c, 45ab^{4}c^{3}$
6. $12a^{6}b^{3}c^{2}, 48abc^{3}, 125a^{2}b^{3}c$
7. $60x^{2}(2x-1)^{3}, 42x(2x-1)^{3}, 6x^{3}(2x-1)$
8. $14y^{5}(y-8)^{2}, 28y^{2}(y-8), 35y(y-8)^{3}$
9. $10a^{2}b^{3}(a+b)^{5}, 48a^{5}b^{2}(a+b)^{2}, 26ab^{5}(a+b)^{3}$
10. $45ab^{7}(a-b)^{7}, 36a^{2}b^{2}(a-b)^{3}, 63a^{4}b^{3}(a-b)^{2}$

Answer

1. $3x^{2}$

3. $5xy^{2}$

5. $15ab^{2}c$

7. $6x(2x-1)$

9. $2ab^{2}(a+b)^{2}$

Exercise 4.2.5

Determine the missing factor.

1. $18x^4 - 6x^3 + 2x^2 = 2x^2(\ ?\ )$
2. $6x^5 - 9x^3 - 3x = 3x(\ ?\ )$
3. $-10y^6 + 6y^4 - 4y^2 = -2y^2(\ ?\ )$
4. $-27y^9 - 9y^6 + 3y^3 = -3y^3(\ ?\ )$
5. $12x^3y^2 - 8x^2y^3 + 8xy = 4xy(\ ?\ )$
6. $10x^4y^3 - 50x^3y^2 + 15x^2y^2 = 5xy(\ ?\ )$
7. $14a^4b^5 - 21a^3b^4 - 7a^2b^3 = 7a^2b^3(\ ?\ )$
8. $15a^5b^4 + 9a^4b^2 - 3a^2b = 3a^2b(\ ?\ )$
9. $x^{3n} + x^{2n} + x^n = x^n(\ ?\ )$
10. $y^{4n} + y^{3n} - y^{2n} = y^{2n}(\ ?\ )$

Answer

1. $(9x^2 - 3x + 1)$
3. $(5y^4 - 3y^2 + 2)$
5. $(3x^2y - 2xy^2 + 2)$
7. $(2a^2b^2 - 3ab - 1)$
9. $(x^{2n} + x^n + 1)$

Exercise 4.2.6

Factor out the GCF.

1. $12x^4 - 16x^3 + 4x^2$
2. $15x^5 - 10x^4 - 5x^3$
3. $20y^8 + 28y^6 + 40y^3$
4. $18y^7 - 24y^5 - 30y^3$
5. $2a^4b^3 - 6a^3b^2 + 8a^2b$
6. $28a^3b^3 - 21a^2b^4 - 14ab^5$
7. $2x^3y^5 - 4x^4y^4 + x^2y^3$
8. $3x^5y - 2x^4y^2 + x^3y^3$
9. $5x^2(2x + 3) - 3(2x + 3)$
10. $y^2(y - 1) + 9(y - 1)$
11. $9x^2(3x - 1) + (3x - 1)$
12. $7y^2(5y + 2) - (5y + 2)$
13. $x^{5n} - x^{3n} + x^n$
14. $y^{6n} - y^{3n} - y^{2n}$

Answer

1. $4x^2\left(3x^2 - 4x + 1\right)$
3. $4y^3\left(5y^5 + 7y^3 + 10\right)$
5. $2a^2b\left(a^2b^2 - 3ab + 4\right)$
7. $x^2y^3\left(2xy^2 - 4x^2y + 1\right)$
9. $(2x + 3)\left(5x^2 - 3\right)$
11. $(3x - 1)\left(9x^2 + 1\right)$
13. $x^n\left(x^{4n} - x^{2n} + 1\right)$

Exercise $4.2.7$

Factor by grouping.

1. $2x^3+3x^2+2x+3$
2. $5x^3+25x^2+x+5$
3. $6x^3-3x^2+4x-2$
4. $3x^3-2x^2-15x+10$
5. x^3-x^2-3x+3
6. $6x^3-15x^2-2x+5$
7. $2x^3+7x^2-10x-35$
8. $3x^3-x^2+24x-8$
9. $14y^4+10y^3-7y-5$
10. $5y^4+2y^3+20y+8$
11. $x^{4n}+x^{3n}+2x^n+2$
12. $x^{5n}+x^{3n}+3x^{2n}+3$
13. $x^3-x^2y+xy^2-y^3$
14. $x^3+x^2y-2xy^2-2y^3$
15. $3x^3y^2+9x^2y^3-x-3y$
16. $2x^3y^3-x^2y^3+2x-y$
17. $a^2b-4ab^2-3a+12b$
18. $a^2b+3ab^2+5a+15b$
19. $a^4+a^2b^3+a^2b+b^4$
20. $a^3b+2a^2+3ab^4+6b^3$
21. $3ax+10by-5ay-6bx$
22. $a^2x-5b^2y-5a^2y+b^2x$
23. $x^4y^2-x^3y^3+x^2y^4-xy^5$
24. $2x^5y^2+4x^4y^2+18x^3y+36x^2y$
25. $a^5b^2+a^4b^4+a^3b^3+a^2b^5$
26. $3a^6b+3a^5b^2+9a^4b^2+9a^3b^3$

Answer

1. $(2x+3)\left(x^2+1\right)$

3. $(2x-1)\left(3x^2+2\right)$

5. $(x-1)\left(x^2-3\right)$

7. $(2x+7)\left(x^2-5\right)$

9. $(7y+5)\left(2y^3-1\right)$

11. $\left(x^n+1\right)\left(x^{3n}+2\right)$

13. $(x-y)\left(x^2+y^2\right)$

15. $(x+3y)\left(3x^2y^2-1\right)$

17. $(a-4b)(ab-3)$

19. $\left(a^2+b\right)\left(a^2+b^3\right)$

21. $(a-2b)(3x-5y)$

23. $xy^2(x-y)\left(x^2+y^2\right)$

25. $a^2b^2\left(a^2+b\right)\left(a+b^2\right)$

Exercise $4.2.8$

Factor.

1. $x^2 - 64$
2. $x^2 - 100$
3. $9 - 4y^2$
4. $25 - y^2$
5. $x^2 - 81y^2$
6. $x^2 - 49y^2$
7. $a^2b^2 - 4$
8. $1 - 9a^2b^2$
9. $a^2b^2 - c^2$
10. $4a^2 - b^2c^2$
11. $x^4 - 64$
12. $36 - y^4$
13. $(2x+5)^2 - x^2$
14. $(3x-5)^2 - x^2$
15. $y^2 - (y-3)^2$
16. $y^2 - (2y+1)^2$
17. $(2x+5)^2 - (x-3)^2$
18. $(3x-1)^2 - (2x-3)^2$
19. $x^4 - 16$
20. $81x^4 - 1$
21. $x^4y^4 - 1$
22. $x^4 - y^4$
23. $x^8 - y^8$
24. $y^8 - 1$
25. $x^{2n} - y^{2n}$
26. $x^{2n}y^{2n} - 4$
27. $x^{4n} - y^{4n}$
28. $x^{4n}y^{4n} - 16$
29. $x^3 - 27$
30. $8x^3 - 125$
31. $8y^3 + 27$
32. $64x^3 + 343$
33. $x^3 - y^3$
34. $x^3 + y^3$
35. $8a^3b^3 + 1$
36. $27a^3 - 8b^3$
37. $x^3y^3 - 125$
38. $216x^3 + y^3$
39. $x^3 + (x+3)^3$
40. $y^3 - (2y-1)^3$
41. $(2x+1)^3 - x^3$
42. $(3y-5)^3 - y^3$
43. $x^{3n} - y^{3n}$
44. $x^{3n} + y^{3n}$
45. $a^6 + 64$
46. $64a^6 - 1$
47. $x^6 - y^6$
48. $x^6 + y^6$
49. $x^{6n} - y^{6n}$

50. $x^{6n}+y^{6n}$
51. Given $f(x)=2x-1$, show that $(f+f)(x)=2f(x)$.
52. Given $f(x)=x^2-3x+2$, show that $(f+f)(x)=2f(x)$.
53. Given $f(x)=mx+b$, show that $(f+f)(x)=2f(x)$.
54. Given $f(x)=ax^2+bx+c$, show that $(f+f)(x)=2f(x)$.
55. Given $f(x)=ax^2+bx+c$, show that $(f-f)(x)=0$.
56. Given $f(x)=mx+b$, show that $(f-f)(x)=0$.

Answer

1. $(x+8)(x-8)$

3. $(3+2y)(3-2y)$

5. $(x+9y)(x-9y)$

7. $(ab+2)(ab-2)$

9. $(ab+c)(ab-c)$

11. $\left(x^2+8\right)\left(x^2-8\right)$

13. $(3x+5)(x+5)$

15. $3(2y-3)$

17. $(3x+2)(x+8)$

19. $\left(x^2+4\right)(x+2)(x-2)$

21. $\left(x^2y^2+1\right)(xy+1)(xy-1)$

23. $\left(x^4+y^4\right)\left(x^2+y^2\right)(x+y)(x-y)$

25. $\left(x^n+y^n\right)\left(x^n-y^n\right)$

27. $\left(x^{2n}+y^{2n}\right)\left(x^n+y^n\right)\left(x^n-y^n\right)$

29. $(x-3)\left(x^2+3x+9\right)$

31. $(2y+3)\left(4y^2-6y+9\right)$

33. $(x-y)\left(x^2+xy+y^2\right)$

35. $(2ab+1)\left(4a^2b^2-2ab+1\right)$

37. $(xy-5)\left(x^2y^2+5xy+25\right)$

39. $(2x+3)\left(x^2+3x+9\right)$

41. $(x+1)\left(7x^2+5x+1\right)$

43. $\left(x^n-y^n\right)\left(x^{2n}+x^ny^n+y^{2n}\right)$

45. $\left(a^2+4\right)\left(a^4-4a^2+16\right)$

47. $(x+y)\left(x^2-xy+y^2\right)(x-y)\left(x^2+xy+y^2\right)$

49. $\left(x^n+y^n\right)\left(x^{2n}-x^ny^n+y^{2n}\right)\times\left(x^n-y^n\right)\left(x^{2n}+x^ny^n+y^{2n}\right)$

51. Answer may vary

53. Answer may vary

55. Answer may vary

Exercise $4.2.9$

1. What can be said about the degree of a factor of a polynomial? Give an example.
2. If a binomial falls into both categories, difference of squares and difference of cubes, which would be best to use for factoring, and why? Create an example that illustrates this situation and factor it using both formulas.
3. Write your own examples for each of the three special types of binomial. Factor them and share your results.

Answer

1. Answer may vary

3. Answer may vary

Footnotes

[5] The process of writing a number or expression as a product.

[6] Any combination of factors, multiplied together, resulting in the product.

[7] Any of the numbers or expressions that form a product.

[8] The product of the common variable factors and the GCF of the coefficients.

[9] Expressions that share no common factors other than 1.

[10] The process of rewriting a polynomial as a product using the GCF of all of its terms.

[11] A polynomial with integer coefficients that cannot be factored as a product of polynomials with integer coefficients other than 1 and itself.

[12] A technique for factoring polynomials with four terms.

[13] $a^2-b^2=(a+b)(a-b)$, where a and b represent algebraic expressions.

[14] a^2+b^2, where a and b represent algebraic expressions. This does not have a general factored equivalent.

[15] $a^3+b^3=(a+b)(a^2-ab+b^2)$ where a and b represent algebraic expressions.

[16] $a^3-b^3=(a-b)(a^2+ab+b^2)$ where a and b represent algebraic expressions.

[17] A polynomial that is prime or written as a product of prime polynomials.

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4.3: Factoring Trinomials

Learning Objectives

- Factor trinomials of the form $x^2 + bx + c$.
- Factor trinomials of higher degree.
- Factor trinomials of the form $ax^2 + bx + c$.
- Factor trinomials using the AC method.

Factoring Trinomials Whose Leading Coefficient Is One

(Factoring Trinomials of the Form $x^2 + bx + c$)

Some trinomials of the form $x^2 + bx + c$ can be factored as a product of binomials. If a trinomial of this type factors, then we have:

$$\begin{aligned} x^2 + bx + c &= (x + m)(x + n) \\ &= x^2 + nx + mx + mn \\ &= x^2 + (n + m)x + mn \end{aligned}$$

This gives us

$$b = n + m \quad \text{and} \quad c = mn$$

In short, if the leading coefficient of a factorable trinomial is 1, then the factors of the last term must add up to the coefficient of the middle term. This observation is the key to factoring trinomials using the technique known as the **trial and error (or guess and check) method**[18].

Example $4.3.1$:

Factor $x^2 + 12x + 20$.

Solution

We begin by writing two sets of blank parentheses. If a trinomial of this form factors, then it will factor into two linear binomial factors.

$$x^2 + 12x + 20 = (\quad)(\quad)$$

Write the factors of the first term in the first space of each set of parentheses. In this case, factor $x^2 = x \cdot x$.

$$x^2 + 12x + 20 = (x \quad)(x \quad)$$

Determine the factors of the last term whose sum equals the coefficient of the middle term. To do this, list all of the factorizations of 20 and search for factors whose sum equals 12.

$$\begin{aligned} 20 &= 1 \cdot 20 \;\rightarrow\; 1 + 20 = 21 \\ &= 2 \cdot 10 \;\rightarrow\; 2 + 10 = 12 \\ &= 4 \cdot 5 \;\rightarrow\; 4 + 5 = 9 \end{aligned}$$

Choose $20 = 2 \cdot 10$ because $2 + 10 = 12$. Write in the last term of each binomial using the factors determined in the previous step.

$$x^2 + 12x + 20 = (x + 2)(x + 10)$$

This can be visually interpreted as follows:

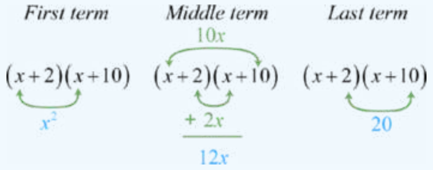


Figure $4.3.1$

Check by multiplying the two binomials.

$$\begin{aligned} (x + 2)(x + 10) &= x^2 + 10x + 2x + 20 \\ &= x^2 + 12x + 20 \ \checkmark \end{aligned}$$

Answer

$$(x+2)(x+10)$$

Since multiplication is commutative, the order of the factors does not matter.

$$\begin{aligned} x^2+12x+20 &= (x+2)(x+10) \\ &= (x+10)(x+2) \end{aligned}$$

If the last term of the trinomial is positive, then either both of the constant factors must be negative or both must be positive.

Example $4.3.2$:

Factor $x^2y^2-7xy+12$.

Solution

First, factor $x^2y^2 = xy \cdot xy$.

$$x^2y^2-7xy+12 = (xy \quad ?)(xy \quad ?)$$

Next, search for factors of 12 whose sum is -7.

$$\begin{aligned} 12 &= 1\cdot 12 & \rightarrow \quad & -1+(-12)=-13 \\ &= 2\cdot 6 & \rightarrow \quad & -2+(-6)=-8 \\ &= 3\cdot 4 & \rightarrow \quad & -3+(-4)=-7 \end{aligned}$$

In this case, choose -3 and -4 because $(-3)(-4)=+12$ and $-3+(-4)=-7$.

$$\begin{aligned} x^2y^2-7xy+12 &= (xy \quad ?)(xy \quad ?) \\ &= (xy-3)(xy-4) \end{aligned}$$

Check

$$\begin{aligned} (xy-3)(xy-4) &= x^2y^2-4xy-3xy+12 \\ &= x^2y^2-7xy+12 \ \checkmark \end{aligned}$$

Answer

$(xy-3)(xy-4)$

If the last term of the trinomial is negative, then one of its factors must be negative.

Example $4.3.3$:

Factor: $x^2-4xy-12y^2$.

Solution

Begin by factoring the first term $x^2 = x\cdot x$.

$$x^2-4xy-12y^2 = (\ x \quad ?\ )(\ x \quad ?\ )$$

The factors of 12 are listed below. In this example, we are looking for factors whose sum is -4.

$$\begin{aligned} 12 &= 1\cdot 12 & \rightarrow \quad & 1+(-12)=-11 \\ &= 2\cdot 6 & \rightarrow \quad & 2+(-6)=-4 \\ &= 3\cdot 4 & \rightarrow \quad & 3+(-4)=-1 \end{aligned}$$

Therefore, the coefficient of the last term can be factored as $-12=2(-6)$, where $2+(-6)=-4$. Because the last term has a variable factor of y^2, use $-12y^2=2y(-6y)$ and factor the trinomial as follows:

$$\begin{aligned} x^2-4xy-12y^2 &= (x \quad ?)(x \quad ?) \\ &= (x+2y)(x-6y) \end{aligned}$$

Multiply to check.

$$\begin{aligned} (x+2y)(x-6y) &= x^2-6xy+2yx-12y^2 \\ &= x^2-6xy+2xy-12y^2 \\ &= x^2-4xy-12y^2 \ \checkmark \end{aligned}$$

Answer

$(x+2y)(x-6y)$

Often our first guess will not produce a correct factorization. This process may require repeated trials. For this reason, the check is very important and is not optional.

Example 4.3.4:

Factor $a^2+10a-24$.

Solution

The first term of this trinomial, a^2, factors as $a \cdot a$.

$$a^2+10a-24=(\ a\quad ?\ )(\ a\quad ?\ )$$

Consider the factors of 24:

$$\begin{aligned} 24 &= 1\cdot 24 \\ &= 2\cdot 12 \\ &= 3\cdot 8 \\ &= 4\cdot 6 \end{aligned}$$

Suppose we choose the factors 4 and 6 because $4+6=10$, the coefficient of the middle term. Then we have the following incorrect factorization:

$$a^2+10a-24 \overset{?}{=} (a+4)(a+6) \quad \textit{Incorrect Factorization}$$

When we multiply to check, we find the error.

$$\begin{aligned} (a+4)(a+6) &= a^2+6a+4a+24 \\ &= a^2+10a+24 \ ✗ \end{aligned}$$

In this case, the middle term is correct but the last term is not. Since the last term in the original expression is negative, we need to choose factors that are opposite in sign. Therefore, we must try again. This time we choose the factors -2 and 12 because $-2+12=10$.

$$a^2+10a-24=(a-2)(a+12)$$

Now the check shows that this factorization is correct.

$$\begin{aligned} (a-2)(a+12) &= a^2+12a-2a-24 \\ &= a^2+10a-24 \ ✓ \end{aligned}$$

Answer

$(a-2)(a+12)$

If we choose the factors wisely, then we can reduce much of the guesswork in this process. However, if a guess is not correct, do not get discouraged; just try a different set of factors. Keep in mind that some polynomials are prime. For example, consider the trinomial $x^2+3x+20$ and the factors of 20:

$$\begin{aligned} 20 &= 1\cdot 20 \\ &= 2\cdot 10 \\ &= 4\cdot 5 \end{aligned}$$

There are no factors of 20 whose sum is 3. Therefore, the original trinomial cannot be factored as a product of two binomials with integer coefficients. The trinomial is prime.

Factoring Trinomials of Higher Degree

We can use the trial and error technique to factor trinomials of higher degree.

Example 4.3.5:

Factor x^4+6x^2+5.

Solution

Begin by factoring the first term $x^4=x^2\cdot x^2$.

$$x^4+6x^2+5=(x^2\quad ?)\,(x^2\quad ?)$$

Since 5 is prime and the coefficient of the middle term is positive, choose $+1$ and $+5$ as the factors of the last term.

$$\begin{aligned} x^4+6x^2+5 &= (x^2\quad ?)\,(x^2\quad ?) \\ &= (x^2+1)\,(x^2+5) \end{aligned}$$

Notice that the variable part of the middle term is x^2 and the factorization checks out.

$$\begin{aligned} (x^2+1)\,(x^2+5) &= x^4+5x^2+x^2+5 \\ &= x^4+6x^2+5\ \checkmark \end{aligned}$$

Answer

$(x^2+1)\,(x^2+5)$

Example $4.3.6$:

Factor: $x^{2n}+4x^n-21$ where n is a positive integer.

Solution

Begin by factoring the first term $x^{2n}=x^n\cdot x^n$.

$$x^{2n}+4x^n-21=(\ x^n\quad ?\ )\,(\ x^n\quad ?\ )$$

Factor $-21\ =7(-3)$ because $7+(-3)=+4$ and write

$$\begin{aligned} x^{2n}+4x^n-21 &= (x^n\quad ?)\,(x^n\quad ?) \\ &= (x^n+7)\,(x^n-3) \end{aligned}$$

Answer

$$(x^n+7)\,(x^n-3)$$

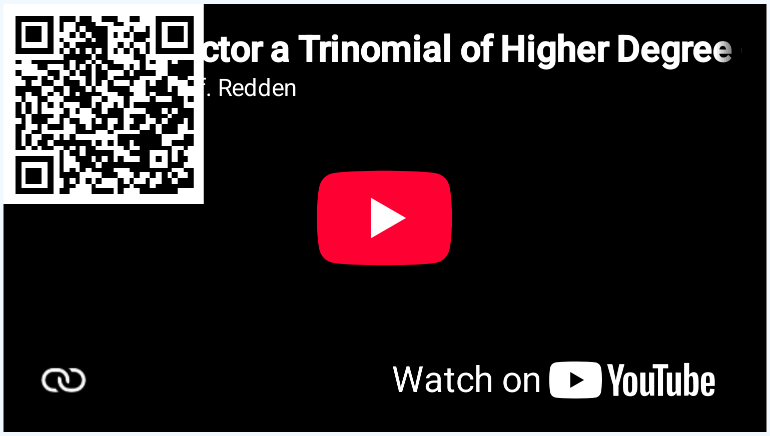
The check is left to the reader.

Exercise $4.3.1$

Factor x^6-x^3-42.

Answer

$$(x^3+6)\,(x^3-7)$$



Factoring Trinomials Whose Leading Coefficient Is Not One

(Factoring Trinomials of the Form ax^2+bx+c)

Factoring trinomials of the form ax^2+bx+c can be challenging because the middle term is affected by the factors of both a and c. In general,

$$\begin{aligned} ax^2+bx+c &= (px+m)(qx+n) \\ &= pqx^2+pnx+qmx+mn \\ &= pqx^2+(pn+qm)x+mn \end{aligned}$$

This gives us,

$$a=pq \quad \text{and} \quad b=pn+qm, \quad \text{where} \quad c=mn$$

In short, when the leading coefficient of a trinomial is something other than 1, there will be more to consider when determining the factors using the trial and error method. The key lies in the understanding of how the middle term is obtained. Multiply $(5x+3)(2x+3)$ and carefully follow the formation of the middle term.

$$\begin{aligned} (5x+3)(2x+3) &= 5x\cdot 2x \underbrace{+5x\cdot 3+3\cdot 2x}_{\text{middle term}} +3\cdot 3 \\ &= 10x^2+15x+6x+9 \\ &= 15x^2+21x+9 \end{aligned}$$

Figure $4.3.2$

As we have seen before, the product of the first terms of each binomial is equal to the first term of the trinomial. The middle term of the trinomial is the sum of the products of the outer and inner terms of the binomials. The product of the last terms of each binomial is equal to the last term of the trinomial. Visually, we have the following:

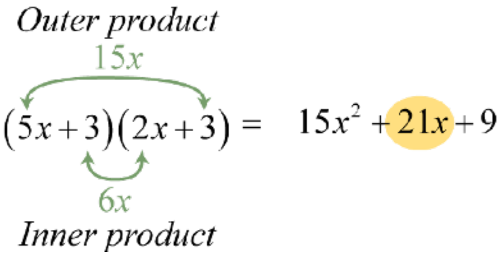


Figure $4.3.3$

For this reason, we need to look for products of the factors of the first and last terms whose sum is equal to the coefficient of the middle term. For example, to factor $6x^2+29x+35$, look at the factors of 6 and 35.

$$\begin{aligned} 6 &= 1\cdot 635 = 1\cdot 35 \\ &= 2\cdot 3 \quad = 5\cdot 7 \end{aligned}$$

The combination that produces the coefficient of the middle term is $2\cdot 7+3\cdot 5=14+15=29$. Make sure that the outer terms have coefficients 2 and 7, and that the inner terms have coefficients 5 and 3. Use this information to factor the trinomial.

$$\begin{aligned} 6x^2+29x+35 &= (2x\quad ?)(3x\quad ?) \\ &= (2x+5)(3x+7) \end{aligned}$$

We can always check by multiplying; this is left to the reader.

Example $4.3.7$

Factor $5x^2+16xy+3y^2$.

Solution

Since the leading coefficient and the last term are both prime, there is only one way to factor each.

$$5=1\cdot 5 \quad \text{and} \quad 3=1\cdot 3$$

Begin by writing the factors of the first term, $5x^2$, as follows:

$$5x^2+16xy+3y^2=(x\quad ?)(5x\quad ?)$$

The middle and last term are both positive; therefore, the factors of 3 are chosen as positive numbers. In this case, the only choice is in which grouping to place these factors.

$$(x+y)(5x+3y) \qquad \text{or} \qquad (x+3y)(5x+y)$$

Determine which grouping is correct by multiplying each expression.

$$\begin{aligned}(x+y)(5x+3y) &= 5x^2+3xy+5xy+3y^2\\ &= 5x^2+8xy+3y^2x \ ✗\\ (x+3y)(5x+y) &= 5x^2+xy+15xy+3y^2\\ &= 5x^2+16xy+3y^2 \ ✓\end{aligned}$$

Answer

$(x+3y)(5x+y)$

Example 4.3.8

Factor: $18a^2b^2-ab-4$.

Solution

First, consider the factors of the coefficients of the first and last terms.

$$\begin{aligned}18 &= 1\cdot 18 \quad & 4 &= 1\cdot 4\\ &= 2\cdot 9 & &= 2\cdot 2\\ &= 3\cdot 6\end{aligned}$$

We are searching for products of factors whose sum equals the coefficient of the middle term, -1. After some thought, we can see that the sum of 8 and -9 is -1 and the combination that gives this follows:

$$2(4)+9(-1)=8-9=-1$$

Factoring begins at this point with two sets of blank parentheses.

$$18a^2b^2-ab-4=(\quad)(\quad)$$

Use $2ab$ and $9ab$ as factors of $18a^2b^2$.

$$18a^2b^2-ab-4=(2ab \quad ?)\,(9ab \quad ?)$$

Next use the factors 1 and 4 in the correct order so that the inner and outer products are $-9ab$ and $8ab$ respectively.

$$18a^2b^2-ab-4=(2ab-1)(9ab+4)$$

Answer

$(2ab-1)(9ab+4)$. The complete check is left to the reader.

It is a good practice to first factor out the GCF, if there is one. Doing this produces a trinomial factor with smaller coefficients. As we have seen, trinomials with smaller coefficients require much less effort to factor. This commonly overlooked step is worth identifying early.

Example 4.3.9:

Factor $12y^3-26y^2-10y$.

Solution

Begin by factoring out the GCF.

$$12y^3-26y^2-10y=2y\left(6y^2-13y-5\right)$$

After factoring out $2y$, the coefficients of the resulting trinomial are smaller and have fewer factors. We can factor the resulting trinomial using $6=2(3)$ and $5=(5)(1)$. Notice that these factors can produce -13 in two ways:

$$\begin{aligned}&2(-5)+3(-1)=-10-3=-13\\ &2(1)+3(-5)=2-15=-13\end{aligned}$$

Because the last term is -5, the correct combination requires the factors 1 and 5 to be opposite signs. Here we use $2(1)=2$ and $3(-5)=-15$ because the sum is -13 and the product of $(1)(-5)=-5$.

$$\begin{aligned}12y^3-26y^2-10y &= 2y\left(6y^2-13y-5\right)\\ &= 2y(2y \quad ?)(3y \quad ?)\\ &= 2y(2y-5)(3y+1)\end{aligned}$$

Check.

$$\begin{aligned} 2y(2y-5)(3y+1) &= 2y\left(6y^2+2y-15y-5\right) \\ &= 2y\left(6y^2-13y-5\right) \\ &= 12y^3-26y^2-10y\checkmark \end{aligned}$$

The factor $2y$ is part of the factored form of the original expression; be sure to include it in the answer.

Answer

$2y(2y-5)(3y+1)$

It is a good practice to consistently work with trinomials where the leading coefficient is positive. If the leading coefficient is negative, factor it out along with any GCF. Note that sometimes the factor will be -1.

Example 4.3.10

Factor: $-18x^6-69x^4+12x^2$.

Solution

In this example, the GCF is $3x^2$. Because the leading coefficient is negative we begin by factoring out $-3x^2$.

$$-18x^6-69x^4+12x^2=-3x^2\left(6x^4+23x^2-4\right)$$

At this point, factor the remaining trinomial as usual, remembering to write the $-3x^2$ as a factor in the final answer. Use 6 $= 1(6)$ and $-4=4(-1)$because $1(-1)+6(4)=23$. Therefore,

$$\begin{aligned} -18x^6-69x^4+12x^2 &= -3x^2\left(6x^4+23x^2-4\right) \\ &= -3x^2\left(x^2\quad\right)\left(6x^2\quad\right) \\ &= -3x^2\left(x^2+4\right)\left(6x^2-1\right) \end{aligned}$$

Answer

$-3x^2\left(x^2+4\right)\left(6x^2-1\right)$. The check is left to the reader.

Exercise 4.3.2

Factor: $-12a^5b+a^3b^3+ab^5$.

Answer

$-ab\left(3a^2-b^2\right)\left(4a^2+b^2\right)$

www.youtube.com/v/aigcuuTobk4

Factoring Using the AC Method

An alternate technique for factoring trinomials, called the **AC method**[19], makes use of the grouping method for factoring four-term polynomials. If a trinomial in the form ax^2+bx+c can be factored, then the middle term, bx, can be replaced with two terms with coefficients whose sum is b and product is ac. This substitution results in an equivalent expression with four terms that can be factored by grouping.

Example 4.3.11:

Factor using the AC method: $18x^2-31x+6$.

Solution

Here $a=18, b=-31$, and $c=6$.

$$\begin{aligned} ac &= 18(6) \\ &= 108 \end{aligned}$$

Factor 108, and search for factors whose sum is -31.

$$\begin{aligned} 108 &= -1(-108) \\ &= -2(-54) \end{aligned}$$

$$\begin{aligned} &= -3(-36) \\ &= -4(-27)\checkmark \\ &= -6(-18) \\ &= -9(-12) \end{aligned} \tag{4.3.1}$$

In this case, the sum of the factors -27 and -4 equals the middle coefficient, -31. Therefore, $-31x = -27x - 4x$, and we can write

$$18x^2 - 31x + 6 = 18x^2 - 27x - 4x + 6$$

Factor the equivalent expression by grouping.

$$\begin{aligned} 18x^2 - 31x + 6 &= 18x^2 - 27x - 4x + 6 \\ &= 9x(2x-3) - 2(2x-3) \\ &= (2x-3)(9x-2) \end{aligned}$$

Answer

$(2x-3)(9x-2)$

Example 4.3.12:

Factor using the AC method: $4x^2y^2 - 7xy - 15$.

Solution

Here $a = 4, b = -7$, and $c = -15$.

$$\begin{aligned} ac &= 4(-15) \\ &= -60 \end{aligned}$$

Factor -60 and search for factors whose sum is -7.

$$\begin{aligned} -60 &= 1(-60) \\ &= 2(-30) \\ &= 3(-20) \\ &= 4(-15) \\ &= 5(-12)\ \checkmark \\ &= 6(-10) \end{aligned}$$

The sum of factors 5 and -12 equals the middle coefficient, -7. Replace $-7xy$ with $5xy - 12xy$.

$$\begin{aligned} 4x^2y^2 - 7xy - 15 &= 4x^2y^2 + 5xy - 12xy - 15 \quad \textit{Factor by grouping.} \\ &= xy(4xy+5) - 3(4xy+5) \\ &= (4xy+5)(xy-3) \end{aligned}$$

Answer

$$(4xy+5)(xy-3). \tag{4.3.2}$$

The check is left to the reader.

If factors of ac cannot be found to add up to b then the trinomial is prime.

Key Takeaways

- If a trinomial of the form $x^2 + bx + c$ factors into the product of two binomials, then the coefficient of the middle term is the sum of factors of the last term.
- If a trinomial of the form $ax^2 + bx + c$ factors into the product of two binomials, then the coefficient of the middle term will be the sum of certain products of factors of the first and last terms.
- If the trinomial has a greatest common factor, then it is a best practice to first factor out the GCF before attempting to factor it into a product of binomials.
- If the leading coefficient of a trinomial is negative, then it is a best practice to first factor that negative factor out before attempting to factor the trinomial.
- Factoring is one of the more important skills required in algebra. For this reason, you should practice working as many problems as it takes to become proficient.

Exercise 4.3.3

Factor.

1. x^2+5x-6
2. x^2+5x+6
3. $x^2+4x-12$
4. $x^2+3x-18$
5. $x^2-14x+48$
6. $x^2-15x+54$
7. $x^2+11x-30$
8. $x^2-2x+24$
9. $x^2-18x+81$
10. $x^2-22x+121$
11. $x^2-xy-20y^2$
12. $x^2+10xy+9y^2$
13. $x^2y^2+5xy-50$
14. $x^2y^2-16xy+48$
15. $a^2-6ab-72b^2$
16. $a^2-21ab+80b^2$
17. $u^2+14uv-32v^2$
18. $m^2+7mn-98n^2$
19. $(x+y)^2-2(x+y)-8$
20. $(x-y)^2-2(x-y)-15$
21. x^4-7x^2-8
22. x^4+13x^2+30
23. x^4-8x^2-48
24. x^4+25x^2+24
25. y^4-20y^2+100
26. y^4+14y^2+49
27. $x^4+3x^2y^2+2y^4$
28. $x^4-8x^2y^2+15y^4$
29. $a^4b^4-4a^2b^2+4$
30. $a^4+6a^2b^2+9b^4$
31. x^6-18x^3-40
32. x^6+18x^3+45
33. $x^6-x^3y^3-6y^6$
34. $x^6+x^3y^3-20y^6$
35. $x^6y^6+2x^3y^3-15$
36. $x^6y^6+16x^3y^3+48$
37. $x^{2n}+12x^n+32$
38. $x^{2n}+41x^n+40$
39. $x^{2n}+2ax^n+a^2$
40. $x^{2n}-2ax^n+a^2$

Answer

1. $(x-1)(x+6)$

3. $(x-2)(x+6)$

5. $(x-6)(x-8)$

7. Prime

9. $(x-9)^2$

11. $(x-5y)(x+4y)$

13. $(xy-5)(xy+10)$

15. $(a+6b)(a-12b)$

17. $(u-2v)(u+16v)$

19. $(x+y-4)(x+y+2)$

21. $\left(x^2-8\right)\left(x^2+1\right)$

23. $\left(x^2+4\right)\left(x^2-12\right)$

25. $\left(y^2-10\right)^2$

27. $\left(x^2+y^2\right)\left(x^2+2y^2\right)$

29. $\left(a^2b^2-2\right)^2$

31. $\left(x^3-20\right)\left(x^3+2\right)$

33. $\left(x^3+2y^3\right)\left(x^3-3y^3\right)$

35. $\left(x^3y^3-3\right)\left(x^3y^3+5\right)$

37. $\left(x^n+4\right)\left(x^n+8\right)$

39. $\left(x^n+a\right)^2$

Exercise 4.3.4

Factor.

1. $3x^2+20x-7$
2. $2x^2-9x-5$
3. $6a^2+13a+6$
4. $4a^2+11a+6$
5. $6x^2+7x-10$
6. $4x^2-25x+6$
7. $24y^2-35y+4$
8. $10y^2-23y+12$
9. $14x^2-11x+9$
10. $9x^2+6x+8$
11. $4x^2-28x+49$
12. $36x^2-60x+25$
13. $27x^2-6x-8$
14. $24x^2+17x-20$
15. $6x^2+23xy-4y^2$
16. $10x^2-21xy-27y^2$
17. $8a^2b^2-18ab+9$
18. $12a^2b^2-ab-20$
19. $8u^2-26uv+15v^2$
20. $24m^2-26mn+5n^2$
21. $4a^2-12ab+9b^2$
22. $16a^2+40ab+25b^2$
23. $5(x+y)^2-9(x+y)+4$
24. $7(x-y)^2+15(x-y)-18$
25. $7x^4-22x^2+3$
26. $5x^4-41x^2+8$
27. $4y^6-3y^3-10$
28. $12y^6+4y^3-5$
29. $5a^4b^4-a^2b^2-18$

30. $21a^4b^4 + 5a^2b^2 - 4$
31. $6x^6y^6 + 17x^3y^3 + 10$
32. $16x^6y^6 + 46x^3y^3 + 15$
33. $8x^{2n} - 10x^n - 25$
34. $30x^{2n} - 11x^n - 6$
35. $36x^{2n} + 12ax^n + a^2$
36. $9x^{2n} - 12ax^n + 4a^2$
37. $-3x^2 + 14x + 5$
38. $-2x^2 + 13x - 20$
39. $-x^2 - 10x + 24$
40. $-x^2 + 8x + 48$
41. $54 - 12x - 2x^2$
42. $60 + 5x - 5x^2$
43. $4x^3 + 16x^2 + 20x$
44. $2x^4 - 12x^3 + 14x^2$
45. $2x^3 - 8x^2y - 24xy^2$
46. $6x^3 - 9x^2y - 6xy^2$
47. $4a^3b - 4a^2b^2 - 24ab^3$
48. $15a^4b - 33a^3b^2 + 6a^2b^3$
49. $3x^5y + 30x^3y^3 + 75xy^5$
50. $45x^5y^2 - 60x^3y^4 + 20xy^6$

Answer

1. $(3x-1)(x+7)$

3. $(2a+3)(3a+2)$

5. $(6x-5)(x+2)$

7. $(8y-1)(3y-4)$

9. Prime

11. $(2x-7)^2$

13. $(9x+4)(3x-2)$

15. $(6x-y)(x+4y)$

17. $(4ab-3)(2ab-3)$

19. $(2u-5v)(4u-3v)$

21. $(2a-3b)^2$

23. $(x+y-1)(5x+5y-4)$

25. $(x^2-3)(7x^2-1)$

27. $(y^3-2)(4y^3+5)$

29. $(a^2b^2-2)(5a^2b^2+9)$

31. $(6x^3y^3+5)(x^3y^3+2)$

33. $(2x^n-5)(4x^n+5)$

35. $(6x^n+a)^2$

37. $-(x-5)(3x+1)$

39. $-(x-2)(x+12)$

41. $-2(x-3)(x+9)$

43. $4x\left(x^{2}+4x+5\right)$

45. $2x(x+2y)(x-6y)$

47. $4ab(a-3b)(a+2b)$

49. $3xy\left(x^{2}+5y^{2}\right)^{2}$

Exercise 4.3.5

Factor.

1. $4-25x^{2}$
2. $8x^{3}-y^{3}$
3. $9x^{2}-12xy+4y^{2}$
4. $30a^{2}-57ab-6b^{2}$
5. $10a^{2}-5a-6ab+3b$
6. $3x^{3}-4x^{2}+9x-12$
7. $x^{2}+4y^{2}$
8. $x^{2}-x+2$
9. $15a^{3}b^{2}+6a^{2}b^{3}-3ab^{4}$
10. $54x^{2}-63x$

Answer

1. $(2-5x)(2+5x)$

3. $(3x-2y)^{2}$

5. $(2a-1)(5a-3b)$

7. Prime

9. $3ab^{2}\left(5a^{2}+2ab-b^{2}\right)$

Exercise 4.3.6

1. Create your own trinomial of the form $ax^{2}+bx+c$ that factors. Share it, along with the solution, on the discussion board.
2. Create a trinomial of the form $ax^{2}+bx+c$ that does not factor and share it along with the reason why it does not factor.

Answer

1. Answer may vary

Footnotes

[18] Describes the method of factoring a trinomial by systematically checking factors to see if their product is the original trinomial.

[19] Method used for factoring trinomials by replacing the middle term with two terms that allow us to factor the resulting four-term polynomial by grouping.

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4.4: Solve Polynomial Equations by Factoring

Learning Objectives

- Review general strategies for factoring.
- Solve polynomial equations by factoring.
- Find roots of a polynomial function.
- Find polynomial equations given the solutions.

Reviewing General Factoring Strategies

We have learned various techniques for factoring polynomials with up to four terms. The challenge is to identify the type of polynomial and then decide which method to apply. The following outlines a general guideline for factoring polynomials.

general guidelines for factoring polynomials

Step 1: Check for common factors. If the terms have common factors, then factor out the greatest common factor (GCF).

Step 2: Determine the number of terms in the polynomial.

1. Factor four-term polynomials by grouping.
2. Factor trinomials (3 terms) using “trial and error” or the AC method.
3. Factor binomials (2 terms) using the following special products:
    - **Difference of squares**:$a^{2}-b^{2}=(a+b)(a-b)$
    - **Sum of squares**: $a^{2}+b^{2}$ no general formula
    - **Difference of cubes**: $a^{3}-b^{3}=(a-b)(a^{2}+ab+b^{2})$
    - **Sum of cubes**: $a^{3}+b^{3}=(a+b)(a^{2}-ab+b^{2})$

Step 3: Look for factors that can be factored further.

Step 4: Check by multiplying.

If a binomial is both a difference of squares and a difference cubes, then first factor it as difference of squares. This will result in a more complete factorization. In addition, not all polynomials with integer coefficients factor. When this is the case, we say that the polynomial is prime.

If an expression has a GCF, then factor this out first. Doing so is often overlooked and typically results in factors that are easier to work with. Furthermore, look for the resulting factors to factor further; many factoring problems require more than one step. A polynomial is completely factored when none of the factors can be factored further.

Example $4.4.1$:

Factor $54x^{4}-36x^{3}-24x^{2}+16x$.

Solution

This four-term polynomial has a GCF of $2x$. Factor this out first.

$$54x^{4}-36x^{3}-24x^{2}+16x=2x\left(27x^{3}-18x^{2}-12x+8\right)$$

Now factor the resulting four-term polynomial by grouping and look for resulting factors to factor further.

$$\begin{aligned} 54x^4-36x^3-24x^2+16x &= 2x\left(\underbrace{27x^3-18x^2}_{\text{group}}\underbrace{-12x+8}_{\text{group}}\right) \\ &= 2x\left(9x^2(3x-2)-4(3x-2)\right) \\ &= 2x(3x-2)\underbrace{\left(9x^2-4\right)}_{\substack{\text{difference}\\ \text{of squares}}} \\ &= 2x(3x-2)(3x-2)(3x+2) \end{aligned}$$

Figure 4.4.1

Answer

$2x(3x-2)^2(3x+2)$. The check is left to the reader.

Example 4.4.2:

Factor: x^4-3x^2-4.

Solution

This trinomial does not have a GCF.

$$\begin{aligned} x^4-3x^2-4 &= \left(x^2 \quad\right)\left(x^2 \quad\right) \\ &= \left(x^2+1\right)\left(x^2-4\right) \quad \color{blue}{Difference\ of\ squares} \\ &= \left(x^2+1\right)(x+2)(x-2) \end{aligned}$$

The factor $\left(x^2+1\right)$ is prime and the trinomial is completlely factored.

Answer:

$\left(x^2+1\right)(x+2)(x-2)$

Example 4.4.3:

Factor: x^6+6x^3-16.

Solution

Begin by factoring $x^6=x^3\cdot x^3$ and look for the factors of 16 that add to 6.

$$\begin{aligned} x^6+6x^3-16 &= \left(x^3 \quad\right)\left(x^3 \quad\right) \\ &= \left(x^3-2\right)\left(x^3+8\right) \quad \color{blue}{sum\ of\ cubes} \\ &= \left(x^3-2\right)(x+2)\left(x^2-2x+4\right) \end{aligned}$$

The factor $\left(x^3-2\right)$ cannot be factored any further using integers and the factorization is complete.

Answer:

$\left(x^3-2\right)(x+2)\left(x^2+2x+4\right)$

Exercise 4.4.1

Factor: $9x^4+17x^2-2$

Answer

$(3x+1)(3x-1)\left(x^2+2\right)$

www.youtube.com/v/lsAP8_UgUx0

Solving Polynomial Equations by Factoring

In this section, we will review a technique that can be used to solve certain polynomial equations. We begin with the *zero-product property*[20]:

$$a \cdot b=0 \quad \text{if and only if} \quad a=0 \quad \text{or} \quad b=0$$

The zero-product property is true for any number of factors that make up an equation. In other words, if any product is equal to zero, then at least one of the variable factors must be equal to zero. If an expression is equal to zero and can be factored into linear factors, then we will be able to set each factor equal to zero and solve for each equation.

Example 4.4.4

Solve: $2x(x-4)(5x+3)=0$.

Solution

Set each variable factor equal to zero and solve.

$$\begin{aligned} 2x&=0\\ \frac{2x}{2}&=\frac{0}{2}\\ x&=0 \end{aligned} \quad \text{or} \quad \begin{aligned} x-4&=0\\ x&=4 \end{aligned} \quad \text{or} \quad \begin{aligned} 5x+3&=0\\ \frac{5x}{5}&=\frac{-3}{5}\\ x&=-\frac{3}{5} \end{aligned}$$

To check that these are solutions we can substitute back into the original equation to see if we obtain a true statement. Note that each solution produces a zero factor. This is left to the reader.

Answer:

The solutions are $0, 4$, and $\frac{-3}{5}$.

Of course, most equations will not be given in factored form.

Example 4.4.5

Solve: $4x^3-x^2-100x+25=0$.

Solution

Begin by factoring the left side completely.

$$\begin{aligned} 4x^3-x^2-100x+25&=0 && \textit{Factor by grouping.}\\ x^2(4x-1)-25(4x-1)&=0\\ (4x-1)\left(x^2-25\right)&=0 && \textit{Factor as a difference of squares.}\\ (4x-1)(x+5)(x-5)&=0 \end{aligned}$$

Set each factor equal to zero and solve.

$$\begin{aligned} 4x-1&=0\\ 4x&=1\\ x&=\tfrac{1}{4} \end{aligned} \quad \text{or} \quad \begin{aligned} x+5&=0\\ x&=-5 \end{aligned} \quad \text{or} \quad \begin{aligned} x-5&=0\\ x&=5 \end{aligned}$$

Answer:

The solutions are $\frac{1}{4}, -5$, and 5.

Using the zero-product property after factoring an equation that is equal to zero is the key to this technique. However, the equation may not be given equal to zero, and so there may be some preliminary steps before factoring. The steps required to **solve by factoring**[21] are outlined in the following example.

Example 4.4.6

Solve: $15x^2+3x-8=5x-7$.

Solution

Step 1: Express the equation in standard form, equal to zero. In this example, subtract $5x$ from and add 7 to both sides.

$$\begin{aligned} 15x^2+3x-8&=5x-7\\ 15x^2-2x-1&=0 \end{aligned}$$

Step 2: Factor the expression.

$$(3x-1)(5x+1)=0$$

Step 3: Apply the zero-product property and set each variable factor equal to zero.

$$3x-1=0 \quad \text{or} \quad 5x+1=0$$

Step 4: Solve the resulting linear equations.

$$\begin{aligned} 3x-1&=0 \\ 3x&=1 \\ x&=\tfrac{1}{3} \end{aligned} \quad \text{or} \quad \begin{aligned} 5x+1&=0 \\ 5x&=-1 \\ x&=-\frac{1}{5} \end{aligned}$$

Answer:

The solutions are $\frac{1}{3}$ and $\frac{-1}{5}$. The check is optional.

Example 4.4.7:

Solve: $(3x+2)(x+1)=4$.

Solution

This quadratic equation appears to be factored; hence it might be tempting to set each factor equal to 4. However, this would lead to incorrect results. We must rewrite the equation equal to zero, so that we can apply the zero-product property.

$$\begin{aligned} (3x+2)(x+1)&=4\\ 3x^2+3x+2x+2&=4\\ 3x^2+5x+2&=4\\ 3x^2+5x-2&=0 \end{aligned}$$

Once it is in standard form, we can factor and then set each factor equal to zero.

$$\begin{aligned} (3x-1)(x+2)&=0 \\ 3x-1=0 \quad &\text{or} \quad x+2=0 \\ 3x=1 \quad & \quad\quad\;\; x=-2 \\ x=\tfrac{1}{3} \quad & \end{aligned}$$

Answer:

The solutions are $\frac{1}{3}$ and -2.

Finding Roots of Functions

Recall that any polynomial with one variable is a function and can be written in the form,

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$$

A *root*[22] of a function is a value in the domain that results in zero. In other words, the roots occur when the function is equal to zero, $f(x)=0$.

Example 4.4.8

Find the roots: $f(x)=(x+2)^2-4$.

Solution

To find roots we set the function equal to zero and solve.

$$\begin{aligned} f(x)&=0\\ (x+2)^2-4&=0\\ x^2+4x+4-4&=0\\ x^2+4x&=0\\ x(x+4)&=0 \end{aligned}$$

Next, set each factor equal to zero and solve.

$$x = 0 \quad \text{or} \quad \begin{aligned} x+4&=0 \\ x&=-4 \end{aligned}$$

We can show that these x-values are roots by evaluating.

$$\begin{aligned} f(0) &= (0+2)^2-4 \\ &= 4-4 \\ &= 0\checkmark \end{aligned} \qquad \begin{aligned} f(-4) &= (-4+2)^2-4 \\ &= (-2)^2-4 \\ &= 4-4 \\ &= 0\checkmark \end{aligned}$$

Answer:

The roots are 0 and -4.

If we graph the function in the previous example we will see that the roots correspond to the x-intercepts of the function. Here the function f is a basic parabola shifted 2 units to the left and 4 units down.

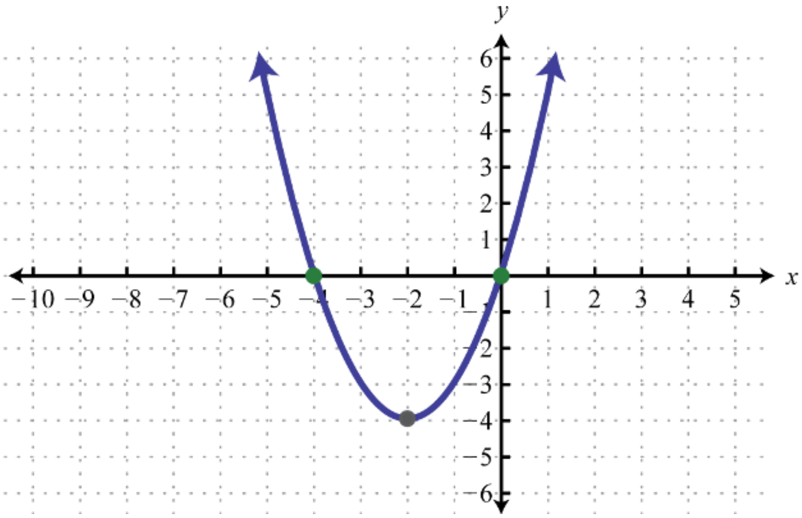

Figure $4.4.2$

Example $4.4.9$

Find the roots: $f(x)=x^4-5x^2+4$.

Solution

To find roots we set the function equal to zero and solve.

$$\begin{aligned} f(x)&=0 \\ x^4-5x^2+4&=0 \\ \left(x^2-1\right)\left(x^2-4\right)&=0 \\ (x+1)(x-1)(x+2)(x-2)&=0 \end{aligned}$$

Next, set each factor equal to zero and solve.

$$\begin{aligned} x+1&=0 \\ x&=-1 \end{aligned} \text{ or } \begin{aligned} x-1&=0 \\ x&=1 \end{aligned} \text{ or } \begin{aligned} x+2&=0 \\ x&=-2 \end{aligned} \text{ or } \begin{aligned} x-2&=0 \\ x&=2 \end{aligned}$$

Answer:

The roots are $-1, 1, -2$, and 2.

Graphing the previous function is not within the scope of this course. However, the graph is provided below:

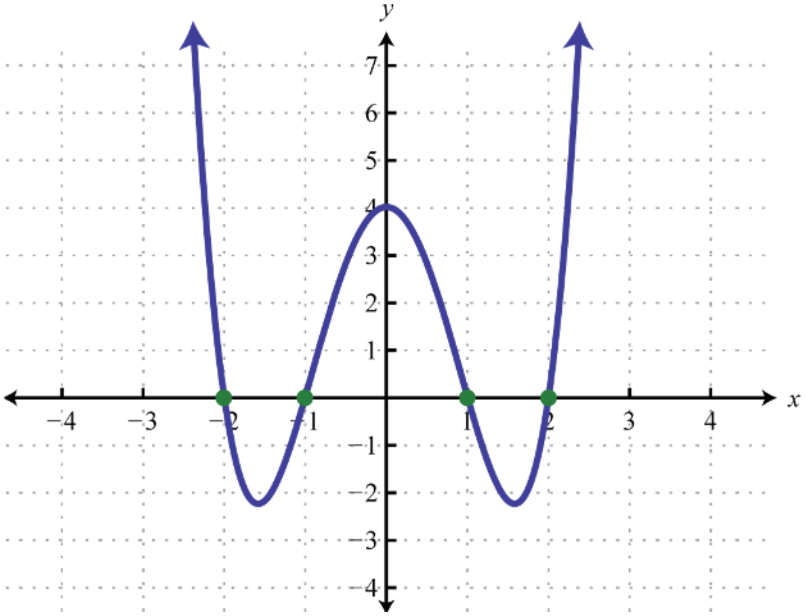

Figure $4.4.3$

Notice that the degree of the polynomial is 4 and we obtained four roots. In general, for any polynomial function with one variable of degree n, the **fundamental theorem of algebra**[23] guarantees n real roots or fewer. We have seen that many polynomials do not factor. This does not imply that functions involving these unfactorable polynomials do not have real roots. In fact, many polynomial functions that do not factor do have real solutions. We will learn how to find these types of roots as we continue in our study of algebra.

Example $4.4.10$

Find the roots: $f(x)=-x^{2}+10x-25$.

Solution

To find roots we set the function equal to zero and solve.

$$\begin{aligned} f(x)&=0\\ -x^{2}+10x-25&=0\\ -\left(x^{2}-10x+25\right)&=0\\ -(x-5)(x-5)&=0 \end{aligned}$$

Next, set each variable factor equal to zero and solve.

$$\begin{aligned} x-5&=0 \quad\text{or}\quad & x-5&=0\\ &=5 & x&=5 \end{aligned}$$

A solution that is repeated twice is called a **double root**[24]. In this case, there is only one solution.

Answer:

The root is 5.

The previous example shows that a function of degree 2 can have one root. From the factoring step, we see that the function can be written

$$f(x)=-(x-5)^{2}$$

In this form, we can see a reflection about the x-axis and a shift to the right 5 units. The vertex is the x-intercept, illustrating the fact that there is only one root.

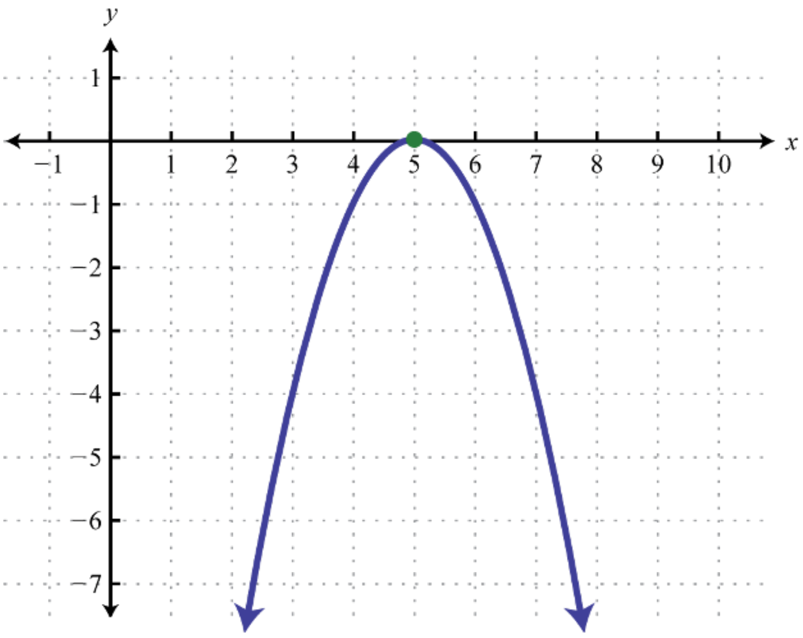

Figure $4.4.4$

Exercise $4.4.2$

Find the roots of $f(x)=x^3+3x^2-x-3$.

Answer

$\pm 1,-3$

www.youtube.com/v/t1ShqqhoaVE

Example $4.4.11$

Assuming dry road conditions and average reaction times, the safe stopping distance in feet is given by $d(x)=\frac{1}{20}x^2+x$, where x represents the speed of the car in miles per hour. Determine the safe speed of the car if you expect to stop in 40 feet.

Solution

We are asked to find the speed x where the safe stopping distance $d(x)=40$ feet.

$$\begin{aligned} d(x)&=40\\ \frac{1}{20}x^2+x&=40\end{aligned}$$

To solve for x, rewrite the resulting equation in standard form. In this case, we will first multiply both sides by 20 to clear the fraction.

$$\begin{aligned} {\color{#00a0e0}20}\left(\frac{1}{20}x^2+x\right)&={\color{#00a0e0}20}(40)\\ x^2+20x&=800\\ x^2+20x-800&=0\end{aligned}$$

Next factor and then set each factor equal to zero.

$$\begin{aligned} x^2+20x-800&=0\\ (x+40)(x-20)&=0\\ x+40=0 \quad &or \quad x-20=0\\ x=-40 \quad & \qquad\quad x=20\end{aligned}$$

The negative answer does not make sense in the context of this problem. Consider $x=20$ miles per hour to be the only solution.

Answer:

20 miles per hour

Finding Equations with Given Solutions

We can use the zero-product property to find equations, given the solutions. To do this, the steps for solving by factoring are performed in reverse.

Example 4.4.12

Find a quadratic equation with integer coefficients, given solutions $\frac{-3}{2}$ and $\frac{1}{3}$.

Solution

Given the solutions, we can determine two linear factors. To avoid fractional coefficients, we first clear the fractions by multiplying both sides by the denominator.

$$\begin{aligned} x &= -\frac{3}{2} \\ 2x &= -3 \\ 2x+3 &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x &= \frac{1}{3} \\ 3x &= 1 \\ 3x-1 &= 0 \end{aligned}$$

The product of these linear factors is equal to zero when $x = \frac{-3}{2}$ or $x = \frac{1}{3}$.

$$(2x+3)(3x-1) = 0$$

Multiply the binomials and present the equation in standard form.

$$\begin{aligned} 6x^2 - 2x + 9x - 3 &= 0 \\ 6x^2 + 7x - 3 &= 0 \end{aligned}$$

We may check our equation by substituting the given answers to see if we obtain a true statement. Also, the equation found above is not unique and so the check becomes essential when our equation looks different from someone else's. This is left as an exercise.

Answer:

$6x^2 + 7x - 3 = 0$

Example 4.4.13

Find a polynomial function with real roots $1, -2$, and 2.

Solution

Given solutions to $f(x) = 0$ we can find linear factors.

$$\begin{aligned} x &= 1 \\ x-1 &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x &= -2 \\ x+2 &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x &= 2 \\ x-2 &= 0 \end{aligned}$$

Apply the zero-product property and multiply.

$$\begin{aligned} (x-1)(x+2)(x-2) &= 0 \\ (x-1)\left(x^2-4\right) &= 0 \\ x^3 - 4x - x^2 + 4 &= 0 \\ x^3 - x^2 - 4x + 4 &= 0 \end{aligned}$$

Answer:

$f(x) = x^3 - x^2 - 4x + 4$

Exercise 4.4.3

Find a polynomial equation with integer coefficients, given solutions $\frac{1}{2}$ and $\frac{-3}{4}$.

Answer

$8x^2 + 2x - 3 = 0$

www.youtube.com/v/o4cuUWWEGdU

Key Takeaways

- Factoring and the zero-product property allow us to solve equations.
- To solve a polynomial equation, first write it in standard form. Once it is equal to zero, factor it and then set each variable factor equal to zero. The solutions to the resulting equations are the solutions to the original.
- Not all polynomial equations can be solved by factoring. We will learn how to solve polynomial equations that do not factor later in the course.
- A polynomial function can have at most a number of real roots equal to its degree. To find roots of a function, set it equal to zero and solve.
- To find a polynomial equation with given solutions, perform the process of solving by factoring in reverse.

Exercise 4.4.4

Factor completely.

1. $50x^2-18$
2. $12x^3-3x$
3. $10x^3+65x^2-35x$
4. $15x^4+7x^3-4x^2$
5. $6a^4b-15a^3b^2-9a^2b^3$
6. $8a^3b-44a^2b^2+20ab^3$
7. $36x^4-72x^3-4x^2+8x$
8. $20x^4+60x^3-5x^2-15x$
9. $3x^5+2x^4-12x^3-8x^2$
10. $10x^5-4x^4-90x^3+36x^2$
11. x^4-23x^2-50
12. $2x^4-31x^2-16$
13. $-2x^5-6x^3+8x$
14. $-36x^5+69x^3+27x$
15. $54x^5-78x^3+24x$
16. $4x^6-65x^4+16x^2$
17. x^6-7x^3-8
18. x^6-25x^3-54
19. $3x^6+4x^3+1$
20. $27x^6-28x^3+1$

Answer

1. $2(5x+3)(5x-3)$

3. $5x(x+7)(2x-1)$

5. $3a^2b(2a+b)(a-3b)$

7. $4x(x-2)(3x+1)(3x-1)$

9. $x^2(3x+2)(x+2)(x-2)$

11. $\left(x^2+2\right)(x+5)(x-5)$

13. $-2x\left(x^2+4\right)(x-1)(x+1)$

15. $6x(x+1)(x-1)(3x+2)(3x-2)$

17. $(x+1)\left(x^2-x+1\right)(x-2)\left(x^2+2x+4\right)$

19. $\left(3x^3+1\right)(x+1)\left(x^2-x+1\right)$

Exercise 4.4.5

Solve.

1. $(6x-5)(x+7)=0$
2. $(x+9)(3x-8)=0$
3. $5x(2x-5)(3x+1)=0$
4. $4x(5x-1)(2x+3)=0$
5. $(x-1)(2x+1)(3x-5)=0$
6. $(x+6)(5x-2)(2x+9)=0$
7. $(x+4)(x-2)=16$
8. $(x+1)(x-7)=9$
9. $(6x+1)(x+1)=6$
10. $(2x-1)(x-4)=39$
11. $x^2-15x+50=0$
12. $x^2+10x-24=0$
13. $3x^2+2x-5=0$
14. $2x^2+9x+7=0$
15. $\frac{1}{10}x^2-\frac{7}{15}x-\frac{1}{6}=0$
16. $\frac{1}{4}-\frac{4}{9}x^2=0$
17. $6x^2-5x-2=30x+4$
18. $6x^2-9x+15=20x-13$
19. $5x^2-23x+12=4(5x-3)$
20. $4x^2+5x-5=15(3-2x)$
21. $(x+6)(x-10)=4(x-18)$
22. $(x+4)(x-6)=2(x+4)$
23. $4x^3-14x^2-30x=0$
24. $9x^3+48x^2-36x=0$
25. $\frac{1}{3}x^3-\frac{3}{4}x=0$
26. $\frac{1}{2}x^3-\frac{1}{50}x=0$
27. $-10x^3-28x^2+48x=0$
28. $-2x^3+15x^2+50x=0$
29. $2x^3-x^2-72x+36=0$
30. $4x^3-32x^2-9x+72=0$
31. $45x^3-9x^2-5x+1=0$
32. $x^3-3x^2-x+3=0$
33. $x^4-5x^2+4=0$
34. $4x^4-37x^2+9=0$

Answer

1. $-7, \frac{5}{6}$

3. $0, \frac{5}{2}, -\frac{1}{3}$

5. $-\frac{1}{2}, 1, \frac{5}{3}$

7. $-6, 4$

9. $-\frac{5}{3}, \frac{1}{2}$

11. $5, 10$

13. $-\frac{5}{3}, 1$

15. $-\frac{1}{3}, 5$

17. $-\frac{1}{6}, 6$

19. $\frac{3}{5}, 8$

21. $2, 6$

23. $0, -\frac{3}{2}, 5$

25. $0, \pm\frac{3}{2}$

27. $-4, 0, \frac{6}{5}$

29. $\pm 6, \frac{1}{2}$

31. $\pm\frac{1}{3}, \frac{1}{5}$

33. $\pm 1, \pm 2$

Exercise 4.4.6

Find the roots of the given functions.

1. $f(x) = x^2 + 10x - 24$
2. $f(x) = x^2 - 14x + 48$
3. $f(x) = -2x^2 + 7x + 4$
4. $f(x) = -3x^2 + 14x + 5$
5. $f(x) = 16x^2 - 40x + 25$
6. $f(x) = 9x^2 - 12x + 4$
7. $g(x) = 8x^2 + 3x$
8. $g(x) = 5x^2 - 30x$
9. $p(x) = 64x^2 - 1$
10. $q(x) = 4x^2 - 121$
11. $f(x) = \frac{1}{5}x^3 - 1x^2 - \frac{1}{20}x + \frac{1}{4}$
12. $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{4}{3}x - 2$
13. $g(x) = x^4 - 13x^2 + 36$
14. $g(x) = 4x^4 - 13x^2 + 9$
15. $f(x) = (x+5)^2 - 1$
16. $g(x) = -(x+5)^2 + 9$
17. $f(x) = -(3x-5)^2$
18. $g(x) = -(x+2)^2 + 4$

Answer

1. $2, -12$

3. $-\frac{1}{2}, 4$

5. $\frac{5}{4}$

7. $-\frac{3}{8}, 0$

9. $\pm\frac{1}{8}$

11. $\pm\frac{1}{2}, 5$

13. $\pm 2, \pm 3$

15. $-6, -4$

17. $\frac{5}{3}$

Exercise 4.4.7

Given the graph of a function, determine the real roots.

1.

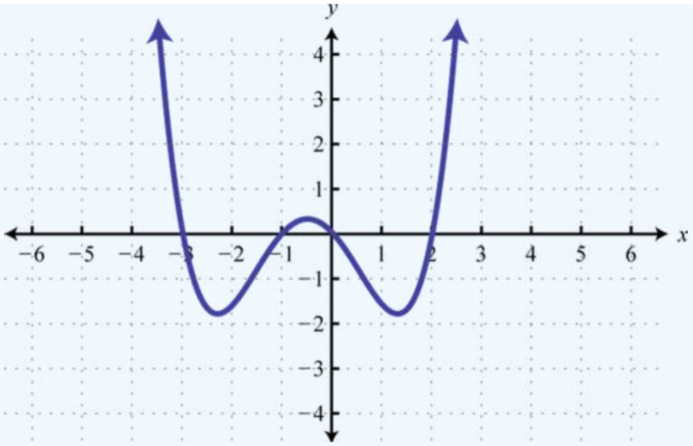

Figure 4.4.5

2.

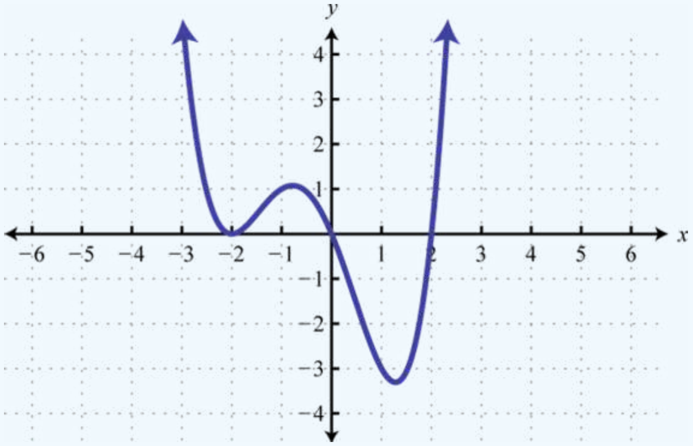

Figure 4.4.6

3.

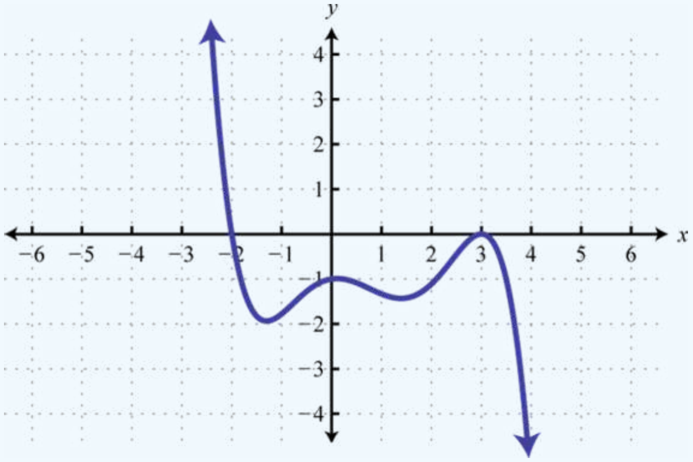

Figure 4.4.7

4.

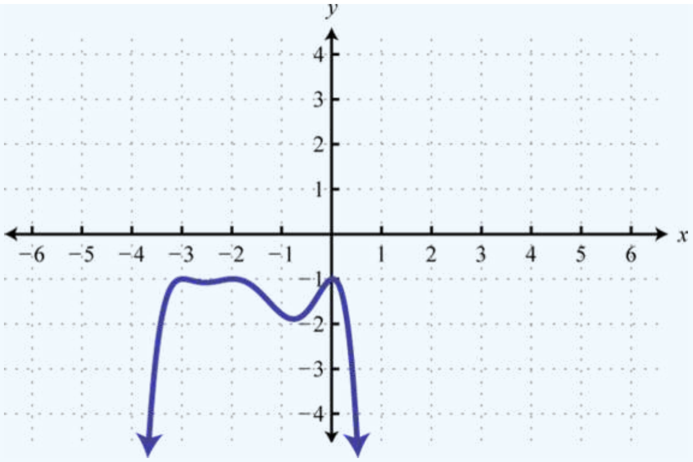

Figure 4.4.8

5. The sides of a square measure $x-2$ units. If the area is 36 square units, then find x.

6. The sides of a right triangle have lengths that are consecutive even integers. Find the lengths of each side. (Hint: Apply the Pythagorean theorem)

7. The profit in dollars generated by producing and selling n bicycles per week is given by the formula $P(n)=-5n^2+400n-6000$. How many bicycles must be produced and sold to break even?

8. The height in feet of an object dropped from the top of a 64-foot building is given by $h(t)=-16t^2+64$ where t represents the time in seconds after it is dropped. How long will it take to hit the ground?

9. A box can be made by cutting out the corners and folding up the edges of a square sheet of cardboard. A template for a cardboard box of height 2 inches is given.

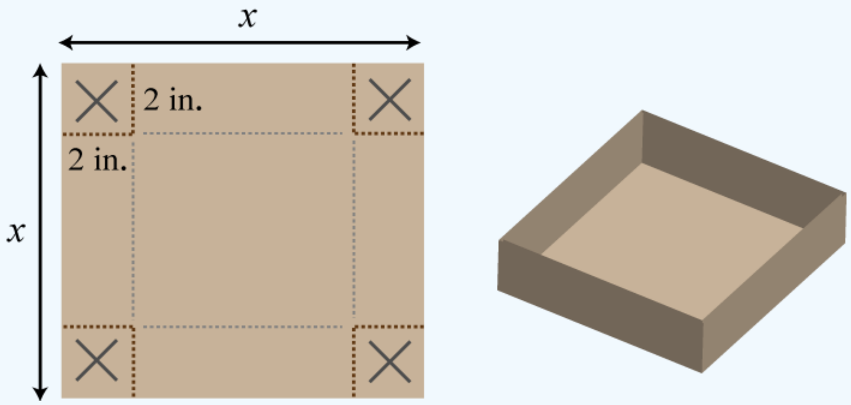

Figure 4.4.9

What is the length of each side of the cardboard sheet if the volume of the box is to be 98 cubic inches?

10. The height of a triangle is 4 centimeters less than twice the length of its base. If the total area of the triangle is 48 square centimeters, then find the lengths of the base and height.

11. A uniform border is to be placed around an 8×10 inch picture.

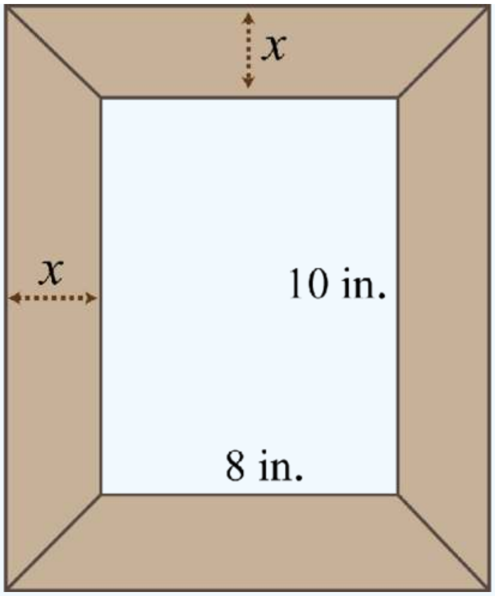

Figure 4.4.10

If the total area including the border must be 168 square inches, then how wide should the border be?

12. The area of a picture frame including a 3-inch wide border is 120 square inches.

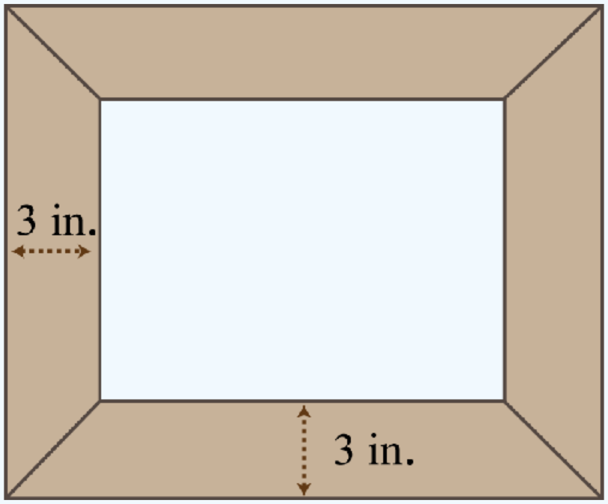

Figure 4.4.11

If the width of the inner area is 2 inches less than its length, then find the dimensions of the inner area.

13. Assuming dry road conditions and average reaction times, the safe stopping distance in feet is given by $d(x) = \frac{1}{20}x^2 + x$ where x represents the speed of the car in miles per hour. Determine the safe speed of the car if you expect to stop in 75 feet.

14. A manufacturing company has determined that the daily revenue in thousands of dollars is given by the formula $R(n) = 12n - 0.6n^2$ where n represents the number of palettes of product sold $(0 \leq n < 20)$. Determine the number of palettes sold in a day if the revenue was 45 thousand dollars.

Answer

1. $-3, -1, 0, 2$

3. $-2, 3$

5. 8 units

7. 20 or 60 bicycles

9. 11 in

11. 2 inches

13. 30 miles per hour

Exercise 4.4.8

Find a polynomial equation with the given solutions.

1. $-3, 5$
2. $-1, 8$
3. $2, \frac{1}{3}$
4. $-\frac{3}{4}, 5$
5. $0, -4$
6. $0, 7$
7. ± 7
8. ± 2
9. $-3, 1, 3$
10. $-5, -1, 1$

Answer

1. $x^2 - 2x - 15 = 0$

3. $3x^2 - 7x + 2 = 0$

5. $x^2 + 4x = 0$

7. $x^2 - 49 = 0$

9. $x^3 - x^2 - 9x + 9 = 0$

Exercise 4.4.9

Find a function with the given roots.

1. $\frac{1}{2}, \frac{2}{3}$
2. $\frac{2}{5}, -\frac{1}{3}$
3. $\pm\frac{3}{4}$
4. $\pm\frac{5}{2}$
5. 5 double root
6. -3 double root
7. $-1, 0, 3$
8. $-5, 0, 2$

Answer

1. $f(x) = 6x^2 - 7x + 2$

3. $f(x) = 16x^2 - 9$

5. $f(x) = x^2 - 10x + 25$

7. $f(x) = x^3 - 2x^2 - 3x$

Exercise 4.4.10

Recall that if $|X| = p$, then $X = -p$ or $X = p$. Use this to solve the following absolute value equations.

1. $|x^2 - 8| = 8$
2. $|2x^2 - 9| = 9$

3. $\left|x^2-2x-1\right|=2$
4. $\left|x^2-8x+14\right|=2$
5. $\left|2x^2-4x-7\right|=9$
6. $\left|x^2-3x-9\right|=9$

Answer

1. $\pm4, 0$

3. $\pm1, 3$

5. $-2, 1, 4$

Exercise $4.4.11$

1. Explain to a beginning algebra student the difference between an equation and an expression.
2. What is the difference between a root and an x-intercept? Explain.
3. Create a function with three real roots of your choosing. Graph it with a graphing utility and verify your results. Share your function on the discussion board.
4. Research and discuss the fundamental theorem of algebra.

Answer

1. Answer may vary

3. Answer may vary

Footnotes

[20] A product is equal to zero if and only if at least one of the factors is zero.

[21] The process of solving an equation that is equal to zero by factoring it and then setting each variable factor equal to zero.

[22] A value in the domain of a function that results in zero.

[23] Guarantees that there will be as many (or fewer) roots to a polynomial function with one variable as its degree.

[24] A root that is repeated twice.

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4.5: Rational Functions - Multiplication and Division

Learning Objectives

- Identify restrictions to the domain of a rational function.
- Simplify rational functions.
- Multiply and divide rational functions.

Identifying Restrictions and Simplifying Rational Functions

Rational functions[25] have the form

$$r(x)=\frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials and $q(x)\neq 0$. The *domain of a rational function*[26] consists of all real numbers x except those where the denominator $q(x)=0$. *Restrictions*[27] are the real numbers for which the expression is not defined. We often express the domain of a rational function in terms of its restrictions. For example, consider the function

$$f(x)=\frac{x^2-4x+3}{x^2-5x+6}$$

which can be written in factored form

$$f(x)=\frac{(x-1)(x-3)}{(x-2)(x-3)}$$

Because rational expressions are undefined when the denominator is 0, we wish to find the values for x that make it 0. To do this, apply the zero-product property. Set each factor in the denominator equal to 0 and solve.

$$\begin{array}{c}(x-2)\,(x-3)=0\\ x-2=0 \quad \text{or} \quad x-3=0\\ x=3 \qquad\qquad x=3\end{array}$$

Therefore, the original function is defined for any real number except 2 and 3. We can express its domain using notation as follows:

$$\begin{array}{ll}\textit{Set}-\textit{builder notation} & \textit{Interval notation}\\ \{x|x\neq 2,3\} & (-\infty,2)\cup(2,3)\cup(3,\infty)\end{array}$$

The restrictions to the domain of a rational function are determined by the denominator. Once the restrictions are determined we can cancel factors and obtain an equivalent function as follows:

$$f(x)=\frac{(x-1)\cancel{(x-3)}}{(x-2)\cancel{(x-3)}}=\frac{x-1}{x-2}$$

Figure $4.5.1$

It is important to note that 1 is *not* a restriction to the domain because the expression is defined as 0 when the numerator is 0. In fact, $x=1$ is a root. This function is graphed below:

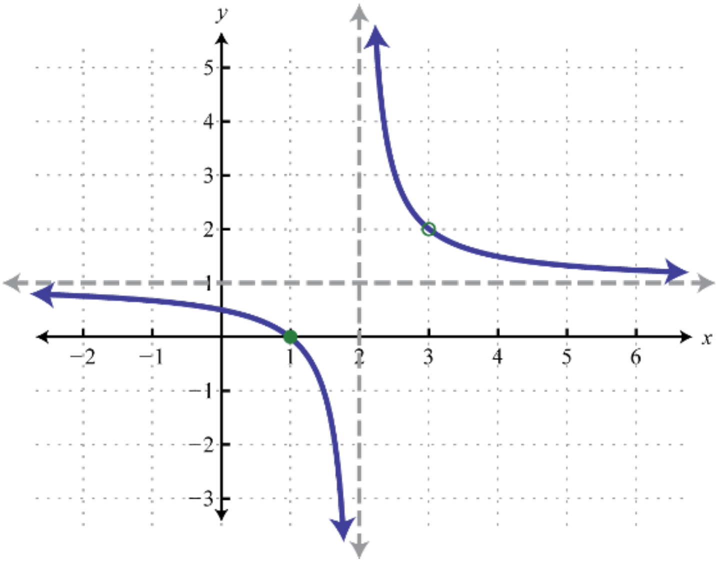

Figure $4.5.2$

Notice that there is a vertical asymptote at the restriction $x = 2$ and the graph is left undefined at the restriction $x = 3$ as indicated by the open dot, or hole, in the graph. Graphing rational functions in general is beyond the scope of this textbook. However, it is useful at this point to know that the restrictions are an important part of the graph of rational functions.

Example $4.5.1$:

State the restrictions and simplify: $g(x) = \frac{24x^7}{6x^5}$

Solution

In this example, the function is undefined where x is 0.

$$g(0) = \frac{24(0)^7}{6(0)^5} = \frac{0}{0} \quad undefined$$

Therefore, the domain consists of all real numbers x, where $x \neq 0$. With this understanding, we can simplify by reducing the rational expression to lowest terms. Cancel common factors.

$$g(x) = \frac{\overset{4}{\cancel{24}}\ \overset{x^2}{\cancel{x^7}}}{\cancel{6}\ \cancel{x^5}} = 4x^2$$

Answer

$g(x) = 4x^2$, where $x \neq 0$

Example $4.5.2$:

State the restrictions and simplify: $f(x) = \frac{2x^2 + 5x - 3}{4x^2 - 1}$.

Solution

First, factor the numerator and denominator.

$$f(x) = \frac{2x^2 + 5x - 3}{4x^2 - 1} = \frac{(2x - 1)(x + 3)}{(2x + 1)(2x - 1)}$$

Any x-value that makes the denominator zero is a restriction. To find the restrictions, first set the denominator equal to zero and then solve

$$(2x + 1)(2x - 1) = 0$$

$$\begin{aligned} 2x+1&=0 & \text{or} \quad 2x-1&=0 \\ 2x&=-1 & 2x&=1 \\ x&=-\frac{1}{2} & x&=\frac{1}{2} \end{aligned}$$

Therefore, $x \neq \pm\frac{1}{2}$. With this understanding, we can cancel any common factors.

$$\begin{aligned} f(x) &= \frac{\cancel{(2x-1)}\,(x+3)}{(2x+1)\,\cancel{(2x-1)}} \\ &= \frac{x+3}{2x+1} \end{aligned}$$

Answer:

$f(x)=\frac{x+3}{2x+1}$, where $x \neq \pm\frac{1}{2}$

We define the opposite of a polynomial P to be $-P$. Finding the opposite of a polynomial requires the application of the distributive property. For example, the opposite of the polynomial $(x-3)$ is written as

$$\begin{aligned} -(x-3) &= -1 \cdot (x-3) \\ &= -x+3 \\ &= 3-x \end{aligned}$$

This leads us to the **opposite binomial property**[28], $-(a-b)=(b-a)$. Care should be taken not to confuse this with the fact that $(a+b)=(b+a)$. This is the case because addition is commutative. In general,

$$\begin{array}{c|c} -(a-b)=(b-a) & (a-b)=(a+b) \\ \text{or} & \text{or} \\ \frac{b-a}{a-b}=-1 & \frac{b+a}{a+b}=1 \end{array}$$

Also, it is important to recall that

$$\frac{-a}{b}=-\frac{a}{b}=\frac{a}{-b}$$

In other words, a negative fraction is shown by placing the negative sign in either the numerator, in front of the fraction bar, or in the denominator. Generally, negative denominators are avoided.

Example $4.5.3$:

State the restrictions and simplify: $\frac{25-x^2}{x^2-10x+25}$.

Solution

Begin by factoring the numerator and denominator.

$$\begin{aligned} \frac{25-x^2}{x^2-10x+25} &= \frac{(5-x)(5+x)}{(x-5)(x-5)} \\ &= \frac{-1\cdot(x-5)(5+x)}{(x-5)(x-5)} && \textit{Opposite binomial property.} \\ &= \frac{-1\cdot\cancel{(x-5)}\,(5+x)}{\cancel{(x-5)}\,(x-5)} && \textit{Cancel.} \\ &= -\frac{x+5}{x-5} \end{aligned}$$

Answer:

$-\frac{x+5}{x-5}$, where $x \neq 5$

It is important to remember that we can only cancel factors of a product. A common mistake is to cancel terms. For example,

$$\frac{\cancel{x^2}+7x-30}{\cancel{x^2}-7x+12}$$

incorrect!

$$\frac{\cancel{x}+10}{\cancel{x}-4} \qquad \frac{2\,\cancel{x-1}}{\cancel{x-1}}$$

incorrect!

incorrect!

Exercise 4.5.1

State the restrictions and simplify: $\frac{x-2x^2}{4x^4-x^2}$.

Answer

$-\frac{1}{x(2x+1)}$, where $x \neq 0, \pm\frac{1}{2}$

www.youtube.com/v/Ek2qw7OgxbY

In some examples, we will make a broad assumption that the denominator is nonzero. When we make that assumption, we do not need to determine the restrictions.

Example 4.5.4:

Simplify: $\frac{x^3-2x^2y+4xy^2-8y^3}{x^4-16y^4}$. (Assume all denominators are nonzero.)

Solution

Factor the numerator by grouping. Factor the denominator using the formula for a difference of squares.

$$\begin{aligned}\frac{x^3+4xy^2-2x^2y-8y^3}{x^4-16y^4} &= \frac{x\left(x^2+4y^2\right)-2y\left(x^2+4y^2\right)}{\left(x^2+4y^2\right)\left(x^2-4y^2\right)}\\ &= \frac{\left(x^2+4y^2\right)(x-2y)}{\left(x^2+4y^2\right)(x+2y)(x-2y)}\end{aligned}$$

Next, cancel common factors.

$$\begin{aligned}&= \frac{\overset{1}{\cancel{\left(x^2+4y^2\right)}}\ \overset{1}{\cancel{(x-2y)}}}{\cancel{\left(x^2+4y^2\right)}\,(x+2y)\,\cancel{(x-2y)}}\\ &= \frac{1}{x+2y}\end{aligned}$$

Note

When the entire numerator or denominator cancels out a factor of 1 always remains.

Answer:

$\frac{1}{x+2y}$

Example 4.5.5:

Given $f(x)=x^2-2x+5$, simplify $\frac{f(x)-f(3)}{x-3}$.

Solution

Begin by calculating $f(3)$.

$$\begin{aligned}f(3) &= (3)^2-2(3)+5\\ &= 9-6+5\\ &= 3+5\\ &= 8\end{aligned}$$

Next, substitute into the quotient that is to be simplified.

$$\begin{aligned}\frac{f(x)-f(3)}{x-3} &= \frac{x^2-2x+5-8}{x-3} \\ &= \frac{x^2-2x-3}{x-3} \\ &= \frac{(x+1)(x-3)}{(x-3)} \\ &= x+1\end{aligned}$$

Answer:

$x+1$, where $x \neq 3$

An important quantity in higher level mathematics is the **difference quotient**[29]:

$$\frac{f(x+h)-f(x)}{h}, \text{ where } h \neq 0$$

This quantity represents the slope of the line connecting two points on the graph of a function. The line passing through the two points is called a **secant line**[30].

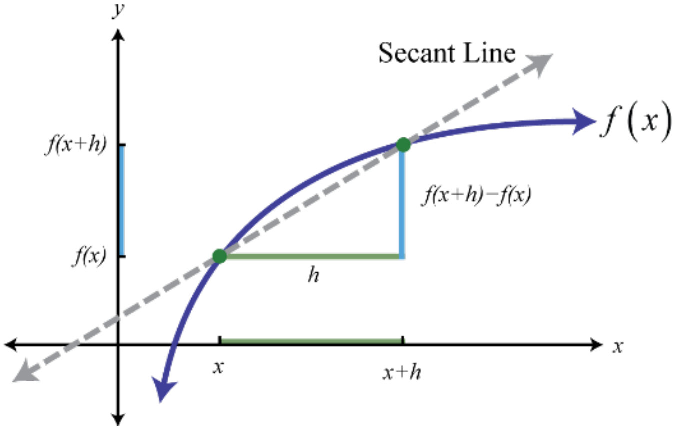

Figure 4.5.3

Calculating the difference quotient for many different functions is an important skill to learn in intermediate algebra. We will encounter this quantity often as we proceed in this textbook. When calculating the difference quotient we assume the denominator is nonzero.

Example 4.5.6:

Given $g(x) = -2x^2+1$, simplify $\frac{g(x+h)-g(x)}{h}$.

Solution

$$\begin{aligned}\frac{g(x+h)-g(x)}{h} &= \frac{(-2(x+h)^2+1)-(-2x^2+1)}{h} \\ &= \frac{-2\left(x^2+2xh+h^2\right)+1+2x^2-1}{h} \\ &= \frac{-2x^2-4xh-2h^2+1+2x^2-1}{h} \\ &= \frac{-4xh-2h^2}{h} \\ &= -4x-2h\end{aligned}$$

Answer:

$-4x-2h$

Exercise 4.5.2

Given $f(x)=x^2-x-1$, simplify $\dfrac{f(x+h)-f(x)}{h}$.

Answer

$2x-1+h$

www.youtube.com/v/xYofIrdNwWs

Multiplying and Dividing Rational Functions

When multiplying fractions, we can multiply the numerators and denominators together and then reduce. Multiplying rational expressions is performed in a similar manner. In general, given polynomials P, Q, R, and S, where $Q\neq 0$ and $S\neq 0$, we have

$$\frac{P}{Q}\cdot\frac{R}{S}=\frac{PR}{QS}$$

The restrictions to the domain of a product consist of the restrictions of each function.

Example 4.5.7:

Given $f(x)=\dfrac{9x^2-25}{x-5}$ and $g(x)=\dfrac{x^2-2x-15}{3x+5}$, find $(f\cdot g)(x)$ and determine the restrictions to the domain.

Solution

In this case, the domain of f consists of all real numbers except 5, and the domain of g consists of all real numbers except $-\dfrac{5}{3}$. Therefore, the domain of the product consists of all real numbers except 5 and $-\dfrac{5}{3}$. Multiply the functions and then simplify the result.

$$\begin{aligned}(f\cdot g)(x)&=f(x)\cdot g(x)\\&=\frac{9x^2-25}{x-5}\cdot\frac{x^2-2x-15}{3x+5}\\&=\frac{(3x+5)(3x-5)}{x-5}\cdot\frac{(x-5)(x+3)}{3x+5} && \color{blue}{Factor.}\\&=\frac{\cancel{(3x+5)}(3x-5)\cancel{(x-5)}(x+3)}{\cancel{(x-5)}\cancel{(3x+5)}} && \color{blue}{Cancel.}\\&=(3x-5)(x+3)\end{aligned}$$

Answer:

$(f\cdot g)(x)=(3x-5)(x+3)$, where $x\neq 5,-\dfrac{5}{3}$

To divide two fractions, we multiply by the reciprocal of the divisor. Dividing rational expressions is performed in a similar manner. In general, given polynomials P, Q, R, and S, where $Q\neq 0, R\neq 0$, and $S\neq 0$, we have

$$\frac{P}{Q}\div\frac{R}{S}=\frac{P}{Q}\cdot\frac{S}{R}=\frac{PS}{QR}$$

The restrictions to the domain of a quotient will consist of the restrictions of each function as well as the restrictions on the reciprocal of the divisor.

Example 4.5.8:

Given $f(x)=\dfrac{2x^2+13x-7}{x^2-4x-21}$ and $g(x)=\dfrac{2x^2+5x-3}{49-x^2}$, find $(f/g)(x)$ and determine the restrictions.

Solution

$$\begin{aligned}(f/g)(x) &= f(x) \quad \div g(x)\\ &= \frac{2x^2+13x-7}{x^2-4x-21} \div \frac{2x^2+5x-3}{49-x^2}\\ &= \frac{2x^2+13x-7}{x^2-4x-21} \cdot \frac{49-x^2}{2x^2+5x-3} && \textit{Multiply by the reciprocal}\\ &= \frac{(2x-1)(x+7)}{(x+3)(x-7)} \cdot \frac{(7+x)(7-x)}{(2x-1)(x+3)} && \textit{Factor.}\\ &= \frac{\cancel{(2x-1)}\,(x+7)(7+x)(-1)\,\cancel{(x-7)}}{(x+3)\,\cancel{(x-7)}\,\cancel{(2x-1)}\,(x+3)} && \textit{Cancel.}\\ &= -\frac{(x+7)^2}{(x+3)^2}\end{aligned}$$

In this case, the domain of f consists of all real numbers except -3 and 7, and the domain of g consists of all real numbers except 7 and -7. In addition, the reciprocal of $g(x)$ has a restriction of -3 and $\frac{1}{2}$. Therefore, the domain of this quotient consists of all real numbers except -3, $\frac{1}{2}$, and ± 7.

Answer:

$(f/g)(x) = -\frac{(x+7)^2}{(x+3)^2}$, where $x \neq -3, \frac{1}{2}, \pm 7$

Recall that multiplication and division operations are to be performed from left to right.

Example 4.5.9:

Simplify: $\frac{4x^2-1}{6x^2+3x} \div \frac{2x+1}{x^2+2x+1} \cdot \frac{27x^4}{2x^2+x-1}$ (Assume all denominators are nonzero.)

Solution

Begin by replacing the factor that is to be divided by multiplication of its reciprocal.

$$\begin{aligned}&\frac{4x^2-1}{6x^2+3x} \div \frac{2x+1}{x^2+2x+1} \cdot \frac{27x^4}{2x^2+x-1}\\ &= \frac{4x^2-1}{6x^2+3x} \cdot \frac{x^2+2x+1}{2x+1} \cdot \frac{27x^4}{2x^2+x-1}\\ &= \frac{(2x+1)(2x-1)}{3x(2x+1)} \cdot \frac{(x+1)(x+1)}{(2x+1)} \cdot \frac{27x^3}{(2x-1)(x+1)}\\ &= \frac{\cancel{(2x+1)}\,\cancel{(2x-1)}\,\cancel{(x+1)}\,(x+1) \cdot \overset{9}{\cancel{27}}\,\overset{x^3}{\cancel{x^4}}}{\cancel{3}\,\cancel{x}\,\cancel{(2x+1)}\,(2x+1)\,\cancel{(2x-1)}\,\cancel{(x+1)}}\\ &= \frac{9x^3(x+1)}{(2x+1)}\end{aligned}$$

Answer:

$$\frac{9x^3(x+1)}{(2x+1)}$$

Exercise 4.5.3

Given $f(x) = \frac{2x+5}{3x^2+14x-5}$ and $g(x) = \frac{6x^2+13x-5}{x+5}$, calculate $(f/g)(x)$ and determine the restrictions.

Answer

$(f/g)(x) = \frac{1}{(3x-1)^2}$, where $x \neq -5, -\frac{5}{2}, \frac{1}{3}$

www.youtube.com/v/UCPL1ZQIvWE

If a cost function C represents the cost of producing x units, then the **average cost**[31] $\overline{C}$ is the cost divided by the number of units produced.

$$\overline{C}(x)=\frac{C(x)}{x}$$

Example 4.5.10:

A manufacturer has determined that the cost in dollars of producing sweaters is given by \9C (x) = 0.01 x ^ { 2 } - 3 x + 1200\), where x represents the number of sweaters produced daily. Determine the average cost of producing $100, 200$, and 300 sweaters per day.

Solution

Set up a function representing the average cost.

$$\overline{C}(x)=\frac{C(x)}{x}=\frac{0.01x^2-3x+1200}{x}$$

Next, calculate $\overline{C}(100)$, $\overline{C}(200)$, and $\overline{C}(300)$.

$$\begin{aligned}
\overline{C}(100)&=\frac{0.01(100)^2-3(100)+1200}{(100)}=\frac{100-300+1200}{100}=\frac{1000}{100}=10.00\\
\overline{C}(200)&=\frac{0.01(200)^2-3(200)+1200}{(200)}=\frac{400-600+1200}{200}=\frac{1000}{200}=5.00\\
\overline{C}(300)&=\frac{0.01(300)^2-3(300)+1200}{(300)}=\frac{900-900+1200}{300}=\frac{1200}{300}=4.00
\end{aligned}$$

Answer:

The average cost of producing 100 sweaters per day is $\$10.00$ per sweater. If 200 sweaters are produced, the average cost per sweater is $\$5.00$. If 300 are produced, the average cost per sweater is $\$4.00$.

Key Takeaways

- Simplifying rational expressions is similar to simplifying fractions. First, factor the numerator and denominator and then cancel the common factors. Rational expressions are simplified if there are no common factors other than 1 in the numerator and the denominator.
- Simplified rational functions are equivalent for values in the domain of the original function. Be sure to state the restrictions unless the problem states that the denominators are assumed to be nonzero.
- After multiplying rational expressions, factor both the numerator and denominator and then cancel common factors. Make note of the restrictions to the domain. The values that give a value of 0 in the denominator for all expressions are the restrictions.
- To divide rational expressions, multiply the numerator by the reciprocal of the divisor.
- The restrictions to the domain of a product consist of the restrictions to the domain of each factor.

Exercise 4.5.4

Simplify the function and state its domain using interval notation.

1. $f(x)=\frac{25x^9}{5x^5}$
2. $f(x)=\frac{64x^8}{16x^3}$
3. $f(x)=\frac{x^2-64}{x^2+16x+64}$
4. $f(x)=\frac{x^2+x-20}{x^2-25}$
5. $g(x)=\frac{9-4x^2}{2x^2-5x+3}$
6. $g(x)=\frac{x-3x^2}{9x^2-6x+1}$
7. $g(x)=\frac{2x^2-8x-42}{2x^2+5x-3}$
8. $g(x)=\frac{6x^2+5x-4}{3x^2+x-4}$

9. $h(x)=\dfrac{x^3+x^2-x-1}{x^2+2x+1}$
10. $h(x)=\dfrac{2x^3-5x^2-8x+20}{2x^2-9x+10}$

Answer

1. $f(x)=5x^4$; Domain: $(-\infty,0)\cup(0,\infty)$
3. $f(x)=\dfrac{x-8}{x+8}$; Domain: $(-\infty,-8)\cup(-8,\infty)$
5. $g(x)=-\dfrac{2x+3}{x-1}$; Domain: $(-\infty,1)\cup\left(1,\dfrac{3}{2}\right)\cup\left(\dfrac{3}{2},\infty\right)$
7. $g(x)=\dfrac{2(x-7)}{2x-1}$; Domain: $(-\infty,-3)\cup\left(-3,\dfrac{1}{2}\right)\cup\left(\dfrac{1}{2},\infty\right)$
9. $h(x)=x-1$; Domain: $(-\infty,-1)\cup(-1,\infty)$

Exercise 4.5.5

State the restrictions and simplify the given rational expressions.

1. $\dfrac{66x(2x-5)}{18x^3(2x-5)^2}$
2. $\dfrac{26x^4(5x+2)^3}{20x^5(5x+2)}$
3. $\dfrac{x^2+5x+6}{x^2-5x-14}$
4. $\dfrac{x^2-8x+12}{x^2-2x-24}$
5. $\dfrac{1-x^2}{5x^2+x-6}$
6. $\dfrac{4-9x^2}{3x^2-8x+4}$
7. $\dfrac{4x^2+15x+9}{9-x^2}$
8. $\dfrac{6x^2+13x-5}{25-4x^2}$
9. $\dfrac{x^2-5x+4}{x^3-x^2-16x+16}$
10. $\dfrac{x^4+4x^2}{x^3+3x^2+4x+12}$

Answer

1. $\dfrac{11}{3x^2(2x-5)}; x\neq 0,\dfrac{5}{2}$
3. $\dfrac{x+3}{x-7}; x\neq -2,7$
5. $-\dfrac{x+1}{5x+6}; x\neq -\dfrac{6}{5},1$
7. $-\dfrac{4x+3}{3-x}; x\neq \pm 3$
9. $\dfrac{1}{x+4}; x\neq 1,\pm 4$

Exercise 4.5.6

Simplify the given rational expressions. Assume all variable expressions in the denominator are nonzero.

1. $\dfrac{50ab^3(a+b)^2}{200a^2b^3(a+b)^3}$
2. $\dfrac{36a^5b^7(a-b)^2}{9a^3b(a-b)}$
3. $\dfrac{a^2-b^2}{a^2+2ab+b^2}$

4. $\dfrac{a^2-2ab+b^2}{a^2-b^2}$
5. $\dfrac{6x^2-xy}{6x^2-7xy+y^2}$
6. $\dfrac{y-x}{2x^3-4x^2y+2xy^2}$
7. $\dfrac{x^2y^2-2xy^3}{x^2y^2-xy^3-2y^4}$
8. $\dfrac{x^4y-x^2y^3}{x^3y+2x^2y^2+xy^3}$
9. $\dfrac{x^3-x^2y+xy^2-y^3}{x^4-y^4}$
10. $\dfrac{y^4-x^4}{x^3+x^2y+xy^2+y^3}$
11. $\dfrac{a^2-(b+c)^2}{(a+b)^2-c^2}$
12. $\dfrac{(a+b)^2-c^2}{(a+c)^2-b^2}$
13. $\dfrac{x^3+y^3}{x^2+2xy+y^2}$
14. $\dfrac{x^3y+x^2y^2+xy^3}{x^3-y^3}$

Answer

1. $\dfrac{1}{4a(a+b)}$
3. $\dfrac{a-b}{a+b}$
5. $\dfrac{x}{x-y}$
7. $\dfrac{x}{x+y}$
9. $\dfrac{1}{x+y}$
11. $\dfrac{a-b-c}{a+b-c}$
13. $\dfrac{x^2-xy+y^2}{x+y}$

Exercise $4.5.7$

Given the function, simplify the rational expression.

1. Given $f(x)=x^2-8$, simplify $\dfrac{f(x)-f(5)}{x-5}$.
2. Given $f(x)=x^2+4x-1$, simplify $\dfrac{f(x)-f(2)}{x-2}$.
3. Given $g(x)=x^2-3x+1$, simplify $\dfrac{g(x)-g(-1)}{x+1}$.
4. Given $g(x)=x^2-2x$, simplify $\dfrac{g(x)-g(-4)}{x+4}$.
5. Given $f(x)=4x^2+6x+1$, simplify $\dfrac{f(x)-f\left(\frac{1}{2}\right)}{2x-1}$.
6. Given $f(x)=9x^2+1$, simplify $\dfrac{f(x)-f\left(-\frac{1}{3}\right)}{3x+1}$.

Answer

1. $x+5$, where $x \neq 5$
3. $x-4$, where $x \neg -1$

5. $2(x+2)$, where $x \neq \frac{1}{2}$

Exercise 4.5.8

For the given function, simplify the difference quotient $\frac{f(x+h)-f(x)}{h}$, where $h \neq 0$.

1. $f(x)=5x-3$
2. $f(x)=3-2x$
3. $f(x)=x^2-3$
4. $f(x)=x^2+8x$
5. $f(x)=x^2-x+5$
6. $f(x)=4x^2+3x-2$
7. $f(x)=ax^2+bx+c$
8. $f(x)=ax^2+bx$
9. $f(x)=x^3+1$
10. $f(x)=x^3-x+2$

Answer

1. 5

3. $2x+h$

5. $2x-1+h$

7. $2ax+b+ah$

9. $3x^2+3xh+h^2$

Exercise 4.5.9

Simplify the product $f \cdot g$ and state its domain using interval notation.

1. $f(x)=\frac{52x^4}{(x-2)^2}, g(x)=\frac{(x-2)^3}{12x^5}$
2. $f(x)=\frac{46(2x-1)^3}{15x^6}, g(x)=\frac{25x^3}{23(2x-1)}$
3. $f(x)=\frac{10x^3}{x^2+4x+4}, g(x)=\frac{x^2-4}{50x^4}$
4. $f(x)=\frac{25-x^2}{46x^5}, g(x)=\frac{12x^3}{x^2+10x+25}$
5. $f(x)=\frac{5-3x}{x^2-10x+25}, g(x)=\frac{x^2-6x+5}{3x^2-8x+5}$
6. $f(x)=\frac{1-4x^2}{6x^2+3x}, g(x)=\frac{12x^2}{4x^2-4x+1}$

Answer

1. $(f \cdot g)(x)=\frac{13(x-2)}{3x}$; Domain: $(-\infty,0)\cup(0,2)\cup(2,\infty)$

3. $(f \cdot g)(x)=\frac{x-2}{5x(x+2)}$; Domain: $(-\infty,-2)\cup(-2,0)\cup(0,\infty)$

5. $(f \cdot g)(x)=-\frac{1}{x-5}$; Domain: $(-\infty,1)\cup\left(1,\frac{5}{3}\right)\cup\left(\frac{5}{3},5\right)\cup(5,\infty)$

Exercise 4.5.10

Simplify the quotient f/g and state its domain using interval notation.

1. $f(x)=\frac{12x^3}{5(5x-1)^3}, g(x)=\frac{6x^2}{25(5x-1)^4}$

2. $f(x)=\dfrac{7x^2(x+9)}{(x-8)^2}, g(x)=\dfrac{49x^3(x+9)}{(x-8)^4}$
3. $f(x)=\dfrac{25x^2-1}{3x^2-15x}, g(x)=\dfrac{25x^2+10x+1}{x^3-5x^2}$
4. $f(x)=\dfrac{x^2-x-6}{2x^2+13x+15}, g(x)=\dfrac{x^2-6x+9}{4x^2+12x+9}$
5. $f(x)=\dfrac{x^2-64}{x^2}, g(x)=2x^2+19x+24$
6. $f(x)=2x^2+11x-6, g(x)=36-x^2$

Answer

1. $(f/g)(x)=10x(5x-1)$; Domain: $(-\infty,0)\cup\left(0,\frac{1}{5}\right)\cup\left(\frac{1}{5},\infty\right)$
3. $(f/g)(x)=\dfrac{x(5x-1)}{2(5x+1)}$; Domain: $(-\infty,0)\cup\left(0,\frac{1}{5}\right)\cup\left(\frac{1}{5},\infty\right)$
5. $(f/g)(x)=\dfrac{x-8}{x^2(2x+3)}$; Domain: $(-\infty,-8)\cup\left(-8,-\frac{3}{2}\right)\cup\left(-\frac{3}{2},0\right)\cup(0,\infty)$

Exercise 4.5.11

Multiply or divide as indicated, state the restrictions, and simplify.

1. $\dfrac{14(x+12)^2}{5x^3}\cdot\dfrac{45x^4}{2(x+12)^3}$
2. $\dfrac{27x^6}{20(x-7)^3}\cdot\dfrac{(x-7)^5}{54x^7}$
3. $\dfrac{x^2-64}{36x^4}\cdot\dfrac{12x^3}{x^2+4x-32}$
4. $\dfrac{50x^5}{x^2+6x-27}\cdot\dfrac{x^2-81}{125x^3}$
5. $\dfrac{2x^2+7x+5}{3x^2}\cdot\dfrac{15x^3-30x^2}{2x^2+x-10}$
6. $\dfrac{3x^2+14x-5}{2x^2+11x+5}\cdot\dfrac{4x^2+4x+1}{6x^2+x-1}$
7. $\dfrac{x^2+4x-21}{5x^2+10x}\div\dfrac{x^2-6x+9}{x^2+9x+14}$
8. $\dfrac{x^2-49}{9x^2-24x+16}\div\dfrac{2x^2-13x-7}{6x^2-5x-4}$
9. $\dfrac{5x^2+x-6}{4x^2-7x-15}\div\dfrac{1-x^2}{4x^2+9x+5}$
10. $\dfrac{6x^2-8x-8}{4-9x^2}\div\dfrac{3x^2-4x-4}{9x^2+12x+4}$
11. $\dfrac{x^2+4x-12}{x^2-2x-15}\div\dfrac{2x^2-13x+18}{6x^2-31x+5}$
12. $\dfrac{8x^2+x-9}{25x^2-1}\div\dfrac{2x^2-x-1}{10x^2-3x-1}$

Answer

1. $\dfrac{63x}{x+12}; x\neq-12,0$
3. $\dfrac{x-8}{3x(x-4)}; x\neq-8,0,4$
5. $5(x+1); x\neq-\dfrac{5}{2},0,2$
7. $\dfrac{(x+7)^2}{5x(x-3)}; x\neq-7,-2,0,3$
9. $-\dfrac{5x+6}{x-3}; x\neq-\dfrac{5}{4},-1,1,3$
11. $\dfrac{(x+6)(6x-1)}{(x+3)(2x-9)}; x\neq-3,\dfrac{1}{6},2,\dfrac{9}{2},5$

Exercise 4.5.12

Perform the operations and simplify. Assume all variable expressions in the denominator are nonzero.

1. $\frac{1}{12ab} \cdot \frac{50a^2(a-b)^2}{a^2-b^2} \cdot \frac{6b}{a(a-b)}$
2. $\frac{b^2-a^2}{(a-b)^2} \cdot \frac{12a(a-b)}{36a^2b} \cdot \frac{9ab(a-b)}{a+b}$
3. $\frac{x^3+y^3}{5xy} \cdot \frac{x^2-y^2}{x^2-2xy+y^2} \cdot \frac{25x^2y}{(y+x)^2}$
4. $\frac{3xy^2}{(2y+x)^2} \cdot \frac{2x^2+5xy+2y^2}{9x^2} \cdot \frac{x^3+8y^3}{6xy^2+3y^3}$
5. $\frac{2x+5}{x-3} \cdot \frac{x^2-9}{5x^4} \div \frac{2x^2+15x+25}{25x^5}$
6. $\frac{5x^2-15x}{9x^2-4} \cdot \frac{3x-2}{20x^3} \div \frac{x-3}{3x^2-x-2}$
7. $\frac{x^2+5x-50}{x^2+5x-14} \div \frac{x^2-25}{x^2-49} \cdot \frac{x-2}{x^2+3x-70}$
8. $\frac{x^2-x-56}{4x^2-4x-3} \div \frac{2x^2+11x-21}{25-9x^2} \cdot \frac{4x^2-12x+9}{3x^2-19x-40}$
9. $\frac{20x^2-8x-1}{6x^2+13x+6} \div \frac{1-100x^2}{3x^2-x-2} \cdot \frac{10x-1}{2x^2-3x+1}$
10. $\frac{12x^2-13x+1}{x^2+18x+81} \div (144x^2-1) \cdot \frac{x^2+14x+45}{12x^2-11x-1}$
11. A manufacturer has determined that the cost in dollars of producing bicycles is given by $C(x)=0.5x^2-x+6200$, where x represents the number of bicycles produced weekly. Determine the average cost of producing $50, 100$, and 150 bicycles per week.
12. The cost in dollars of producing custom lighting fixtures is given by the function $C(x)=x^2-20x+1200$, where x represents the number of fixtures produced in a week. Determine the average cost per unit if $20, 40$, and 50 units are produced in a week.
13. A manufacturer has determined that the cost in dollars of producing electric scooters is given by the function $C(x)=3x(x-100)+32,000$, where x represents the number of scooters produced in a month. Determine the average cost per scooter if 50 are produced in a month.
14. The cost in dollars of producing a custom injected molded part is given by $C(n)=1,900+0.01n$, where n represents the number of parts produced. Calculate the average cost of each part if $2,500$ custom parts are ordered.
15. The cost in dollars of an environmental cleanup is given by the function $C(p)=\frac{25,000p}{1-p}$, where p represents the percentage of the area to be cleaned up $(0 \leq p < 1)$. Use the function to determine the cost of cleaning up 50% of an affected area and the cost of cleaning up 80% of the area.
16. The value of a new car is given by the function $V(t)=16,500(t+1)^{-1}$ where t represents the age of the car in years. Determine the value of the car when it is 6 years old.

Answer

1. $\frac{25}{a+b}$
3. $\frac{5x\left(x^2-xy+y^2\right)}{x-y}$
5. $\frac{5x(x+3)}{x+5}$
7. $\frac{1}{x+5}$
9. $-\frac{1}{2x+3}$
11. If 50 bicycles are produced, the average cost per bicycle is $\$148$. If 100 are produced, the average cost is $\$111$. If 150 bicycles are produced, the average cost is $\$115.33$.
13. If 50 scooters are produced, the average cost of each is $\$490$.
15. A 50% cleanup will cost $\$25,000$. An 80% cleanup will cost $\$100,000$.

Exercise $4.5.13$

1. Describe the restrictions to the rational expression $\frac{1}{x^2-y^2}$. Explain.
2. Describe the restrictions to the rational expression $\frac{1}{x^2+y^2}$. Explain.
3. Explain why $x=5$ is a restriction to $\frac{1}{x+5} \div \frac{x-5}{x}$.
4. Explain to a beginning algebra student why we cannot cancel x in the rational expression $\frac{x+2}{x}$.
5. Research and discuss the importance of the difference quotient. What does it represent and in what subject does it appear?

Answer

1. Answer may vary

3. Answer may vary

5. Answer may vary

Footnotes

[25] Functions of the form $r(x)=\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$.

[26] The set of real numbers for which the rational function is defined.

[27] The set of real numbers for which a rational function is not defined.

[28] If given a binomial $a-b$, then the opposite is $-(a-b)=b-a$.

[29] The mathematical quantity $\frac{f(x+h)-f(x)}{h}$, where $h \neq 0$, which represents the slope of a secant line through a function f.

[30] Line that intersects two points on the graph of a function.

[31] The total cost divided by the number of units produced, which can be represented by $\overline{C}(x)=\frac{C(x)}{x}$, where $C(x)$ is a cost function.

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4.6: Rational Functions - Addition and Subtraction

Learning Objectives

- Add and subtract rational functions.
- Simplify complex rational expressions.

Adding and Subtracting Rational Functions

Adding and subtracting rational expressions is similar to adding and subtracting fractions. Recall that if the denominators are the same, we can add or subtract the numerators and write the result over the common denominator. When working with rational expressions, the common denominator will be a polynomial. In general, given polynomials P, Q, and R, where $Q \neq 0$, we have the following:

$$\frac{P}{Q} \pm \frac{R}{Q} = \frac{P \pm R}{Q}$$

The set of restrictions to the domain of a sum or difference of rational expressions consists of the restrictions to the domains of each expression.

Example $4.6.1$:

Subtract: $\frac{4x}{x^2-64} - \frac{3x+8}{x^2-64}$.

Solution

The denominators are the same. Hence we can subtract the numerators and write the result over the common denominator. Take care to distribute the negative 1.

$$\begin{aligned} \frac{4x}{x^2-64} - \frac{3x+8}{x^2-64} &= \frac{4x-(3x+8)}{x^2-64} & & \textit{Subtract the numerators.} \\ &= \frac{4x-3x-8}{x^2-64} & & \textit{Simplify.} \\ &= \frac{\cancel{x-8}}{(x+8)\,\cancel{(x-8)}} & & \textit{Cancel.} \\ &= \frac{1}{x+8} & & \textit{Restrictions } x \neq \pm 8 \end{aligned}$$

Answer:

$\frac{1}{x+8}$, where $x \neq \pm 8$

To add rational expressions with unlike denominators, first find equivalent expressions with common denominators. Do this just as you have with fractions. If the denominators of fractions are relatively prime, then the least common denominator (LCD) is their product. For example,

$$\frac{1}{x} + \frac{1}{y} \quad \Rightarrow \quad \text{LCD} = x \cdot y = xy$$

Multiply each fraction by the appropriate form of 1 to obtain equivalent fractions with a common denominator.

$$\begin{aligned} \frac{1}{x} + \frac{1}{y} &= \frac{1 \cdot y}{x \cdot y} + \frac{1 \cdot x}{y \cdot x} \\ &= \frac{y}{xy} + \frac{x}{xy} & & \textit{Equivalent fractions with a common denominator} \\ &= \frac{y+x}{xy} \end{aligned}$$

In general, given polynomials P, Q, R, and S, where $Q \neq 0$ and $S \neq 0$, we have the following:

$$\frac{P}{Q} \pm \frac{R}{S} = \frac{PS \pm QR}{QS}$$

Example $4.6.2$:

Given $f(x)=\dfrac{5x}{3x+1}$ and $g(x)=\dfrac{2}{x+1}$, find $f+g$ and state the restrictions.

Solution

Here the LCD is the product of the denominators $(3x+1)(x+1)$. Multiply by the appropriate factors to obtain rational expressions with a common denominator before adding.

$$\begin{aligned}
&=(f+g)(x)=f(x)+g(x)\\
&=\frac{5x}{3x+1}+\frac{2}{x+1}\\
&=\frac{5x}{(3x+1)}\cdot\frac{(x+1)}{(x+1)}+\frac{2}{(x+1)}\cdot\frac{(3x+1)}{(3x+1)}\\
&=\frac{5x(x+1)}{(3x+1)(x+1)}+\frac{2(3x+1)}{(x+1)(3x+1)}\\
&=\frac{5x(x+1)+2(3x+1)}{(3x+1)(x+1)}\\
&=\frac{5x^2+5x+6x+2}{(3x+1)(x+1)}\\
&=\frac{5x^2+11x+2}{(3x+1)(x+1)}\\
&=\frac{(5x+1)(x+2)}{(3x+1)(x+1)}
\end{aligned}$$

The domain of f consists all real numbers except $-\dfrac{1}{3}$, and the domain of g consists of all real numbers except -1. Therefore, the domain of $f+g$ consists of all real numbers except -1 and $-\dfrac{1}{3}$.

Answer:

$(f+g)(x)=\dfrac{(5x+1)(x+2)}{(3x+1)(x+1)}$, where $x\neq -1,-\dfrac{1}{3}$

It is not always the case that the LCD is the product of the given denominators. Typically, the denominators are not relatively prime; thus determining the LCD requires some thought. Begin by factoring all denominators. The LCD is the product of all factors with the highest power.

Example $4.6.3$:

Given $f(x)=\dfrac{3x}{3x-1}$ and $g(x)=\dfrac{4-14x}{3x^2-4x+1}$, find $f-g$ and state the restrictions to the domain.

Solution

To determine the LCD, factor the denominator of g.

$$\begin{aligned}
(f-g)(x)&=f(x)-g(x)\\
&=\frac{3x}{3x-1}-\frac{4-14x}{3x^2-4x+1}\\
&=\frac{3x}{(3x-1)}-\frac{4-14x}{(3x-1)(x-1)}
\end{aligned}$$

In this case the LCD $=(3x-1)(x-1)$. Multiply f by 1 in the form of $\dfrac{(x-1)}{(x-1)}$ to obtain equivalent algebraic fractions with a common denominator and then subtract.

$$\begin{aligned}
&=\frac{3x}{(3x-1)}\cdot\frac{(x-1)}{(x-1)}-\frac{4-14x}{(3x-1)(x-1)}\\
&=\frac{3x(x-1)-4+14x}{(3x-1)(x-1)}\\
&=\frac{3x^2+11x-4}{(3x-1)(x-1)}\\
&=\frac{\cancel{(3x-1)}\,(x+4)}{\cancel{(3x-1)}\,(x-1)}\\
&=\frac{(x+4)}{(x-1)}
\end{aligned}$$

The domain of f consists of all real numbers except $\frac{1}{3}$, and the domain of g consists of all real numbers except 1 and $\frac{1}{3}$. Therefore, the domain of $f-g$ consists of all real numbers except 1 and $\frac{1}{3}$.

Answer:

$(f-g)(x)=\frac{x+4}{x-1}$, where $x\neq\frac{1}{3},1$

Example 4.6.4:

Simplify and state the restrictions: $\frac{-2x}{x+6}-\frac{3x}{6-x}-\frac{18(x-2)}{x^2-36}$.

Solution

Begin by applying the opposite binomial property $6-x=-(x-6)$.

$$\begin{aligned}
&\frac{-2x}{x+6}-\frac{3x}{6-x}-\frac{18(x-2)}{x^2-36}\\
&=\frac{-2x}{(x+6)}-\frac{3x}{-1\cdot(x-6)}-\frac{18(x-2)}{(x+6)(x-6)}\\
&=\frac{-2x}{(x+6)}+\frac{3x}{(x-6)}-\frac{18(x-2)}{(x+6)(x-6)}
\end{aligned}$$

Next, find equivalent fractions with the $LCD=(x+6)(x-6)$ and then simplify.

$$\begin{aligned}
&=\frac{-2x}{(x+6)}\cdot\frac{(x-6)}{(x-6)}+\frac{3x}{(x-6)}\cdot\frac{(x+6)}{(x+6)}-\frac{18(x-2)}{(x+6)(x-6)}\\
&=\frac{-2x(x-6)+3x(x+6)-18(x-2)}{(x+6)(x-6)}\\
&=\frac{-2x^2+12x+3x^2+18x-18x+36}{(x+6)(x-6)}\\
&=\frac{x^2+12x+36}{(x+6)(x-6)}\\
&=\frac{\cancel{(x+6)}\,(x+6)}{\cancel{(x+6)}\,(x-6)}\\
&=\frac{x+6}{x-6}
\end{aligned}$$

Answer:

$\frac{x+6}{x-6}$, where $x\neq\pm6$

Exercise 4.6.1

Simplify and state the restrictions: $\frac{x+1}{(x-1)^2}-\frac{2}{x^2-1}-\frac{4}{(x+1)(x-1)^2}$

Answer

$\frac{1}{x-1}$, where $x\neq\pm1$

www.youtube.com/v/UjdOYZxy83s

Rational expressions are sometimes expressed using negative exponents. In this case, apply the rules for negative exponents before simplifying the expression.

Example 4.6.5:

Simplify and state the restrictions: $5a^{-2}+(2a+5)^{-1}$.

Solution

Recall that $x^{-n}=\frac{1}{x^n}$. Begin by rewriting the rational expressions with negative exponents as fractions.

$$5a^{-2}+(2a+5)^{-1}=\frac{5}{a^2}+\frac{1}{(2a+5)^1}$$

Then find the LCD and add.

$$\begin{aligned}\frac{5}{a^2}+\frac{1}{(2a+5)^1}&=\frac{5}{a^2}\cdot\frac{(2a+5)}{(2a+5)}+\frac{1}{(2a+5)}\cdot\frac{a^2}{a^2}\\&=\frac{5(2a+5)}{a^2(2a+5)}+\frac{a^2}{a^2(2a+5)} && \textit{Equivalent expressions with a common denominator.}\\&=\frac{10a+25+a^2}{a^2(2a+5)} && \textit{Add.}\\&=\frac{a^2+10a+25}{a^2(2a+5)} && \textit{Simplify.}\\&=\frac{(a+5)(a+5)}{a^2(2a+5)}\end{aligned}$$

Answer:

$\frac{(a+5)^2}{a^2(2a+5)^2}$, where $a\neq-\frac{5}{2},0$

Simplifying Complex Rational Expressions

A **complex rational expression**[32] is defined as a rational expression that contains one or more rational expressions in the numerator or denominator or both. For example,

$$\frac{4-\frac{12}{x}+\frac{9}{x^2}}{2-\frac{5}{x}+\frac{3}{x^2}}$$

is a complex rational expression. We simplify a complex rational expression by finding an equivalent fraction where the numerator and denominator are polynomials. There are two methods for simplifying complex rational expressions, and we will outline the steps for both methods. For the sake of clarity, assume that variable expressions used as denominators are nonzero.

Method 1: Simplify Using Division

We begin our discussion on simplifying complex rational expressions using division. Before we can multiply by the reciprocal of the denominator, we must simplify the numerator and denominator separately. The goal is to first obtain single algebraic fractions in the numerator and the denominator. The steps for simplifying a complex algebraic fraction are illustrated in the following example.

Example 4.6.6:

Simplify: $\frac{4-\frac{12}{x}+\frac{9}{x^2}}{2-\frac{5}{x}+\frac{3}{x^2}}$.

Solution

Step 1: Simplify the numerator and denominator to obtain a single algebraic fraction divided by another single algebraic fraction. In this example, find equivalent terms with a common denominator in both the numerator and denominator before adding and subtracting.

$$\begin{aligned}\frac{4-\frac{12}{x}+\frac{9}{x^2}}{2-\frac{5}{x}+\frac{3}{x^2}} &= \frac{\frac{4}{1}\cdot\frac{x^2}{x^2}-\frac{12}{x}\cdot\frac{x}{x}+\frac{9}{x^2}}{\frac{2}{1}\cdot\frac{x^2}{x^2}-\frac{5}{x}\cdot\frac{x}{x}+\frac{3}{x^2}} \\ &= \frac{\frac{4x^2}{x^2}-\frac{12x}{x^2}+\frac{9}{x^2}}{\frac{2x^2}{x^2}-\frac{5x}{x^2}+\frac{3}{x^2}} && \textit{Equivalent fractions with common denominators.} \\ &= \frac{\frac{4x^2-12x+9}{x^2}}{\frac{2x^2-5x+3}{x^2}} && \textit{Add the fractions in the numerator and denominator.}\end{aligned}$$

At this point we have a single algebraic fraction divided by another single algebraic fraction.

Step 2: Multiply the numerator by the reciprocal of the denominator.

$$\frac{\frac{4x^2-12x+9}{x^2}}{\frac{2x^2-5x+3}{x^2}}=\frac{4x^2-12x+9}{x^2}\cdot\frac{x^2}{2x^2-5x+3}$$

Step 3: Factor all numerators and denominators completely.

$$=\frac{(2x-3)(2x-3)}{x^2}\cdot\frac{x^2}{(2x-3)(x-1)}$$

Step 4: Cancel all common factors.

$$\begin{aligned}&=\frac{\cancel{(2x-3)}\,(2x-3)}{\cancel{x^2}}\cdot\frac{\cancel{x^2}}{\cancel{(2x-3)}\,(x-1)} \\ &=\frac{2x-3}{x-1}\end{aligned}$$

Answer:

$$\frac{2x-3}{x-1}$$

Example 4.6.7:

Simplify: $\dfrac{\frac{2x}{x-1}+\frac{7}{x+3}}{\frac{2x}{x-1}-\frac{5}{x-3}}$

Solution

Obtain a single algebraic fraction in the numerator and in the denominator.

$$\begin{aligned}\frac{\frac{2x}{x-1}+\frac{7}{x+3}}{\frac{2x}{x-1}-\frac{5}{x-3}} &= \frac{\frac{2x}{x-1}\cdot\frac{(x+3)}{(x+3)}+\frac{7}{x+3}\cdot\frac{(x-1)}{(x-1)}}{\frac{2x}{x-1}\cdot\frac{(x-3)}{(x-3)}-\frac{5}{x-3}\cdot\frac{(x-1)}{(x-1)}} \\ &= \frac{\frac{2x(x+3)+7(x-1)}{(x-1)(x+3)}}{\frac{2x(x-3)-5(x-1)}{(x-1)(x-3)}} \\ &= \frac{\frac{2x^2+6x+7x-7}{(x-1)(x+3)}}{\frac{2x^2-6x-5x+5}{(x-1)(x-3)}} \\ &= \frac{\frac{2x^2+13x-7}{(x-1)(x+3)}}{\frac{2x^2-11x+5}{(x-1)(x-3)}}\end{aligned}$$

Next, multiply the numerator by the reciprocal of the denominator, factor, and then cancel.

$$\begin{aligned} &= \frac{2x^2+13x-7}{(x-1)(x+3)}\cdot\frac{(x-1)(x-3)}{2x^2-11x+5} \\ &= \frac{\cancel{(2x-1)}(x+7)}{\cancel{(x-1)}(x+3)}\cdot\frac{\cancel{(x-1)}(x-3)}{\cancel{(2x-1)}(x-5)} \\ &= \frac{(x+7)(x-3)}{(x+3)(x-5)}\end{aligned}$$

Answer:

$\frac{(x+7)(x-3)}{(x+3)(x-5)}$

Exercise $4.6.2$

Simplify using division: $\frac{\frac{1}{y^2}-\frac{1}{x^2}}{\frac{1}{y}+\frac{1}{x}}$.

Answer

$\frac{x-y}{xy}$

www.youtube.com/v/4DqCL_HbINQ

Sometimes complex rational expressions are expressed using negative exponents.

Example $4.6.8$:

Simplify: $\frac{2y^{-1}-x^{-1}}{x^{-2}-4y^{-2}}$.

Solution

We begin by rewriting the expression without negative exponents.

$$\frac{2y^{-1}-x^{-1}}{x^{-2}-4y^{-2}}=\frac{\frac{2}{y}-\frac{1}{x}}{\frac{1}{x^2}-\frac{4}{y^2}}$$

Obtain single algebraic fractions in the numerator and denominator and then multiply by the reciprocal of the denominator.

$$\begin{aligned}\frac{\frac{2}{y}-\frac{1}{x}}{x^2-\frac{4}{y^2}} &= \frac{\frac{2x-y}{xy}}{\frac{y^2-4x^2}{x^2y^2}}\\ &= \frac{2x-y}{xy}\cdot\frac{x^2y^2}{y^2-4x^2}\\ &= \frac{2x-y}{xy}\cdot\frac{x^2y^2}{(y-2x)(y+2x)}\end{aligned}$$

Apply the opposite binomial property $(y-2x)=-(2x-y)$ and then cancel.

$$\begin{aligned}&= \frac{\cancel{(2x-y)}}{xy}\cdot\frac{\overset{x}{\cancel{x^2}}\ \overset{y}{\cancel{y^2}}}{-\cancel{(2x-y)}\,(y+2x)}\\ &= -\frac{xy}{y+2x}\end{aligned}$$

Answer: $-\dfrac{xy}{y+2x}$

Method 2: Simplify Using the LCD

An alternative method for simplifying complex rational expressions involves clearing the fractions by multiplying the expression by a special form of 1. In this method, multiply the numerator and denominator by the least common denominator (LCD) of all given fractions.

Example $4.6.9$:

Simplify: $\dfrac{4-\frac{12}{x}+\frac{9}{x^2}}{2-\frac{5}{x}+\frac{3}{x^2}}$.

Solution

Step 1: Determine the LCD of all the fractions in the numerator and denominator. In this case, the denominators of the given fractions are $1, x$, and x^2. Therefore, the LCD is x^2.

Step 2: Multiply the numerator and denominator by the LCD. This step should clear the fractions in both the numerator and denominator.

$$\begin{aligned}\frac{4-\frac{12}{x}+\frac{9}{x^2}}{2-\frac{5}{x}+\frac{3}{x^2}} &= \frac{\left(4-\frac{12}{x}+\frac{9}{x^2}\right)\cdot x^2}{\left(2-\frac{5}{x}+\frac{3}{x^2}\right)\cdot x^2} && \textit{Multiply numerator and denominator.}\\ &= \frac{4\cdot x^2-\frac{12}{x}\cdot x^2+\frac{9}{x^2}\cdot x^2}{2\cdot x^2-\frac{5}{x}\cdot x^2+\frac{3}{x^2}\cdot x^2} && \textit{Distribute and then cancel.}\\ &= \frac{4x^2-12x+9}{2x^2-5x+3}\end{aligned}$$

This leaves us with a single algebraic fraction with a polynomial in the numerator and in the denominator.

Step 3: Factor the numerator and denominator completely.

$$\begin{aligned}&= \frac{4x^2-12x+9}{2x^2-5x+3}\\ &= \frac{(2x-3)(2x-3)}{(x-1)(2x-3)}\end{aligned}$$

Step 4: Cancel all common factors.

$$\begin{aligned}&= \frac{(2x-3)\,\cancel{(2x-3)}}{(x-1)\,\cancel{(2x-3)}}\\ &= \frac{2x-3}{x-1}\end{aligned}$$

Note

This was the same problem presented in Example 6 and the results here are the same. It is worth taking the time to compare the steps involved using both methods on the same problem.

Answer: $\frac{2x-3}{x-1}$

It is important to point out that multiplying the numerator and denominator by the same nonzero factor is equivalent to multiplying by 1 and does not change the problem.

Exercise $4.6.3$

Simplify using the LCD: $\dfrac{\frac{1}{y^2}-\frac{1}{x^2}}{\frac{1}{y}+\frac{1}{x}}$.

Answer

$\frac{x-y}{xy}$

www.youtube.com/v/FP7Z1YEHgLE

Key Takeaways

- Adding and subtracting rational expressions is similar to adding and subtracting fractions. A common denominator is required. If the denominators are the same, then we can add or subtract the numerators and write the result over the common denominator.
- The set of restrictions to the domain of a sum or difference of rational functions consists of the restrictions to the domains of each function.
- Complex rational expressions can be simplified into equivalent expressions with a polynomial numerator and polynomial denominator. They are reduced to lowest terms if the numerator and denominator are polynomials that share no common factors other than 1.
- One method of simplifying a complex rational expression requires us to first write the numerator and denominator as a single algebraic fraction. Then multiply the numerator by the reciprocal of the denominator and simplify the result.
- Another method for simplifying a complex rational expression requires that we multiply it by a special form of 1. Multiply the numerator and denominator by the LCD of all the denominators as a means to clear the fractions. After doing this, simplify the remaining rational expression.

Exercise $4.6.4$

State the restrictions and simplify.

1. $\frac{3x}{3x+4}+\frac{2}{3x+4}$
2. $\frac{3x}{2x-1}-\frac{2x+1}{2x-1}$
3. $\frac{x-2}{2x^2-11x-6}+\frac{x+3}{2x^2-11x-6}$
4. $\frac{4x-1}{3x^2+2x-5}-\frac{x-6}{3x^2+2x-5}$
5. $\frac{1}{x}-2x$
6. $\frac{4}{x^3}-\frac{1}{x}$
7. $\frac{1}{x-1}+5$
8. $\frac{1}{x+7}-1$
9. $\frac{1}{x-2}-\frac{1}{3x+4}$
10. $\frac{2}{5x-2}+\frac{x}{x+3}$

11. $\dfrac{1}{x^2}+\dfrac{1}{x-2}$
12. $\dfrac{2x}{x}+\dfrac{2}{x-2}$
13. $\dfrac{3x-7}{x(x-7)}+\dfrac{1}{7-x}$
14. $\dfrac{2}{8-x}+\dfrac{3x^2-1}{x^2(x-8)}$
15. $\dfrac{x-1}{x^2-25}-\dfrac{2}{x^2-10x+25}$
16. $\dfrac{x+1}{2x^2+5x-3}-\dfrac{x}{4x^2-1}$
17. $\dfrac{x}{x^2+4x}-\dfrac{2}{x^2+8x+16}$
18. $\dfrac{2x-1}{4x^2+8x-5}-\dfrac{3}{4x^2+20x+25}$
19. $\dfrac{5-x}{7x+x^2}-\dfrac{x+2}{49-x^2}$
20. $\dfrac{2x}{4x^2+x}-\dfrac{x+1}{8x^2+6x+1}$
21. $\dfrac{x-1}{2x^2-7x-4}+\dfrac{2x-1}{x^2-5x+4}$
22. $\dfrac{2(x+3)}{3x^2-5x-2}+\dfrac{4-x}{3x^2+10x+3}$
23. $\dfrac{x^2}{4+2x^2}-\dfrac{2}{x^4+2x^2}$
24. $\dfrac{3x}{4x^4+6x^3}-\dfrac{2x^2}{6x^3+9x^2}$
25. $\dfrac{3x^2-12}{x^4-8x^2+16}-\dfrac{x^2+2}{4-x^2}$
26. $\dfrac{x^2}{2x^2+1}+\dfrac{6x^2-24}{2x^4-7x^2-4}$

Answer

1. $\dfrac{3x+2}{3x+4}; x\neq -\dfrac{4}{3}$
3. $\dfrac{1}{x-6}; x\neq -\dfrac{1}{2}, 6$
5. $\dfrac{1-2x^2}{x}; x\neq 0$
7. $\dfrac{5x-4}{x-1}; x\neq 1$
9. $\dfrac{2(x+3)}{(x-2)(3x+4)}; x\neq -\dfrac{4}{3}, 2$
11. $\dfrac{(x-1)(x+2)}{x^2(x-2)}; x\neq 0, 2$
13. $\dfrac{2x-7}{x(x-7)}; x\neq 0, 7$
15. $\dfrac{x^2-8x-5}{(x+5)(x-5)^2}; x\neq \pm 5$
17. $\dfrac{x+2}{(x+4)^2}; x\neq 0, -4$
19. $\dfrac{7(5-2x)}{x(7+x)(7-x)}; x\neq -7, 0, 7$
21. $\dfrac{x(5x-2)}{(x-4)(x-1)(2x+1)}; x\neq -\dfrac{1}{2}, 1, 4$
23. $\dfrac{x^2-2}{2x^2}; x\neq 0$
25. $\dfrac{x^2+5}{(x+2)(x-2)}; x\neq \pm 2$

Exercise $4.6.5$

Given f and g, simplify $f+g$ and difference $f-g$. Also, state the domain using interval notation.

1. $f(x)=\frac{1}{x}, g(x)=\frac{5}{x^2}$
2. $f(x)=\frac{1}{x+2}, g(x)=\frac{2}{x-1}$
3. $f(x)=\frac{x-2}{x+2}, g(x)=\frac{x+2}{x-2}$
4. $f(x)=\frac{x}{2x-1}, g(x)=\frac{2x}{2x+1}$
5. $f(x)=\frac{6}{3x^2+x}, g(x)=\frac{18}{9x^2+6x+1}$
6. $f(x)=\frac{x-1}{x^2-8x+16}, g(x)=\frac{x-2}{x^2-4x}$
7. $f(x)=\frac{x}{x^2-25}, g(x)=\frac{x-1}{x^2-4x-5}$
8. $f(x)=\frac{2x-3}{x^2-4}, g(x)=\frac{x}{2x^2+3x-2}$
9. $f(x)=\frac{1}{3x^2-x-2}, g(x)=-\frac{1}{4x^2-3x-1}$
10. $f(x)=\frac{6}{6x^2+13x-5}, g(x)=-\frac{2}{2x^2+x-10}$

Answer

1. $(f+g)(x)=\frac{x+5}{x^2}$; $(f-g)(x)=\frac{x-5}{x^2}$; Domain: $(-\infty, 0) \cup(0, \infty)$
3. $(f+g)(x)=\frac{2\left(x^2+4\right)}{(x+2)(x-2)}$; $(f-g)(x)=-\frac{8x}{(x+2)(x-2)}$; Domain: $(-\infty,-2) \cup(-2,2) \cup(2, \infty)$
5. $(f+g)(x)=\frac{6(6x+1)}{x(3x+1)^2}$; $(f-g)(x)=\frac{6}{x(3x+1)^2}$; Domain: $\left(-\infty,-\frac{1}{3}\right) \cup\left(-\frac{1}{3}, 0\right) \cup(0, \infty)$
7. $(f+g)(x)=\frac{2x^2+5x-5}{(x+1)(x+5)(x-5)}$; $(f-g)(x)=-\frac{3x-5}{(x+1)(x+5)(x-5)}$; Domain: $(-\infty,-5) \cup(-5,-1) \cup(-1,5) \cup(5, \infty)$
9. $(f+g)(x)=\frac{1}{(3x+2)(4x+1)}$; $(f-g)(x)=-\frac{7x+3}{(x-1)(3x+2)(4x+1)}$; Domain: $\left(-\infty,-\frac{2}{3}\right) \cup\left(-\frac{2}{3}, \frac{1}{4}\right) \cup\left(\frac{1}{4}, 1\right) \cup(1, \infty)$

Exercise $4.6.6$

State the restrictions and simplify.

1. $1+\frac{3}{x}-\frac{5x-1}{x^2}$
2. $4+\frac{2}{x}-\frac{6x-1}{x^2}$
3. $\frac{2x}{x-8}-\frac{1}{3x+1}-\frac{2x+9}{3x^2-23x-8}$
4. $\frac{4x}{x-2}-\frac{10}{3x+1}-\frac{19x+18}{3x^2-5x-2}$
5. $\frac{1}{x-1}+\frac{1}{(x-1)^2}-\frac{1}{x^2-1}$
6. $\frac{1}{x-2}-\frac{1}{x^2-4}+\frac{1}{(x-2)^2}$
7. $\frac{2x+1}{x-1}-\frac{3x}{2x^2-3x+1}+\frac{x+1}{x-2x^2}$
8. $\frac{5x^2}{2x^2+2x}-\frac{x^2-4x}{x^2-2x}+\frac{4+2x^2}{4+2x-2x^2}$
9. $\frac{x+2}{2x(3x-2)}+\frac{4x}{(x-2)(3x-2)}-\frac{3x+2}{2x(x-2)}$
10. $\frac{10x}{x(x-5)}-\frac{2x^2}{(2x-5)(x-5)}-\frac{5x}{x(2x-5)}$

Answer

1. $\frac{(x-1)^2}{x^2}; x \neq 0$
3. $\frac{2x-1}{x-8}; x \neq-\frac{1}{3}, 8$

5. $\frac{x^2+1}{(x-1)^2(x+1)}; x \neq \pm 1$
7. $\frac{2x+1}{x}; x \neq 0, \frac{1}{2}, 1$
9. $0; x \neq 0, \frac{2}{3}, 2$

Exercise 4.6.7

Simplify the given algebraic expressions. Assume all variable expressions in the denominator are nonzero.

1. $x^{-2}+y^{-2}$
2. $x^{-2}+(2y)^{-2}$
3. $2x^{-1}+y^{-2}$
4. $x^{-2}-4y^{-1}$
5. $16x^{-2}+y^{2}$
6. $xy^{-1}-yx^{-1}$
7. $3(x+y)^{-1}+x^{-2}$
8. $2(x-y)^{-2}-(x-y)^{-1}$
9. $a^{-2}-(a+b)^{-1}$
10. $(a-b)^{-1}-(a+b)^{-1}$
11. $x^{-n}+y^{-n}$
12. $xy^{-n}+yx^{-n}$

Answer

1. $\frac{y^2+x^2}{x^2y^2}$
3. $\frac{x+2y^2}{xy^2}$
5. $\frac{x^2y^2+16}{x^2}$
7. $\frac{3x^2+x+y}{x^2(x+y)}$
9. $\frac{a+b-a^2}{a^2(a+b)}$
11. $\frac{x^n+y^n}{x^ny^n}$

Exercise 4.6.8

Simplify. Assume all variable expressions in the denominators are nonzero.

1. $\frac{\frac{75x^2}{(x-3)^2}}{\frac{25x^3}{x-3}}$
2. $\frac{\frac{x+5}{36x^3}}{\frac{(x+5)^3}{9x^2}}$
3. $\frac{\frac{x^2-36}{32x^5}}{\frac{x-6}{4x^3}}$
4. $\frac{\frac{x-8}{56x^2}}{\frac{x^2-64}{7x^3}}$

5. $\dfrac{\dfrac{5x+1}{2x^2+x-10}}{\dfrac{25x^2+10x+1}{4x^2-25}}$

6. $\dfrac{\dfrac{4x^2-27x-7}{4x^2-1}}{\dfrac{x-7}{6x^2-x-1}}$

7. $\dfrac{\dfrac{x^2-4x-5}{2x^2+3x+1}}{\dfrac{x^2-10x+25}{2x^2+7x+3}}$

8. $\dfrac{\dfrac{5x^2+9x-2}{x^2+4x+4}}{\dfrac{10x^2+3x-1}{4x^2+7x-2}}$

9. $\dfrac{x^2}{\dfrac{1}{5}-\dfrac{3}{x}}$

10. $\dfrac{\dfrac{4}{x}-3}{2x^2}$

11. $\dfrac{\dfrac{1}{3}-\dfrac{1}{x}}{\dfrac{1}{9}-\dfrac{1}{x^2}}$

12. $\dfrac{\dfrac{2}{5}+\dfrac{1}{x}}{\dfrac{4}{25}-\dfrac{1}{x^2}}$

13. $\dfrac{\dfrac{1}{y^2}-36}{6-\dfrac{1}{y}}$

14. $\dfrac{\dfrac{1}{5}-\dfrac{1}{y}}{\dfrac{1}{y^2}-\dfrac{1}{25}}$

15. $\dfrac{1-\dfrac{6}{x}+\dfrac{8}{x^2}}{3-\dfrac{5}{x}-\dfrac{2}{x^2}}$

16. $\dfrac{2+\dfrac{13}{x}-\dfrac{7}{x^2}}{3+\dfrac{1}{x}-\dfrac{10}{x^2}}$

17. $\dfrac{9-\dfrac{12}{x}+\dfrac{4}{x^2}}{9-\dfrac{4}{x^2}}$

18. $\dfrac{4-\dfrac{25}{x^2}}{4-\dfrac{8}{x}-\dfrac{5}{x^2}}$

19. $\dfrac{\dfrac{1}{x}+\dfrac{5}{3x-1}}{\dfrac{2}{3x-1}-\dfrac{1}{x}}$

20. $\dfrac{\dfrac{2}{x-5}-\dfrac{1}{x}}{\dfrac{1}{x}-\dfrac{3}{x-5}}$

21. $\dfrac{\dfrac{1}{x+1}+\dfrac{2}{x-2}}{\dfrac{2}{x-3}-\dfrac{1}{x-2}}$

22. $$\frac{\frac{4}{x+5}-\frac{1}{x-3}}{\frac{3}{x-3}+\frac{1}{2x-1}}$$

23. $$\frac{\frac{x-1}{3x-1}-\frac{1}{x+1}}{\frac{x-1}{x+1}-\frac{2}{x+1}}$$

24. $$\frac{\frac{x+1}{3x+5}-\frac{1}{x+3}}{\frac{2}{x+3}-\frac{x+1}{x+3}}$$

25. $$\frac{\frac{2x+3}{2x-3}+\frac{2x-3}{2x+3}}{\frac{2x+3}{2x-3}-\frac{2x-3}{2x+3}}$$

26. $$\frac{\frac{x-1}{x+1}-\frac{x+1}{x-1}}{\frac{x+1}{x-1}-\frac{x-1}{x+1}}$$

27. $$\frac{\frac{1}{2x+5}-\frac{1}{2x-5}+\frac{4x}{4x^2-25}}{\frac{1}{2x+5}+\frac{1}{2x-5}+\frac{4x}{4x^2-25}}$$

28. $$\frac{\frac{1}{3x-1}+\frac{1}{3x+1}}{\frac{3x}{3x-1}-\frac{1}{3x+1}-\frac{6x}{9x^2-1}}$$

29. $$\frac{1}{1+\frac{1}{1+\frac{1}{x}}}$$

30. $$\frac{\frac{1}{x}}{1-\frac{1}{1+\frac{1}{x}}}$$

31. $$\frac{\frac{1}{y}-\frac{1}{x}}{\frac{1}{y^2}-\frac{1}{x^2}}$$

32. $$\frac{\frac{2}{y}+\frac{1}{x}}{\frac{4}{y^2}-\frac{1}{x^2}}$$

33. $$\frac{\frac{1}{25y^2}-\frac{1}{x^2}}{\frac{1}{x}-\frac{1}{5y}}$$

34. $$\frac{16y^2-\frac{1}{x^2}}{\frac{1}{x}-4y}$$

35. $$\frac{\frac{1}{b}+\frac{1}{a}}{\frac{1}{b^3}+\frac{1}{a^3}}$$

36. $$\frac{\frac{1}{a}-\frac{1}{b}}{\frac{1}{b^3}-\frac{1}{a^3}}$$

37. $$\frac{\frac{x}{y}-\frac{y}{x}}{\frac{1}{y^2}-\frac{2}{xy}+\frac{1}{x^2}}$$

38. $\dfrac{\dfrac{2}{y}-\dfrac{5}{x}}{4x-\dfrac{25y^2}{x}}$

39. $\dfrac{x^{-1}+y^{-1}}{y^{-2}-x^{-2}}$

40. $\dfrac{y^{-2}-25x^{-2}}{5x^{-1}-y^{-1}}$

41. $\dfrac{1-x^{-1}}{x-x^{-1}}$

42. $\dfrac{16-x^{-2}}{x^{-1}-4}$

43. $\dfrac{1-4x^{-1}-21x^{-2}}{1-2x^{-1}-15x^{-2}}$

44. $\dfrac{x^{-1}-4\left(3x^2\right)^{-1}}{3-8x^{-1}+16\left(3x^2\right)^{-1}}$

45. $\dfrac{(x-3)^{-1}+2x^{-1}}{x^{-1}-3(x-3)^{-1}}$

46. $\dfrac{(4x-5)^{-1}+x^{-2}}{x^{-2}+(3x-10)^{-1}}$

47. Given $f(x)=\dfrac{1}{x}$, simplify $\dfrac{f(b)-f(a)}{b-a}$.

48. Given $f(x)=\dfrac{1}{x-1}$, simplify $\dfrac{f(b)-f(a)}{b-a}$.

49. Given $f(x)=\dfrac{1}{x}$, simplify the difference quotient $\dfrac{f(x+h)-f(x)}{h}$.

50. Given $f(x)=\dfrac{1}{x}+1$, simplify the difference quotient $\dfrac{f(x+h)-f(x)}{h}$.

Answer

1. $\dfrac{3}{x(x-3)}$

3. $\dfrac{x+6}{8x^2}$

5. $\dfrac{2x-5}{(x-2)(5x+1)}$

7. $\dfrac{x+3}{x-5}$

9. $\dfrac{5x^3}{x-15}$

11. $\dfrac{3x}{x+3}$

13. $-\dfrac{6y+1}{y}$

15. $\dfrac{x-4}{3x+1}$

17. $\dfrac{3x-2}{3x+2}$

19. $-\dfrac{8x-1}{x-1}$

21. $\dfrac{3x(x-3)}{(x+1)(x-1)}$

23. $\dfrac{x}{3x-1}$

25. $\dfrac{4x^2+9}{12x}$

27. $\dfrac{2x-5}{4x}$

29. $\dfrac{x+1}{2x+1}$

31. $\dfrac{xy}{x+y}$

33. $-\frac{x+5y}{5xy}$

35. $\frac{a^2b^2}{a^2-ab+b^2}$

37. $\frac{xy(x+y)}{x-y}$

39. $\frac{xy}{x-y}$

41. $\frac{1}{x+1}$

43. $\frac{x-7}{x-5}$

45. $-\frac{3(x-2)}{2x+3}$

47. $-\frac{1}{ab}$

49. $-\frac{1}{x(x+h)}$

Exercise 4.6.9

1. Explain why the domain of a sum of rational functions is the same as the domain of the difference of those functions.
2. Two methods for simplifying complex rational expressions have been presented in this section. Which of the two methods do you feel is more efficient, and why?

Answer

1. Answer may vary

Footnotes

[32] A rational expression that contains one or more rational expressions in the numerator or denominator or both.

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4.7: Solving Rational Equations

Learning Objectives

- Solve rational equations.
- Solve literal equations, or formulas, involving rational expressions.
- Solve applications involving the reciprocal of unknowns.

Solving Rational Equations

A **rational equation**[33] is an equation containing at least one rational expression. Rational expressions typically contain a variable in the denominator. For this reason, we will take care to ensure that the denominator is not 0 by making note of restrictions and checking our solutions. Solving rational equations involves clearing fractions by multiplying both sides of the equation by the least common denominator (LCD).

Example $4.7.1$:

Solve: $\frac{1}{x}+\frac{2}{x^2}=\frac{x+9}{2x^2}$.

Solution

We first make a note of the restriction on \(x, x≠0\). We then multiply both sides by the LCD, which in this case equals $2x^2$.

$$\begin{aligned} 2x^2\cdot\left(\frac{1}{x}+\frac{2}{x^2}\right)&=2x^2\cdot\left(\frac{x+9}{2x^2}\right) && \textit{Multiply both sides by the LCD.}\\ 2x^2\cdot\frac{1}{x}+2x^2\cdot\frac{2}{x^2}&=2x^2\cdot\frac{x+9}{2x^2} && \textit{Distribute.}\\ 2x+4&=x+9 && \textit{Simplify and then solve.}\\ x&=5 \end{aligned}$$

Check your answer**.** Substitute $x=5$ into the original equation and see if you obtain a true statement.

$$\begin{aligned} \frac{1}{x}+\frac{2}{x^2}&=\frac{x+9}{2x^2} && \textit{Original equation}\\ \frac{1}{5}+\frac{2}{5^2}&=\frac{5+9}{2(5)^2} && \textit{Check } x=5.\\ \frac{1}{5}+\frac{2}{25}&=\frac{14}{2\cdot 25}\\ \frac{5}{25}+\frac{2}{25}&=\frac{7}{25}\\ \frac{7}{25}&=\frac{7}{25}\ \checkmark \end{aligned}$$

Answer:

The solution is 5.

After multiplying both sides of the previous example by the LCD, we were left with a linear equation to solve. This is not always the case; sometimes we will be left with quadratic equation.

Example $4.7.2$:

Solve: $\frac{3(x+2)}{x-4}-\frac{x+4}{x-2}=\frac{x-2}{x-4}$.

Solution

In this example, there are two restrictions, $x\neq 4$ and $x\neq 2$. Begin by multiplying both sides by the LCD, $(x-2)(x-4)$.

$$\begin{aligned} (x-2)(x-4)\cdot\left(\frac{3(x+2)}{x-4}-\frac{x+4}{x-2}\right)&=(x-2)(x-4)\cdot\left(\frac{x-2}{x-4}\right)\\ (x-2)\cancel{(x-4)}\cdot\frac{3(x+2)}{\cancel{x-4}}-\cancel{(x-2)}(x-4)\cdot\frac{x+4}{\cancel{x-2}}&=(x-2)\cancel{(x-4)}\cdot\frac{x-2}{\cancel{x-4}}\\ 3(x+2)(x-2)-(x+4)(x-4)&=(x-2)(x-2)\\ 3\left(x^2-4\right)-\left(x^2-16\right)&=x^2-2x-2x+4\\ 3x^2-12-x^2+16&=x^2-4x+4\\ 2x^2+4&=x^2-4x+4 \end{aligned}$$

After distributing and simplifying both sides of the equation, a quadratic equation remains. To solve, rewrite the quadratic equation in standard form, factor, and then set each factor equal to 0.

$$\begin{aligned}&2x^2+4=x^2-4x+4\\&x^2+4x=0\\&x(x+4)=0\\&x=0 \text{ or } x+4=0\\&\qquad\qquad\quad x=-4\end{aligned}$$

Check to see if these values solve the original equation.

$$\frac{3(x+2)}{x-4}-\frac{x+4}{x-2}=\frac{x-2}{x-4}$$

Table 4.7.1

Check $x=0$	Check $x=4$
$\frac{3(0+2)}{0-4}-\frac{0+4}{0-2}=\frac{0-2}{0-4}$ $\frac{6}{-4}-\frac{4}{-2}=\frac{-2}{-4}$ $-\frac{3}{2}+2=\frac{1}{2}$ $-\frac{3}{2}+\frac{4}{2}=\frac{1}{2}$ $\frac{1}{2}=\frac{1}{2}\checkmark$	$\frac{3(-4+2)}{-4-4}-\frac{-4+4}{-4-2}=\frac{-4-2}{-4-4}$ $\frac{3(-2)}{-8}-\frac{0}{-6}=\frac{-6}{-8}$ $\frac{-6}{-8}-0=\frac{3}{4}$ $\frac{3}{4}=\frac{3}{4}\checkmark$

Answer:

The solutions are 0 and -4.

Up to this point, all of the possible solutions have solved the original equation. However, this may not always be the case. Multiplying both sides of an equation by variable factors may lead to **extraneous solutions**[34], which are solutions that do not solve the original equation. A complete list of steps for solving a rational equation is outlined in the following example.

Example 4.7.3

Solve: $\frac{2x}{3x+1}=\frac{1}{x-5}-\frac{4(x-1)}{3x^2-14x-5}$.

Solution

Step 1: Factor all denominators and determine the LCD.

$$\begin{aligned}\frac{2x}{3x+1}&=\frac{1}{x-5}-\frac{4(x-1)}{3x^2-14x-5}\\\frac{2x}{(3x+1)}&=\frac{1}{(x-5)}-\frac{4(x-1)}{(3x+1)(x-5)}\end{aligned}$$

The LCD is $(3x+1)(x-5)$.

Step 2: Identify the restrictions. In this case, $x\neq-\frac{1}{3}$ and $x\neq5$.

Step 3: Multiply both sides of the equation by the LCD. Distribute carefully and then simplify.

$$\begin{aligned}(3x+1)(x-5)\cdot\frac{2x}{(3x+1)}&=(3x+1)(x-5)\cdot\left(\frac{1}{(x-5)}-\frac{4(x-1)}{(3x+1)(x-5)}\right)\\\cancel{(3x+1)}(x-5)\cdot\frac{2x}{\cancel{(3x+1)}}&=(3x+1)\cancel{(x-5)}\cdot\frac{1}{\cancel{(x-5)}}-\cancel{(3x+1)}\cancel{(x-5)}\cdot\frac{4(x-1)}{\cancel{(3x+1)}\cancel{(x-5)}}\\2x(x-5)&=(3x+1)-4(x-1)\end{aligned}$$

Step 4: Solve the resulting equation. Here the result is a quadratic equation. Rewrite it in standard form, factor, and then set each factor equal to 0.

$$\begin{aligned} 2x(x-5) &= (3x+1) - 4(x-1) \\ 2x^2 - 10x &= 3x + 1 - 4x + 4 \\ 2x^2 - 10x &= -x + 5 \\ 2x^2 - 9x - 5 &= 0 \\ (2x+1)(x-5) &= 0 \end{aligned}$$

$$\begin{aligned} 2x+1 &= 0 \quad \text{or} \quad & x-5 &= 0 \\ 2x &= -1 & x &= 5 \\ x &= -\frac{1}{2} \end{aligned}$$

Step 5: Check for extraneous solutions. Always substitute into the original equation, or the factored equivalent. In this case, choose the factored equivalent to check:

$$\frac{2x}{(3x+1)} = \frac{1}{(x-5)} - \frac{4(x-1)}{(3x+1)(x-5)}$$

Table $4.7.2$

Check $x = -\frac{1}{2}$	Check $x = 5$
$\frac{2\left(-\frac{1}{2}\right)}{\left(3\left(-\frac{1}{2}\right)+1\right)} = \frac{1}{\left(\left(-\frac{1}{2}\right)-5\right)} - \frac{4\left(\left(-\frac{1}{2}\right)-1\right)}{\left(3\left(-\frac{1}{2}\right)+1\right)\left(\left(-\frac{1}{2}\right)-5\right)}$ $\frac{-1}{\left(-\frac{1}{2}\right)} = \frac{1}{\left(-\frac{11}{2}\right)} - \frac{4\left(-\frac{3}{2}\right)}{\left(-\frac{1}{2}\right)\left(-\frac{11}{2}\right)}$ $2 = -\frac{2}{11} - \frac{-6}{\left(\frac{11}{4}\right)}$ $2 = -\frac{2}{11} + \frac{24}{11}$ $2 = \frac{22}{11}$ $2 = 2$ ✓	$\frac{2(5)}{(3(5)+1)} = \frac{1}{((5)-5)} - \frac{4((5)-1)}{(3(5)+1)((5)-5)}$ $\frac{10}{16} = \frac{1}{0} - \frac{16}{0}$
	Undefined

Here 5 is an extraneous solution and is not included in the solution set. It is important to note that 5 is a restriction.

Answer:

The solution is -12.

If this process produces a solution that happens to be a restriction, then disregard it as a solution.

Exercise $4.7.1$

Solve: $\frac{4(x-3)}{36-x^2} = \frac{1}{6-x} + \frac{2x}{6+x}$.

Answer

$-\frac{3}{2}$

www.youtube.com/v/Mxln5c53J1M

Sometimes all potential solutions are extraneous, in which case we say that there is no solution to the original equation. In the next two examples, we demonstrate two ways in which rational equation can have no solutions.

Example $4.7.4$

Solve: $1 + \frac{5x+22}{x^2+3x-4} = \frac{x+4}{x-1}$

Solution

To identify the LCD, first factor the denominators.

$$1 + \frac{5x+22}{x^2+3x-4} = \frac{x+4}{x-1}$$
$$1 + \frac{5x+22}{(x+4)(x-1)} = \frac{x+4}{(x-1)}$$

Multiply both sides by the LCD, $(x+4)(x-1)$, distributing carefully.

$$\begin{aligned}
(x+4)(x-1)\cdot\left(1+\frac{5x+22}{(x+4)(x-1)}\right) &= (x+4)(x-1)\cdot\frac{x+4}{(x-1)} \\
(x+4)(x-1)\cdot 1+(x+4)(x-1)\cdot\frac{(5x+22)}{(x+4)(x-1)} &= (x+4)(x-1)\cdot\frac{(x+4)}{(x-1)} \\
(x+4)(x-1)+(5x+22) &= (x+4)(x+4) \\
x^2-x+4x-4+5x+22 &= x^2+4x+4x+16 \\
x^2+8x+18 &= x^2+8x+16 \\
18 &= 16 \;\; \textit{False}
\end{aligned}$$

The equation is a contradiction and thus has no solution.

Answer:

No solution, $\emptyset$

Example 4.7.5:

Solve: $\frac{3x}{2x-3}-\frac{3(4x+3)}{4x^2-9}=\frac{x}{2x+3}$.

Solution

First, factor the denominators.

$$\frac{3x}{(2x-3)}-\frac{3(4x+3)}{(2x+3)(2x-3)}=\frac{x}{(2x+3)}$$

Take note that the restrictions on the domain are $x \neq \pm\frac{3}{2}$. To clear the fractions, multiply by the LCD, $(2x+3)(2x-3)$.

$$\begin{aligned}
\frac{3x\cdot(2x+3)(2x-3)}{(2x-3)}-\frac{3(4x+3)\cdot(2x+3)(2x-3)}{(2x+3)(2x-3)} &= \frac{x\cdot(2x+3)(2x-3)}{(2x+3)} \\
3x(2x+3)-3(4x+3) &= x(2x-3) \\
6x^2+9x-12x-9 &= 2x^2-3x \\
6x^2-3x-9 &= 2x^2-3x \\
4x^2-9 &= 0 \\
(2x+3)(2x-3) &= 0
\end{aligned}$$

$$\begin{array}{rclcrcl}
2x+3 &=& 0 & \text{or} & 2x-3 &=& 0 \\
2x &=& -3 & & 2x &=& 3 \\
x &=& -\dfrac{3}{2} & & x &=& \dfrac{3}{2}
\end{array}$$

Both of these values are restrictions of the original equation; hence both are extraneous.

Answer:

No solution, $\emptyset$

It is important to point out that this technique for clearing algebraic fractions only works for equations. Do not try to clear algebraic fractions when simplifying expressions. As a reminder, an example of each is provided below.

Table 4.7.3

Expression	Equation
$\frac{1}{x}+\frac{x}{2x+1}$	$\frac{1}{x}+\frac{x}{2x+1}=0$

Expressions are to be simplified and equations are to be solved. If we multiply the expression by the LCD, $x(2x+1)$, we obtain another expression that is not equivalent.

Table 4.7.4

Incorrect	Correct

Incorrect	Correct
$\frac{1}{x}+\frac{x}{2x+1}$ $\neq x(2x+1)\cdot\left(\frac{1}{x}+\frac{x}{2x+1}\right)$ $=2x+1+x^2$✗	$\frac{1}{x}+\frac{x}{2x+1}=0$ $x(2x+1)\cdot(\frac{1}{x}+\frac{x}{2x+1})=x(2x+1)\cdot 0$ $2x+1+x^2=0$ $x^2+2x+1=0$✓

Rational equations are sometimes expressed using negative exponents.

Example 4.7.6

Solve: $6+x^{-1}=x^{-2}$.

Solution:

Begin by removing the negative exponents.

$$\begin{aligned} 6+x^{-1}&=x^{-2}\\ 6+\frac{1}{x}&=\frac{1}{x^2}\end{aligned}$$

Here we can see the restriction, $x\neq 0$. Next, multiply both sides by the LCD, x^2.

$$\begin{aligned} x^2\cdot\left(6+\frac{1}{x}\right)&=x^2\cdot\left(\frac{1}{x^2}\right)\\ x^2\cdot 6+x^2\cdot\frac{1}{x}&=x^2\cdot\frac{1}{x^2}\\ 6x^2+x&=1\\ 6x^2+x-1&=0\\ (3x-1)(2x+1)&=0\end{aligned}$$

$$\begin{aligned} 3x-1&=0 &\text{or}\qquad 2x+1&=0\\ 3x&=1 & 2x&=-1\\ x&=\frac{1}{3} & x&=-\frac{1}{2}\end{aligned}$$

Answer:

$-\frac{1}{2},\frac{1}{3}$

A **proportion**[35] is a statement of equality of two ratios.

$$\frac{a}{b}=\frac{c}{d}$$

This proportion is often read "a is to b as c is to d." Given any nonzero real numbers a,b,c, and d that satisfy a proportion, multiply both sides by the product of the denominators to obtain the following:

$$\begin{aligned}\frac{a}{b}&=\frac{c}{d}\\ bd\cdot\frac{a}{b}&=bd\cdot\frac{c}{d}\\ ad&=bc\end{aligned}$$

This shows that cross products are equal, and is commonly referred to as **cross multiplication**[36].

$$\text{If }\frac{a}{b}=\frac{c}{d}\text{ then }\frac{a}{d}=\frac{b}{c}$$

Cross multiply to solve proportions where terms are unknown.

Example 4.7.7

Solve: $\frac{5n-1}{5}=\frac{3n}{2}$.

Solution

When cross multiplying, be sure to group $5n-1$.

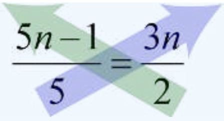

Figure 4.7.1

$$(5n-1)\cdot 2 = 5\cdot 3n$$

Apply the distributive property in the next step.

$$\begin{aligned}(5n-1)\cdot 2 &= 5\cdot 3n\\ 10n-2 &= 15n \quad \textit{Distribute.}\\ -2 &= 5n \quad \textit{Solve.}\\ \frac{-2}{5} &= n\end{aligned}$$

Answer:

$n=-\frac{2}{5}$

Cross multiplication can be used as an alternate method for solving rational equations. The idea is to simplify each side of the equation to a single algebraic fraction and then cross multiply.

Example 4.7.8

Solve: $\frac{1}{2}-\frac{4}{x}=-\frac{x}{8}$.

Solution

Obtain a single algebraic fraction on the left side by subtracting the equivalent fractions with a common denominator.

$$\begin{aligned}\frac{1}{2}\cdot\frac{x}{x}-\frac{4}{x}\cdot\frac{2}{2} &= -\frac{x}{8}\\ \frac{x}{2x}-\frac{8}{2x} &= -\frac{x}{8}\\ \frac{x-8}{2x} &= -\frac{x}{8}\end{aligned}$$

Note that $x\neq 0$, cross multiply, and then solve for x.

$$\begin{aligned}\frac{x-8}{2x} &= \frac{-x}{8}\\ 8(x-8) &= -x\cdot 2x\\ 8x-64 &= -2x^2\\ 2x^2+8x-64 &= 0\\ 2\left(x^2+4x-32\right) &= 0\\ 2(x-4)(x+8) &= 0\end{aligned}$$

Next, set each variable factor equal to zero.

$$\begin{aligned}x-4 &= 0 \quad \text{or} \quad & x+8 &= 0\\ x &= 4 & x &= -8\end{aligned}$$

The check is left to the reader.

Answer:

$-8, 4$

Exercise 4.7.2

Solve: $\frac{2(2x-5)}{x-1}=-\frac{x-4}{2x-5}$.

Answer

Answer: $2, 3$

www.youtube.com/v/KYzJwk_dxHc

Solving Literal Equations and Applications Involving Reciprocals

Literal equations, or formulas, are often rational equations. Hence the techniques described in this section can be used to solve for particular variables. Assume that all variable expressions in the denominator are nonzero.

Example 4.7.9:

The reciprocal of the combined resistance R of two resistors R_1 and R_2 in parallel is given by the formula $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$. Solve for R in terms of R_1 and R_2.

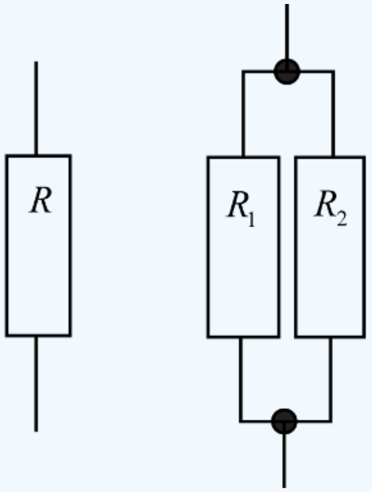

Figure 4.7.2

Solution

The goal is to isolate R on one side of the equation. Begin by multiplying both sides of the equation by the LCD, RR_1R_2.

$$\begin{aligned} RR_1R_2\cdot\frac{1}{R} &= RR_1R_2\cdot\frac{1}{R_1}+RR_1R_2\cdot\frac{1}{R_2} \\ R_1R_2 &= RR_2+RR_1 \\ R_1R_2 &= R\left(R_2+R_1\right) \\ \frac{R_1R_2}{R_2+R_1} &= R \end{aligned}$$

Answer:

$R=\frac{R_1R_2}{R_1+R_2}$

Exercise 4.7.3

Solve for $y: x=\frac{2y+5}{y-3}$

Answer

$y=\frac{3x+5}{x-2}$

www.youtube.com/v/YsSxYxGZBvA

Recall that the reciprocal of a nonzero number n is $\frac{1}{n}$. For example, the reciprocal of 5 is $\frac{1}{5}$ and $5\cdot\frac{1}{5}=1$. In this section, the applications will often involve the key word "reciprocal." When this is the case, we will see that the algebraic setup results in a rational equation.

Example 4.7.10:

A positive integer is 3 less than another. If the reciprocal of the smaller integer is subtracted from twice the reciprocal of the larger, then the result is $\frac{1}{20}$. Find the two integers.

Solution

Let n represent the larger positive integer.

Let $n-3$ represent the smaller positive integer.

Set up an algebraic equation.

$$\overbrace{2\left(\frac{1}{n}\right)}^{\text{twice the reciprocal of the larger}} \overbrace{-\qquad\frac{1}{n-3}}^{\text{the reciprocal of the smaller is subtracted}} \overbrace{=}^{\text{the result is}} \frac{1}{20}$$

Figure $4.7.3$

Solve this rational expression by multiplying both sides by the LCD. The LCD is $20n(n-3)$.

$$\begin{aligned}
\frac{2}{n}-\frac{1}{n-3}&=\frac{1}{20}\\
20n(n-3)\cdot\left(\frac{2}{n}-\frac{1}{n-3}\right)&=20n(n-3)\cdot\left(\frac{1}{20}\right)\\
20n(n-3)\cdot\frac{2}{n}-20n(n-3)\cdot\frac{1}{n-3}&=20n(n-3)\cdot\left(\frac{1}{20}\right)\\
40(n-3)-20n&=n(n-3)\\
40n-120-20n&=n^2-3n\\
20n-120&=n^2-3n\\
0&=n^2-23n+120\\
0&=(n-8)(n-15)
\end{aligned}$$

$$\begin{aligned}
n-8&=0 &\text{or}\qquad n-15&=0\\
n&=8 & n&=15
\end{aligned}$$

Here we have two viable possibilities for the larger integer n. For this reason, we will we have two solutions to this problem.

If $n=8$, then $n-3=8-3=5$.

If $n=15$, then $n-3=15-3=12$.

As a check, perform the operations indicated in the problem.

$$2\left(\frac{1}{n}\right)-\frac{1}{n-3}=\frac{1}{20}$$

Check 8 and 5	Check 15 and 12
$2\left(\frac{1}{8}\right)-\frac{1}{5}=\frac{1}{4}-\frac{1}{5}$ $=\frac{5}{20}-\frac{4}{20}$ $=\frac{1}{20}$ ✓	$2\left(\frac{1}{15}\right)-\frac{1}{12}=\frac{2}{15}-\frac{1}{12}$ $=\frac{8}{60}-\frac{5}{60}$ $=\frac{3}{60}=\frac{1}{20}$ ✓

Answer:

Two sets of positive integers solve this problem: $\{5,8\}$ and $\{12,15\}$.

Exercise $4.7.4$

When the reciprocal of the larger of two consecutive even integers is subtracted from 4 times the reciprocal of the smaller, the result is $\frac{5}{6}$. Find the integers.

Answer

$4,6$

www.youtube.com/v/z1t5VIN8yCI

Key Takeaways

- Begin solving rational equations by multiplying both sides by the LCD. The resulting equivalent equation can be solved using the techniques learned up to this point.
- Multiplying both sides of a rational equation by a variable expression introduces the possibility of extraneous solutions. Therefore, we must check the solutions against the set of restrictions. If a solution is a restriction, then it is not part of the domain and is extraneous.
- When multiplying both sides of an equation by an expression, distribute carefully and multiply each term by that expression.
- If all of the resulting solutions are extraneous, then the original equation has no solutions.

Exercise $4.7.5$

Solve

1. $\frac{3}{x}+2=\frac{1}{3x}$
2. $5-\frac{1}{2x}=-\frac{1}{x}$
3. $\frac{7}{x^2}+\frac{3}{2x}=\frac{1}{x^2}$
4. $\frac{4}{3x^2}+\frac{1}{2x}=\frac{1}{3x^2}$
5. $\frac{1}{6}+\frac{2}{3x}=\frac{7}{2x^2}$
6. $\frac{1}{12}-\frac{1}{3x}=\frac{1}{x^2}$
7. $2+\frac{3}{x}+\frac{7}{x(x-3)}=0$
8. $\frac{20}{x}-\frac{x+44}{x(x+2)}=3$
9. $\frac{2x}{2x-3}+\frac{4}{x}=\frac{x-18}{x(2x-3)}$
10. $\frac{2x}{x-5}+\frac{2(4x+7)}{x(x-5)}=-\frac{1}{x}$
11. $\frac{4}{4x-1}-\frac{1}{x-1}=\frac{2}{4x-1}$
12. $\frac{5}{2x-3}-\frac{1}{x+3}=\frac{2}{2x-3}$
13. $\frac{4x}{x-3}+\frac{4}{x^2-2x-3}=-\frac{1}{x+1}$
14. $\frac{2x}{x-2}-\frac{15}{x+4}=\frac{24}{x^2+2x-8}$
15. $\frac{x}{x-8}-\frac{8}{x-1}=\frac{56}{x^2-9x+8}$
16. $\frac{2x}{x-1}+\frac{9}{3x-1}+\frac{11}{3x^2-4x+1}=0$
17. $\frac{3x}{x-2}-\frac{14}{2x^2-x-6}=\frac{2}{2x+3}$
18. $\frac{x}{x-4}-\frac{4}{x-5}=-\frac{4}{x^2-9x+20}$
19. $\frac{2x}{5+x}-\frac{1}{5-x}=\frac{2x}{x^2-25}$
20. $\frac{2x}{2x+3}-\frac{1}{2x-3}=\frac{6}{9-4x^2}$
21. $1+\frac{1}{x+1}=\frac{8}{x-1}-\frac{16}{x^2-1}$
22. $1-\frac{1}{3x+5}=\frac{2x}{3x-5}-\frac{2(6x+5)}{9x^2-25}$
23. $\frac{x}{x-2}-\frac{3}{x+8}=\frac{x+2}{x+8}+\frac{5(x+3)}{x^2+6x-16}$
24. $\frac{2x}{x-10}+\frac{1}{x-3}=\frac{x+3}{x-10}+\frac{x^2-5x+5}{x^2-13x+30}$
25. $\frac{5}{x^2+9x+18}+\frac{x+3}{x^2+7x+6}=\frac{5}{x^2+4x+3}$
26. $\frac{1}{x^2+4x-60}+\frac{x-6}{x^2+16x+60}=\frac{1}{x^2-36}$
27. $\frac{4}{x^2+10x+21}+\frac{2(x+3)}{x^2+6x-7}=\frac{x+7}{x^2+2x-3}$
28. $\frac{x-1}{x^2-11x+28}+\frac{x-1}{x^2-5x+4}=\frac{x-4}{x^2-8x+7}$
29. $\frac{5}{x^2+5x+4}+\frac{x+1}{x^2+3x-4}=\frac{5}{x^2-1}$
30. $\frac{1}{x^2-2x-63}+\frac{x-9}{x^2+10x+21}=\frac{1}{x^2-6x-27}$
31. $\frac{4}{x^2-4}+\frac{2(x-2)}{x^2-4x-12}=\frac{x+2}{x^2-8x+12}$
32. $\frac{x+2}{x^2-5x+4}+\frac{x+2}{x^2+x-2}=\frac{x-1}{x^2-2x-8}$

Answer

1. $-\frac{4}{3}$

3. -4

5. $-7, 3$

7. $-\frac{1}{2}, 2$

9. $-2, -\frac{3}{2}$

11. $-\frac{1}{2}$

13. $-\frac{1}{4}$

15. $\varnothing$

17. $-2, \frac{5}{6}$

19. $\frac{1}{2}$

21. 6

23. $\varnothing$

25. $-8, 2$

27. 5

29. $-6, 4$

31. 10

Exercise 4.7.6

Solve the following equations involving negative exponents.

1. $2x^{-1} = 2x^{-2} - x^{-1}$
2. $3 + x(x+1)^{-1} = 2(x+1)^{-1}$
3. $x^{-2} - 64 = 0$
4. $1 - 4x^{-2} = 0$
5. $x - (x+2)^{-1} = -2$
6. $2x - 9(2x-1)^{-1} = 1$
7. $2x^{-2} + (x-12)^{-1} = 0$
8. $-2x^{-2} + 3(x+4)^{-1} = 0$

Answer

1. $\frac{2}{3}$

3. $\pm\frac{1}{8}$

5. $-3, -1$

7. $-6, 4$

Exercise 4.7.7

Solve by cross multiplying.

1. $\frac{5}{n} = -\frac{3}{n-2}$
2. $\frac{2n-1}{2n} = -\frac{1}{2}$
3. $-3 = \frac{5n+2}{3n}$
4. $\frac{n+1}{2n-1} = \frac{1}{3}$
5. $\frac{x+2}{x-5} = \frac{x+4}{x-2}$
6. $\frac{x+1}{x+5} = \frac{x-5}{x}$
7. $\frac{2x+1}{6x-1} = \frac{x+5}{3x-2}$
8. $\frac{6(2x+3)}{4x-1} = \frac{3x}{x+2}$
9. $\frac{3(x+1)}{1-x} = \frac{x+3}{x+1}$
10. $\frac{8(x-2)}{x+1} = \frac{5-x}{x-2}$
11. $\frac{x+3}{x+7} = \frac{x+3}{3(5-x)}$
12. $\frac{x+1}{x+4} = \frac{-8(x+4)}{x+7}$

Answer

1. $\frac{5}{4}$

3. $-\frac{1}{7}$

5. -16

7. $\frac{1}{10}$

9. $-2, 0$

11. $-3, 2$

Exercise 4.7.8

Simplify or solve, whichever is appropriate.

1. $\frac{1}{x} + \frac{2}{x-3} = -\frac{2}{3}$
2. $\frac{1}{x-3} - \frac{3}{4} = \frac{1}{x}$
3. $\frac{x-2}{3x-1} - \frac{2-x}{x}$
4. $\frac{5}{2} + \frac{x}{2x-1} - \frac{1}{2x}$
5. $\frac{x-1}{3x} + \frac{2}{x+1} - \frac{5}{6}$
6. $\frac{x-1}{3x} + \frac{2}{x+1} = \frac{5}{6}$
7. $\frac{2x+1}{2x-3} + 2 = \frac{1}{2x}$
8. $5 - \frac{3x+1}{2x} + \frac{1}{x+1}$

Answer

1. Solve; $-3, \frac{3}{2}$

3. Simplify; $\frac{(4x-1)(x-2)}{x(3x-1)}$

5. Simplify; $-\frac{(x-2)(3x-1)}{6x(x+1)}$

7. Solve; $\frac{1}{2}$

Exercise 4.7.9

Find the roots of the given function.

1. $f(x) = \frac{2x-1}{x-1}$
2. $f(x) = \frac{3x+1}{x+2}$
3. $g(x) = \frac{x^2-81}{x^2-5x}$
4. $g(x) = \frac{x^2-x-20}{x^2-9}$
5. $f(x) = \frac{4x^2-9}{2x-3}$
6. $f(x) = \frac{3x^2-2x-1}{x^2-1}$
7. Given $f(x) = \frac{1}{x} + 5$, find x when $f(x) = 2$.
8. Given $f(x) = \frac{1}{x-4}$, find x when $f(x) = \frac{1}{2}$.
9. Given $f(x) = \frac{1}{x+3} + 2$, find x when $f(x) = 1$.
10. Given $f(x) = \frac{1}{x-2} + 5$, find x when $f(x) = 3$.

Answer

1. $\frac{1}{2}$

3. ± 9

5. $-\frac{3}{2}$

7. $x = -\frac{1}{3}$

9. $x = -4$

Exercise 4.7.10

Find the x- and y-intercepts.

1. $f(x)=\frac{1}{x+1}+4$
2. $f(x)=\frac{1}{x-2}-6$
3. $f(x)=\frac{1}{x-3}+2$
4. $f(x)=\frac{1}{x+1}-1$
5. $f(x)=\frac{1}{x}-3$
6. $f(x)=\frac{1}{x+5}$

Answer

1. x-intercept: $(-\frac{5}{4},0)$; y-intercept: $(0,5)$
3. x-intercept: $(\frac{5}{2},0)$; y-intercept: $(0,\frac{5}{3})$
5. x-intercept: $(\frac{1}{3},0)$; y-intercept: none

Exercise 4.7.11

Find the points where the given functions coincide. (Hint: Find the points where $f(x)=g(x)$.)

1. $f(x)=\frac{1}{x}, g(x)=x$
2. $f(x)=-\frac{1}{x}, g(x)=-x$
3. $f(x)=\frac{1}{x-2}+3, g(x)=x+1$
4. $f(x)=\frac{1}{x+3}-1, g(x)=x+2$

Answer

1. $(-1,-1)$ and $(1,1)$
3. $(1,2)$ and $(3,4)$

Exercise 4.7.12

Recall that if $|X|=p$, then $X=-p$ or $X=p$. Use this to solve the following absolute value equations.

1. $\left|\frac{1}{x+1}\right|=2$
2. $\left|\frac{2x}{x+2}\right|=1$
3. $\left|\frac{3x-2}{x-3}\right|=4$
4. $\left|\frac{5x-3}{2x+1}\right|=3$
5. $\left|\frac{x^2}{5x+6}\right|=1$
6. $\left|\frac{x^2-48}{x}\right|=2$

Answer

1. $-\frac{3}{2},-\frac{1}{2}$
3. $2,10$
5. $-3,-2,-1,6$

Exercise 4.7.13

Solve for the given variable.

1. Solve for $P: w=\frac{P-2l}{2}$
2. Solve for $A: t=\frac{A-P}{Pr}$
3. Solve for $t: \frac{1}{t_1}+\frac{1}{t_2}=\frac{1}{t}$
4. Solve for $n: P=1+\frac{r}{n}$
5. Solve for $y: m=\frac{y-y_0}{x-x_0}$

6. Solve for $m_1 : F = G\frac{m_1 m_2}{r^2}$
7. Solve for $y : x = \frac{2y-1}{y-1}$
8. Solve for $y : x = \frac{3y+2}{y+3}$
9. Solve for $y : x = \frac{2y}{2y+5}$
10. Solve for $y : x = \frac{5y+1}{3y}$
11. Solve for $x : \frac{a}{x} + \frac{c}{b} = \frac{a}{c}$
12. Solve for $y : \frac{a}{y} - \frac{1}{a} = b$

Answer

1. $P = 2l + 2w$

3. $t = \frac{t_1 t_2}{t_1 + t_2}$

5. $y = m(x - x_0) + y_0$

7. $y = \frac{x-1}{x-2}$

9. $y = -\frac{5x}{2x-2}$

11. $x = \frac{abc}{ab-c^2}$

Exercise 4.7.14

Use algebra to solve the following applications.

1. The value in dollars of a tablet computer is given by the function $V(t) = 460(t+1)^{-1}$, where t represents the age of the tablet. Determine the age of the tablet if it is now worth $\$100$.
2. The value in dollars of a car is given by the function $V(t) = 24,000(0.5t+1)^{-1}$, where t represents the age of the car. Determine the age of the car if it is now worth $\$6,000$.

Answer

1. 3.6 years old

Exercise 4.7.15

Solve for the unknowns.

1. When 2 is added to 5 times the reciprocal of a number, the result is 12. Find the number.
2. When 1 is subtracted from 4 times the reciprocal of a number, the result is 11. Find the number.
3. The sum of the reciprocals of two consecutive odd integers is $\frac{12}{35}$. Find the integers.
4. The sum of the reciprocals of two consecutive even integers is $\frac{9}{40}$. Find the integers.
5. An integer is 4 more than another. If 2 times the reciprocal of the larger is subtracted from 3 times the reciprocal of the smaller, then the result is $\frac{1}{8}$. Find the integers.
6. An integer is 2 more than twice another. If 2 times the reciprocal of the larger is subtracted from 3 times the reciprocal of the smaller, then the result is $\frac{5}{14}$. Find the integers.
7. If 3 times the reciprocal of the larger of two consecutive integers is subtracted from 2 times the reciprocal of the smaller, then the result is $\frac{1}{2}$. Find the two integers.
8. If 3 times the reciprocal of the smaller of two consecutive integers is subtracted from 7 times the reciprocal of the larger, then the result is $\frac{1}{2}$. Find the two integers.
9. A positive integer is 5 less than another. If the reciprocal of the smaller integer is subtracted from 3 times the reciprocal of the larger, then the result is $\frac{1}{12}$. Find the two integers.
10. A positive integer is 6 less than another. If the reciprocal of the smaller integer is subtracted from 10 times the reciprocal of the larger, then the result is $\frac{3}{7}$. Find the two integers.

Answer

1. $\frac{1}{2}$

3. $5, 7$
5. $\{-8, -4\}$ and $\{12, 16\}$
7. $\{1, 2\}$ or $\{-4, -3\}$
9. $\{4, 9\}$ or $\{15, 20\}$

Exercise $4.7.16$

1. Explain how we can tell the difference between a rational expression and a rational equation. How do we treat them differently? Give an example of each.
2. Research and discuss reasons why multiplying both sides of a rational equation by the LCD sometimes produces extraneous solutions.

Answer

1. Answer may vary

Footnotes

[33] An equation containing at least one rational expression.

[34] A solution that does not solve the original equation.

[35] A statement of equality of two ratios.

[36] If $\frac{a}{b} = \frac{c}{d}$ then $ad = bc$.

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4.8: Applications and Variation

Learning Objectives

- Solve applications involving uniform motion (distance problems).
- Solve work-rate applications.
- Set up and solve applications involving direct, inverse, and joint variation.

Solving Uniform Motion Problems

Uniform motion (or distance)[37] problems involve the formula $D = rt$, where the distance D is given as the product of the average rate r and the time t traveled at that rate. If we divide both sides by the average rate r, then we obtain the formula

$$t = \frac{D}{r}$$

For this reason, when the unknown quantity is time, the algebraic setup for distance problems often results in a rational equation. We begin any uniform motion problem by first organizing our data with a chart. Use this information to set up an algebraic equation that models the application.

Example $4.8.1$:

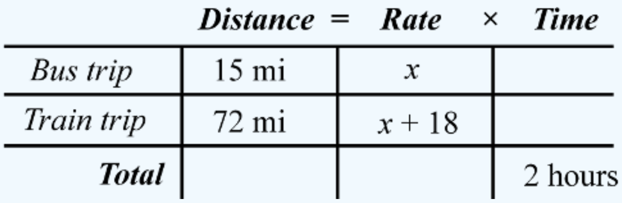
Sally traveled 15 miles on the bus and then another 72 miles on a train. The train was 18 miles per hour faster than the bus, and the total trip took 2 hours. What was the average speed of the train?

Solution

First, identify the unknown quantity and organize the data.

Let x represent the average speed (in miles per hour) of the bus.

Let $x + 18$ represent the average speed of the train.



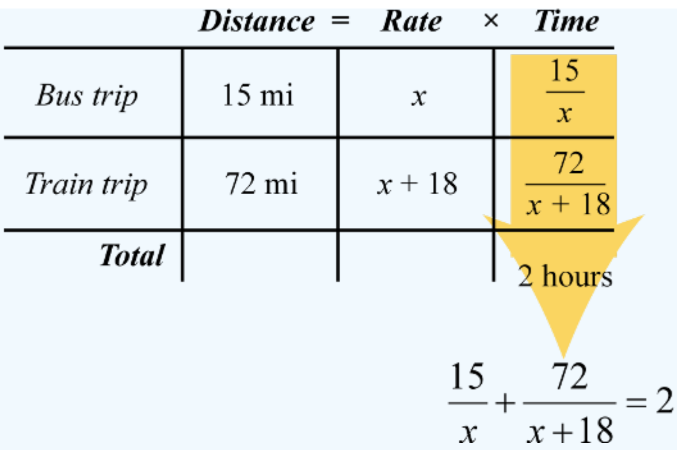
	Distance =	*Rate* ×	*Time*
Bus trip	15 mi	x	
Train trip	72 mi	$x + 18$	
Total			2 hours

Figure $4.8.1$

To avoid introducing two more variables for the time column, use the formula $t = \frac{D}{r}$. The time for each leg of the trip is calculated as follows:

$$\textit{Time spent on the bus }: t = \frac{D}{r} = \frac{15}{x}$$
$$\textit{Time spent on the train }: t = \frac{D}{r} = \frac{72}{x+18}$$

Use these expressions to complete the chart.



	Distance	= *Rate*	× *Time*
Bus trip	15 mi	x	$\frac{15}{x}$
Train trip	72 mi	$x+18$	$\frac{72}{x+18}$
Total			2 hours

$$\frac{15}{x}+\frac{72}{x+18}=2$$

Figure $4.8.2$

The algebraic setup is defined by the time column. Add the time spent on each leg of the trip to obtain a total of 2 hours:

$$\overbrace{\frac{15}{x}}^{\text{time spent on the bus}}+\overbrace{\frac{72}{x+18}}^{\text{time spent on the train}}=\overbrace{2}^{\text{total time of trip}}$$

Figure $4.8.3$

We begin solving this equation by first multiplying both sides by the LCD, $x(x+18)$.

$$\begin{aligned}\frac{15}{x}+\frac{72}{x+18}&=2\\ x(x+18)\cdot\left(\frac{15}{x}+\frac{72}{x+18}\right)&=x(x+18)\cdot 2\\ x(x+18)\cdot\frac{15}{x}+x(x+18)\cdot\frac{72}{x+18}&=x(x+18)\cdot 2\\ 15(x+18)\cdot 72x&=2x(x+18)\\ 15x+270+72x&=2x^2+36x\\ 87x+270&=2x^2+36x\\ 0&=2x^2-51x-270\end{aligned}$$

Solve the resulting quadratic equation by factoring.

$$\begin{aligned}0&=2x^2-51x-270\\ 0&=(2x+9)(x-30)\end{aligned}$$

$$\begin{array}{rcr}2x+9=0 & \text{or} & x-30=0\\ x=-\frac{9}{2} & & x=30\end{array}$$

Since we are looking for an average speed we will disregard the negative answer and conclude the bus averaged 30 mph. Substitute $x=30$ in the expression identified as the speed of the train.

$$x+18=30+18=48$$

Answer:

The speed of the train was 48 mph.

Example $4.8.2$:

A boat can average 12 miles per hour in still water. On a trip downriver the boat was able to travel 29 miles with the current. On the return trip the boat was only able to travel 19 miles in the same amount of time against the current. What was the speed of the current?

Solution

First, identify the unknown quantities and organize the data.

Let c represent the speed of the river current.

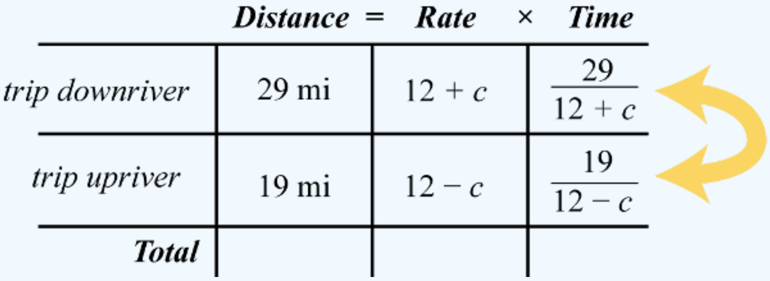
Next, organize the given data in a chart. Traveling downstream, the current will increase the speed of the boat, so it adds to the average speed of the boat. Traveling upstream, the current slows the boat, so it will subtract from the average speed of the boat.

	Distance =	*Rate* ×	*Time*
trip downriver	29 mi	$12 + c$	
trip upriver	19 mi	$12 - c$	
Total			

Figure $4.8.4$

Use the formula $t = \frac{D}{r}$ to fill in the time column.

$$\begin{aligned} \text{trip downriver }: t &= \frac{D}{r} = \frac{29}{12+c} \\ \text{trip upriver }: t &= \frac{D}{r} = \frac{19}{12-c} \end{aligned}$$



	Distance =	*Rate* ×	*Time*
trip downriver	29 mi	$12 + c$	$\frac{29}{12+c}$
trip upriver	19 mi	$12 - c$	$\frac{19}{12-c}$
Total			

Figure $4.8.5$

Because the boat traveled the same amount of time downriver as it did upriver, finish the algebraic setup by setting the expressions that represent the times equal to each other.

$$\frac{29}{12+c} = \frac{19}{12-c}$$

Since there is a single algebraic fraction on each side, we can solve this equation using cross multiplication.

$$\begin{aligned} \frac{29}{12+c} &= \frac{19}{12-c} \\ 29(12-c) &= 19(12+c) \\ 348 - 29c &= 228 + 19c \\ 120 &= 48c \\ \frac{120}{48} &= c \\ \frac{5}{2} &= c \end{aligned}$$

Answer:

The speed of the current was $2\frac{1}{2}$ miles per hour.

Exercise $4.8.1$

A jet aircraft can average 160 miles per hour in calm air. On a trip, the aircraft traveled 600 miles with a tailwind and returned the 600 miles against a headwind of the same speed. If the total round trip took 8 hours, then what was the speed of the wind?

Answer

40 miles per hour

www.youtube.com/v/0NglBthTwss

Solving Work-Rate Problems

The rate at which a task can be performed is called a **work rate**[38]. For example, if a painter can paint a room in 6 hours, then the task is to paint the room, and we can write

$$\frac{1\text{ task}}{6\text{ hours}} \quad \textit{work rate}$$

In other words, the painter can complete $\frac{1}{6}$ of the task per hour. If he works for less than 6 hours, then he will perform a fraction of the task. If he works for more than 6 hours, then he can complete more than one task. For example,

$$\begin{aligned} work-rate \quad &\times \quad time = amount\ of\ task\ completed \\ \frac{1}{6} \quad &\times \quad 3hrs = \frac{1}{2} \quad one-half\ of\ the\ room\ painted \\ \frac{1}{6} \quad &\times \quad 6hrs = 1 \quad one\ whole\ room\ painted \\ \frac{1}{6} \quad &\times \quad 12\text{ hrs } = 2 \quad two\ whole\ rooms\ painted \end{aligned}$$

Obtain the amount of the task completed by multiplying the work rate by the amount of time the painter works. Typically, work-rate problems involve people or machines working together to complete tasks. In general, if t represents the time two people work together, then we have the following **work-rate formula**[39]:

$$\frac{1}{t_1}t + \frac{1}{t_2}t = amount\ of\ task\ completed\ together$$

Here $\frac{1}{t_1}$ and $\frac{1}{t_2}$ are the individual work rates.

Example 4.8.3:

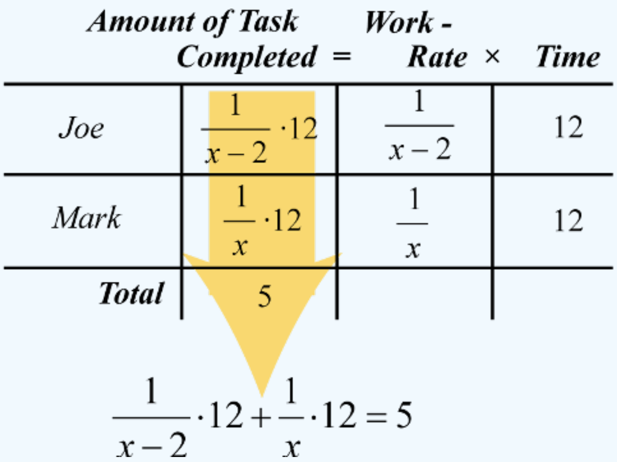
Joe can paint a typical room in 2 hours less time than Mark. If Joe and Mark can paint 5 rooms working together in a 12 hour shift, how long does it take each to paint a single room?

Solution

Let x represent the time it takes Mark to paint a typical room.

Let $x-2$ represent the time it takes Joe to paint a typical room.

Therefore, Mark's individual work-rate is $\frac{1}{x}$ rooms per hour and Joe's is $\frac{1}{x-2}$ rooms per hour. Both men worked for 12 hours. We can organize the data in a chart, just as we did with distance problems.



	Amount of Task Completed =	Work - Rate ×	Time
Joe	$\frac{1}{x-2}\cdot 12$	$\frac{1}{x-2}$	12
Mark	$\frac{1}{x}\cdot 12$	$\frac{1}{x}$	12
Total	5		

$$\frac{1}{x-2}\cdot 12+\frac{1}{x}\cdot 12=5$$

Figure $4.8.6$

Working together, they can paint 5 total rooms in 12 hours. This leads us to the following algebraic setup:

$$\frac{12}{x-2}+\frac{12}{x}=5$$

Multiply both sides by the LCD, $x(x-2)$.

$$\begin{aligned} x(x-2)\cdot\left(\frac{12}{x-2}+\frac{12}{x}\right) &= x(x-2)\cdot 5 \\ x(x-2)\cdot\frac{12}{x-2}+x(x-2)\cdot\frac{12}{x} &= x(x-2)\cdot 5 \\ 12x+12(x-2) &= 5x(x-2) \\ 12x+12x-24 &= 5x^2-10x \\ 0 &= 5x^2-34x+24 \end{aligned}$$

Solve the resulting quadratic equation by factoring.

$$\begin{aligned} 0 &= 5x^2-34x+24 \\ 0 &= (5x-4)(x-6) \end{aligned}$$

$$\begin{array}{rcr} 5x-4=0 & \text{or} & x-6=0 \\ 5x=4 & & x=6 \\ x=\frac{4}{5} & & \end{array}$$

We can disregard $\frac{4}{5}$ because back substituting into $x-2$ would yield a negative time to paint a room. Take $x=6$ to be the only solution and use it to find the time it takes Joe to paint a typical room.

$$x-2=6-2=4$$

Answer:

Joe can paint a typical room in 4 hours and Mark can paint a typical room in 6 hours. As a check we can multiply both work rates by 12 hours to see that together they can paint 5 rooms.

$$\left.\begin{array}{l} Joe\ \frac{1\text{ room}}{4\text{hrs}}\cdot 12\text{ hrs } = 3\text{ rooms} \\ Mark\ \frac{1\text{ room}}{6\text{hrs}}\cdot 12\text{hrs} = 2\text{ rooms} \end{array}\right\}\ Total\ 5\ rooms\ \checkmark$$

Example $4.8.4$:

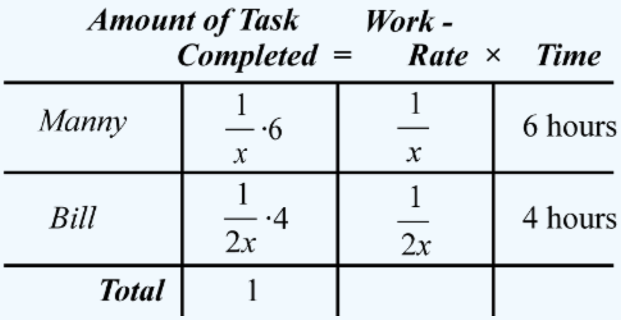
It takes Bill twice as long to lay a tile floor by himself as it does Manny. After working together with Bill for 4 hours, Manny was able to complete the job in 2 additional hours. How long would it have taken Manny working alone?

Solution

Let x represent the time it takes Manny to lay the floor alone.

Let $2x$ represent the time it takes Bill to lay the floor alone.

Manny's work rate is $\frac{1}{x}$ of the floor per hour and Bill's work rate is $\frac{1}{2x}$. Bill worked on the job for 4 hours and Manny worked on the job for 6 hours.



	Amount of Task Completed =	*Work - Rate* ×	*Time*
Manny	$\frac{1}{x}\cdot 6$	$\frac{1}{x}$	6 hours
Bill	$\frac{1}{2x}\cdot 4$	$\frac{1}{2x}$	4 hours
Total	1		

Figure $4.8.7$

This leads us to the following algebraic setup:

$$\frac{1}{x}\cdot 6+\frac{1}{2x}\cdot 4=1$$

Solve.

$$\begin{aligned} \frac{6}{x}+\frac{4}{2x}&=1 \\ x\cdot\left(\frac{6}{x}+\frac{2}{x}\right)&=x\cdot 1 \\ 6+2&=x \\ 8&=x \end{aligned}$$

Answer:

It would have taken Manny 8 hours to complete the floor by himself.

Consider the work-rate formula where one task is to be completed.

$$\frac{1}{t_1}t+\frac{1}{t_2}t=1$$

Factor out the time t and then divide both sides by t. This will result in equivalent specialized work-rate formulas:

$$\begin{aligned} t\left(\frac{1}{t_1}+\frac{1}{t_2}\right)&=1 \\ \frac{1}{t_1}+\frac{1}{t_2}&=\frac{1}{t} \end{aligned}$$

In summary, we have the following equivalent work-rate formulas:

$$\text{Work rate formulas}$$
$$\frac{1}{t_1}t+\frac{1}{t_2}t=1 \quad \text{or} \quad \frac{t}{t_1}+\frac{t}{t_2}=1 \quad \text{or} \quad \frac{1}{t_1}+\frac{1}{t_2}=\frac{1}{t}$$

Exercise $4.8.2$

Matt can tile a countertop in 2 hours, and his assistant can do the same job in 3 hours. If Matt starts the job and his assistant joins him 1 hour later, then how long will it take to tile the countertop?

Answer

$1\frac{3}{5}$ hours

www.youtube.com/v/5g6sSFWGb7M

Solving Problems involving Direct, Inverse, and Joint variation

Many real-world problems encountered in the sciences involve two types of functional relationships. The first type can be explored using the fact that the distance s in feet an object falls from rest, without regard to air resistance, can be approximated using the following formula:

$$s=16t^2$$

Here t represents the time in seconds the object has been falling. For example, after 2 seconds the object will have fallen $s=16(2)^2=16\cdot 4=64$ feet.

Table $4.8.1$

Time t in seconds	Distance $s=16t^2$ in feet
0	0
1	16
2	64
3	144
4	256

In this example, we can see that the distance varies over time as the product of a constant 16 and the square of the time t. This relationship is described as **direct variation**[40] and 16 is called the **constant of variation**[41]. Furthermore, if we divide both sides of $s=16t^2$ by t^2 we have

$$\frac{s}{t^2} = 16$$

In this form, it is reasonable to say that s is proportional to t^2, and 16 is called the **constant of proportionality**[42]. In general, we have

Table $4.8.2$

Key words	Translation
"y varies x directly as x"	$y = kx$
"y is **directly proportional**[43] to x"	
"y is proportional to x"	

Here k is nonzero and is called the constant of variation or the constant of proportionality. Typically, we will be given information from which we can determine this constant.

Example $4.8.5$:

An object's weight on Earth varies directly to its weight on the Moon. If a man weighs 180 pounds on Earth, then he will weigh 30 pounds on the Moon. Set up an algebraic equation that expresses the weight on Earth in terms of the weight on the Moon and use it to determine the weight of a woman on the Moon if she weighs 120 pounds on Earth.

Solution

Let y represent weight on Earth.

Let x represent weight on the Moon.

We are given that the "weight on Earth varies directly to the weight on the Moon."

$$y = kx$$

To find the constant of variation k, use the given information. A 180-lb man on Earth weighs 30 pounds on the Moon, or $y = 180$ when $x = 30$.

$$180 = k \cdot 30$$

Solve for k.

$$\frac{180}{30} = k$$
$$6 = k$$

Next, set up a formula that models the given information.

$$y = 6x$$

This implies that a person's weight on Earth is 6 times his weight on the Moon. To answer the question, use the woman's weight on Earth, $y = 120$ lbs, and solve for x.

$$120 = 6x$$
$$\frac{120}{6} = x$$
$$20 = x$$

Answer:

The woman weighs 20 pounds on the Moon.

The second functional relationship can be explored using the formula that relates the intensity of light I to the distance from its source d.

$$I = \frac{k}{d^2}$$

Here k represents some constant. A foot-candle is a measurement of the intensity of light. One foot-candle is defined to be equal to the amount of illumination produced by a standard candle measured one foot away. For example, a 125-Watt fluorescent growing light is advertised to produce 525 foot-candles of illumination. This means that at a distance $d = 1$ foot, $I = 525$ foot-candles and we have:

$$\begin{aligned} 525 &= \frac{k}{(1)^2} \\ 525 &= k \end{aligned}$$

Using $k = 525$ we can construct a formula which gives the light intensity produced by the bulb:

$$I = \frac{525}{d^2}$$

Here d represents the distance the growing light is from the plants. In the following chart, we can see that the amount of illumination fades quickly as the distance from the plants increases.

Table $4.8.3$

Distance t in feet	Light Intensity $I = \frac{525}{d^2}$
1	525
2	131.25
3	58.33
4	32.81
5	21

This type of relationship is described as an **inverse variation**[44]. We say that I is **inversely proportional**[45] to the square of the distance d, where 525 is the constant of proportionality. In general, we have

Table $4.8.4$

Key words	Translation
"y varies inversely as x"	$y = \frac{k}{x}$
"y is inversely proportional to x"	

Again, k is nonzero and is called the constant of variation or the constant of proportionality.

Example $4.8.6$:

The weight of an object varies inversely as the square of its distance from the center of Earth. If an object weighs 100 pounds on the surface of Earth (approximately $4,000$ miles from the center), how much will it weigh at $1,000$ miles above Earth's surface?

Solution

Let w represent the weight of the object.

Let d represent the object's distance from the center of Earth.

Since "w varies inversely as the square of d," we can write

$$w = \frac{k}{d^2}$$

Use the given information to find k. An object weighs 100 pounds on the surface of Earth, approximately $4,000$ miles from the center. In other words, $w = 100$ when $d = 4,000$:

$$100 = \frac{k}{(4,000)^2}$$

Solve for k.

$$\begin{aligned} {\color{#1a9ad6}(4,000)^2} \cdot 100 &= {\color{#1a9ad6}(4,000)^2} \cdot \frac{k}{(4,000)^2} \\ 1,600,000,000 &= k \\ 1.6 \times 10^9 &= k \end{aligned}$$

Therefore, we can model the problem with the following formula:

$$w=\frac{1.6\times10^9}{d^2}$$

To use the formula to find the weight, we need the distance from the center of Earth. Since the object is $1,000$ miles above the surface, find the distance from the center of Earth by adding $4,000$ miles:

$$d=4,000+1,000=5,000 \text{ miles}$$

To answer the question, use the formula with $d=5,000$.

$$\begin{aligned} y &= \frac{1.6\times10^9}{(5,000)^2} \\ &= \frac{1.6\times10^9}{25,000,000} \\ &= \frac{1.6\times10^9}{2.5\times10^9} \\ &= 0.64\times10^2 \\ &= 64 \end{aligned}$$

Answer:

The object will weigh 64 pounds at a distance $1,000$ miles above the surface of Earth.

Lastly, we define relationships between multiple variables, described as **joint variation**[46]. In general, we have

Table $4.8.5$

Key Words	Translation
"y varies jointly as x and z"	$y=kxz$
"y is **jointly proportional**[47] to x and z"	

Here k is nonzero and is called the constant of variation or the constant of proportionality.

Example $4.8.7$:

The area of an ellipse varies jointly as a, half of the ellipse's major axis, and b, half of the ellipse's minor axis as pictured. If the area of an ellipse is $300\pi\ cm^2$, where $a=10$ cm and $b=30$ cm, what is the constant of proportionality? Give a formula for the area of an ellipse.

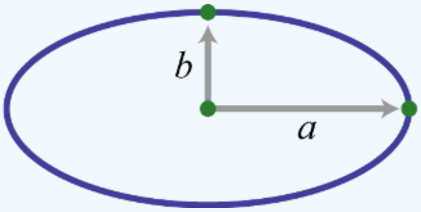


Figure $4.8.8$

Solution

If we let A represent the area of an ellipse, then we can use the statement "area varies jointly as a and b" to write

$$A=kab$$

To find the constant of variation k, use the fact that the area is 300π when $a=10$ and $b=30$.

$$\begin{aligned} 300\pi &= k(10)(30) \\ 300\pi &= 300k \\ \pi &= k \end{aligned}$$

Therefore, the formula for the area of an ellipse is

$$A=\pi ab$$

Answer:

The constant of proportionality is π and the formula for the area of an ellipse is $A=ab\pi$.

Exercise $4.8.3$

Given that y varies directly as the square of x and inversely with z, where $y = 2$ when $x = 3$ and $z = 27$, find y when $x = 2$ and $z = 16$.

Answer

$\frac{3}{2}$

www.youtube.com/v/ee3AFf7b6Kg

Key Takeaways

- When solving distance problems where the time element is unknown, use the equivalent form of the uniform motion formula, $t = \frac{D}{r}$, to avoid introducing more variables.
- When solving work-rate problems, multiply the individual work rate by the time to obtain the portion of the task completed. The sum of the portions of the task results in the total amount of work completed.
- The setup of variation problems usually requires multiple steps. First, identify the key words to set up an equation and then use the given information to find the constant of variation k. After determining the constant of variation, write a formula that models the problem. Once a formula is found, use it to answer the question.

Exercise $4.8.4$

Use algebra to solve the following applications.

1. Every morning Jim spends 1 hour exercising. He runs 2 miles and then he bikes 16 miles. If Jim can bike twice as fast as he can run, at what speed does he average on his bike?
2. Sally runs 3 times as fast as she walks. She ran for $\frac{3}{4}$ of a mile and then walked another $3\frac{1}{2}$ miles. The total workout took $1\frac{1}{2}$ hours. What was Sally’s average walking speed?
3. On a business trip, an executive traveled 720 miles by jet and then another 80 miles by helicopter. If the jet averaged 3 times the speed of the helicopter, and the total trip took 4 hours, what was the average speed of the jet?
4. A triathlete can run 3 times as fast as she can swim and bike 6 times as fast as she can swim. The race consists of a $\frac{1}{4}$ mile swim, 3 mile run, and a 12 mile bike race. If she can complete all of these events in $1\frac{5}{8}$ hour, then how fast can she swim, run and bike?
5. On a road trip, Marty was able to drive an average 4 miles per hour faster than George. If Marty was able to drive 39 miles in the same amount of time George drove 36 miles, what was Marty’s average speed?
6. The bus is 8 miles per hour faster than the trolley. If the bus travels 9 miles in the same amount of time the trolley can travel 7 miles, what is the average speed of each?
7. Terry decided to jog the 5 miles to town. On the return trip, she walked the 5 miles home at half of the speed that she was able to jog. If the total trip took 3 hours, what was her average jogging speed?
8. James drove the 24 miles to town and back in 1 hour. On the return trip, he was able to average 20 miles per hour faster than he averaged on the trip to town. What was his average speed on the trip to town?
9. A light aircraft was able to travel 189 miles with a 14 mile per hour tailwind in the same time it was able to travel 147 miles against it. What was the speed of the aircraft in calm air?
10. A jet flew 875 miles with a 30 mile per hour tailwind. On the return trip, against a 30 mile per hour headwind, it was able to cover only 725 miles in the same amount of time. How fast was the jet in calm air?
11. A helicopter averaged 90 miles per hour in calm air. Flying with the wind it was able to travel 250 miles in the same amount of time it took to travel 200 miles against it. What is the speed of the wind?
12. Mary and Joe took a road-trip on separate motorcycles. Mary’s average speed was 12 miles per hour less than Joe’s average speed. If Mary drove 115 miles in the same time it took Joe to drive 145 miles, what was Mary’s average speed?
13. A boat averaged 12 miles per hour in still water. On a trip downstream, with the current, the boat was able to travel 26 miles. The boat then turned around and returned upstream 33 miles. How fast was the current if the total trip took 5 hours?
14. If the river current flows at an average 3 miles per hour, a tour boat can make an 18-mile tour downstream with the current and back the 18 miles against the current in $4\frac{1}{2}$ hours. What is the average speed of the boat in still water?

15. Jose drove 10 miles to his grandmother's house for dinner and back that same evening. Because of traffic, he averaged 20 miles per hour less on the return trip. If it took $\frac{1}{4}$ hour longer to get home, what was his average speed driving to his grandmother's house?
16. Jerry paddled his kayak, upstream against a 1 mph current, for 12 miles. The return trip, downstream with the 1 mph current, took one hour less time. How fast did Jerry paddle the kayak in still water?
17. James and Mildred left the same location in separate cars and met in Los Angeles 300 miles away. James was able to average 10 miles an hour faster than Mildred on the trip. If James arrived 1 hour earlier than Mildred, what was Mildred's average speed?
18. A bus is 20 miles per hour faster than a bicycle. If Bill boards a bus at the same time and place that Mary departs on her bicycle, Bill will arrive downtown 5 miles away $\frac{1}{3}$ hour earlier than Mary. What is the average speed of the bus?

Answer

1. 20 miles per hour

3. 240 miles per hour

5. 52 miles per hour

7. 5 miles per hour

9. 112 miles per hour

11. 10 miles per hour

13. 1 mile per hour

15. 40 miles per hour

17. 50 miles per hour

Exercise $4.8.5$

Use algebra to solve the following applications.

1. Mike can paint the office by himself in $4\frac{1}{2}$ hours. Jordan can paint the office in 6 hours. How long will it take them to paint the office working together?
2. Barry can lay a brick driveway by himself in $3\frac{1}{2}$ days. Robert does the same job in 5 days. How long will it take them to lay the brick driveway working together?
3. A larger pipe fills a water tank twice as fast as a smaller pipe. When both pipes are used, they fill the tank in 10 hours. If the larger pipe is left off, how long would it take the smaller pipe to fill the tank?
4. A newer printer can print twice as fast as an older printer. If both printers working together can print a batch of flyers in 45 minutes, then how long would it take the older printer to print the batch working alone?
5. Mary can assemble a bicycle for display in 2 hours. It takes Jane 3 hours to assemble a bicycle. How long will it take Mary and Jane, working together, to assemble 5 bicycles?
6. Working alone, James takes twice as long to assemble a computer as it takes Bill. In one 8-hour shift, working together, James and Bill can assemble 6 computers. How long would it take James to assemble a computer if he were working alone?
7. Working alone, it takes Harry one hour longer than Mike to install a fountain. Together they can install 10 fountains in 12 hours. How long would it take Mike to install 10 fountains by himself?
8. Working alone, it takes Henry 2 hours longer than Bill to paint a room. Working together they painted $2\frac{1}{2}$ rooms in 6 hours. How long would it have taken Henry to paint the same amount if he were working alone?
9. Manny, working alone, can install a custom cabinet in 3 hours less time than his assistant. Working together they can install the cabinet in 2 hours. How long would it take Manny to install the cabinet working alone?
10. Working alone, Garret can assemble a garden shed in 5 hours less time than his brother. Working together, they need 6 hours to build the garden shed. How long would it take Garret to build the shed working alone?
11. Working alone, the assistant-manager takes 2 more hours than the manager to record the inventory of the entire shop. After working together for 2 hours, it took the assistant-manager 1 additional hour to complete the inventory. How long would it have taken the manager to complete the inventory working alone?

12. An older printer can print a batch of sales brochures in 16 minutes. A newer printer can print the same batch in 10 minutes. After working together for some time, the newer printer was shut down and it took the older printer 3 more minutes to complete the job. How long was the newer printer operating?

Answer

1. $2\frac{4}{7}$ hours

3. 30 hours

5. 6 hours

7. 20 hours

9. 3 hours

11. 4 hours

Exercise $4.8.6$

Translate each of the following sentences into a mathematical formula.

1. The distance D an automobile can travel is directly proportional to the time t that it travels at a constant speed.
2. The extension of a hanging spring d is directly proportional to the weight w attached to it.
3. An automobile's braking distance d is directly proportional to the square of the automobile's speed v.
4. The volume V of a sphere varies directly as the cube of its radius r.
5. The volume V of a given mass of gas is inversely proportional to the pressure p exerted on it.
6. Every particle of matter in the universe attracts every other particle with a force F that is directly proportional to the product of the masses m_1 and m_2 of the particles, and it is inversely proportional to the square of the distance d between them.
7. Simple interest I is jointly proportional to the annual interest rate r and the time t in years a fixed amount of money is invested.
8. The time t it takes an object to fall is directly proportional to the square root of the distance d it falls.

Answer

1. $D = kt$

3. $d = kv^2$

5. $V = \frac{k}{p}$

7. $I = krt$

Exercise $4.8.7$

Construct a mathematical model given the following:

1. y varies directly as x, and $y = 30$ when $x = 6$.
2. y varies directly as x, and $y = 52$ when $x = 4$.
3. y is directly proportional to x, and $y = 12$ when $x = 3$.
4. y is directly proportional to x, and $y = 120$ when $x = 20$.
5. y is directly proportional to x, and $y = 3$ when $x = 9$.
6. y is directly proportional to x, and $y = 21$ when $x = 3$.
7. y varies inversely as x, and $y = 2$ when $x = \frac{1}{8}$.
8. y varies inversely as x, and $y = \frac{3}{2}$ when $x = \frac{1}{9}$.
9. y is jointly proportional to x and z, where $y = 2$ when $x = 1$ and $z = 3$.
10. y is jointly proportional to x and z, where $y = 15$ when $x = 3$ and $z = 7$.
11. y varies jointly as x and z, where $y = \frac{2}{3}$ when $x = \frac{1}{2}$ and $z = 12$.
12. y varies jointly as x and z, where $y = 5$ when $x = \frac{3}{2}$ and $z = \frac{2}{9}$.
13. y varies directly as the square of x, where $y = 45$ when $x = 3$.

14. y varies directly as the square of x, where $y=3$ when $x=\frac{1}{2}$.
15. y is inversely proportional to the square of x, where $y=27$ when $x=\frac{1}{3}$.
16. y is inversely proportional to the square of x, where $y=9$ when $x=\frac{2}{3}$.
17. y varies jointly as x and the square of z, where $y=6$ when $x=\frac{1}{4}$ and $z=\frac{2}{3}$.
18. y varies jointly as x and z and inversely as the square of w, where $y=5$ when $z=1, z=3$, and $w=\frac{1}{2}$.
19. y varies directly as the square root of x and inversely as the square of z, where $y=15$ when $x=25$ and $z=2$.
20. y varies directly as the square of x and inversely as z and the square of w, where $y=14$ when $x=4, w=2$ and $z=2$.

Answer

1. $y=5x$

3. $y=4x$

5. $y=\frac{27}{x}$

7. $y=\frac{1}{4x}$

9. $y=\frac{2}{3}xz$

11. $y=\frac{1}{9}xz$

13. $y=5x^2$

15. $y=\frac{3}{x^2}$

17. $y=54xz^2$

19. $y=\frac{12\sqrt{x}}{z^2}$

Exercise 4.8.8

Solve applications involving variation.

1. Revenue in dollars is directly proportional to the number of branded sweatshirts sold. The revenue earned from selling 25 sweatshirts is $318.75. Determine the revenue if 30 sweatshirts are sold.
2. The sales tax on the purchase of a new car varies directly as the price of the car. If an $18,000 new car is purchased, then the sales tax is $1,350. How much sales tax is charged if the new car is priced at $22,000?
3. The price of a share of common stock in a company is directly proportional to the earnings per share (EPS) of the previous 12 months. If the price of a share of common stock in a company is $22.55, and the EPS is published to be $1.10, determine the value of the stock if the EPS increases by $0.20.
4. The distance traveled on a road trip varies directly with the time spent on the road. If a 126-mile trip can be made in 3 hours, then what distance can be traveled in 4 hours?
5. The circumference of a circle is directly proportional to its radius. The circumference of a circle with radius 7 centimeters is measured as 14π centimeters. What is the constant of proportionality?
6. The area of circle varies directly as the square of its radius. The area of a circle with radius 7 centimeters is determined to be 49π square centimeters. What is the constant of proportionality?
7. The surface area of a sphere varies directly as the square of its radius. When the radius of a sphere measures 2 meters, the surface area measures 16π square meters. Find the surface area of a sphere with radius 3 meters.
8. The volume of a sphere varies directly as the cube of its radius. When the radius of a sphere measures 3 meters, the volume is 36π cubic meters. Find the volume of a sphere with radius 1 meter.
9. With a fixed height, the volume of a cone is directly proportional to the square of the radius at the base. When the radius at the base measures 10 centimeters, the volume is 200 cubic centimeters. Determine the volume of the cone if the radius of the base is halved.
10. The distance d an object in free fall drops varies directly with the square of the time t that it has been falling. If an object in free fall drops 36 feet in 1.5 seconds, then how far will it have fallen in 3 seconds?

Answer

1. $382.50

3. $\$26.65$

5. 2π

7. 36π square meters

9. 50 cubic centimeters

Exercise $4.8.9$

Hooke’s law suggests that the extension of a hanging spring is directly proportional to the weight attached to it. The constant of variation is called the spring constant.

Figure $4.8.9$: Robert Hooke (1635-1703)

1. A hanging spring is stretched 5 inches when a 20-pound weight is attached to it. Determine its spring constant.
2. A hanging spring is stretched 3 centimeters when a 2-kilogram weight is attached to it. Determine the spring constant.
3. If a hanging spring is stretched 3 inches when a 2-pound weight is attached, how far will it stretch with a 5-pound weight attached?
4. If a hanging spring is stretched 6 centimeters when a 4-kilogram weight is attached to it, how far will it stretch with a 2-kilogram weight attached?

Answer

1. $\frac{1}{4}$

3. 7.5 inches

Exercise $4.8.10$

The braking distance of an automobile is directly proportional to the square of its speed.

1. It takes 36 feet to stop a particular automobile moving at a speed of 30 miles per hour. How much breaking distance is required if the speed is 35 miles per hour?
2. After an accident, it was determined that it took a driver 80 feet to stop his car. In an experiment under similar conditions, it takes 45 feet to stop the car moving at a speed of 30 miles per hour. Estimate how fast the driver was moving before the accident.

Figure $4.8.10$: Robert Boyle (1627-1691)

Answer

1. 49 feet

Exercise $4.8.11$

Boyle’s law states that if the temperature remains constant, the volume V of a given mass of gas is inversely proportional to the pressure p exerted on it.

1. A balloon is filled to a volume of 216 cubic inches on a diving boat under 1 atmosphere of pressure. If the balloon is taken underwater approximately 33 feet, where the pressure measures 2 atmospheres, then what is the volume of the balloon?

2. A balloon is filled to 216 cubic inches under a pressure of 3 atmospheres at a depth of 66 feet. What would the volume be at the surface, where the pressure is 1 atmosphere?
3. To balance a seesaw, the distance from the fulcrum that a person must sit is inversely proportional to his weight. If a 72-pound boy is sitting 3 feet from the fulcrum, how far from the fulcrum must a 54-pound boy sit to balance the seesaw?
4. The current I in an electrical conductor is inversely proportional to its resistance R. If the current is $\frac{1}{4}$ ampere when the resistance is 100 ohms, what is the current when the resistance is 150 ohms?
5. The amount of illumination I is inversely proportional to the square of the distance d from a light source. If 70 foot-candles of illumination is measured 2 feet away from a lamp, what level of illumination might we expect $\frac{1}{2}$ foot away from the lamp?
6. The amount of illumination I is inversely proportional to the square of the distance d from a light source. If 40 foot-candles of illumination is measured 3 feet away from a lamp, at what distance can we expect 10 foot-candles of illumination?
7. The number of men, represented by y, needed to lay a cobblestone driveway is directly proportional to the area A of the driveway and inversely proportional to the amount of time t allowed to complete the job. Typically, 3 men can lay $1,200$ square feet of cobblestone in 4 hours. How many men will be required to lay $2,400$ square feet of cobblestone in 6 hours?
8. The volume of a right circular cylinder varies jointly as the square of its radius and its height. A right circular cylinder with a 3-centimeter radius and a height of 4 centimeters has a volume of 36π cubic centimeters. Find a formula for the volume of a right circular cylinder in terms of its radius and height.
9. The period T of a pendulum is directly proportional to the square root of its length L. If the length of a pendulum is 1 meter, then the period is approximately 2 seconds. Approximate the period of a pendulum that is 0.5 meter in length.
10. The time t it takes an object to fall is directly proportional to the square root of the distance d it falls. An object dropped from 4 feet will take $\frac{1}{2}$ second to hit the ground. How long will it take an object dropped from 16 feet to hit the ground?

Answer

1. 108 cubic inches

3. 4 feet

5. $1,120$ foot-candles

7. 4 men

9. 1.4 seconds

Exercise $4.8.12$

Newton's universal law of gravitation states that every particle of matter in the universe attracts every other particle with a force F that is directly proportional to the product of the masses m_1 and m_2 of the particles and inversely proportional to the square of the distance d between them. The constant of proportionality is called the gravitational constant.

Figure $4.8.11$: Sir Isaac Newton (1643-1727)

1. If two objects with masses 50 kilograms and 100 kilograms are $\frac{1}{2}$ meter apart, then they produce approximately 1.34×10^{-6} newtons (N) of force. Calculate the gravitational constant.
2. Use the gravitational constant from the previous exercise to write a formula that approximates the force F in newtons between two masses m_1 and m_2, expressed in kilograms, given the distance d between them in meters.
3. Calculate the force in newtons between Earth and the Moon, given that the mass of the Moon is approximately 7.3×10^{22} kilograms, the mass of Earth is approximately 6.0×10^{24} kilograms, and the distance between them is on average 1.5×10^{11} meters.
4. Calculate the force in newtons between Earth and the Sun, given that the mass of the Sun is approximately 2.0×10^{30} kilograms, the mass of Earth is approximately 6.0×10^{24} kilograms, and the distance between them is on average 3.85×10^{8} meters.

5. If y varies directly as the square of x, then how does y change if x is doubled?
6. If y varies inversely as square of t, then how does y change if t is doubled?
7. If y varies directly as the square of x and inversely as the square of t, then how does y change if both x and t are doubled?

Answer

1. $6.7 \times 10^{-11} \mathrm{Nm}^2/\mathrm{kg}^2$

3. $1.98 \times 10^{20} \mathrm{N}$

5. y changes by a factor of 4

7. y remains unchanged

Footnotes

[37] Described by the formula $D = rt$, where the distance D is given as the product of the average rate r and the time t traveled at that rate.

[38] The rate at which a task can be performed.

[39] $\frac{1}{t_1} \cdot t + \frac{1}{t_2} \cdot t = 1$, where $\frac{1}{t_1}$ and $\frac{1}{t_2}$ are the individual work rates and t is the time it takes to complete the task working together.

[40] Describes two quantities x and y that are constant multiples of each other: $y = kx$.

[41] The nonzero multiple k, when quantities vary directly or inversely.

[42] Used when referring to the constant of variation.

[43] Used when referring to direct variation.

[44] Describes two quantities x and y, where one variable is directly proportional to the reciprocal of the other: $y = \frac{k}{x}$.

[45] Used when referring to inverse variation.

[46] Describes a quantity y that varies directly as the product of two other quantities x and $z : y = kxz$.

[47] Used when referring to joint variation.

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4.E: Polynomial and Rational Functions (Exercises)

Exercise $4.E.1$

Evaluate

1. Given $f(x)=2x^2-x+6$, find $f(-3), f(0)$, and $f(10)$.
2. Given $g(x)=-x^2+4x-1$, find $g(-1), g(0)$, and $g(3)$.
3. Given $h(t)=-t^3-2t^2+3$, find $h(-3), h(0)$, and $h(2)$.
4. Given $p(x)=x^4-2x^2+x$, find $p(-1), p(0)$, and $p(2)$.
5. The following graph gives the height $h(t)$ in feet of a projectile over time t in seconds

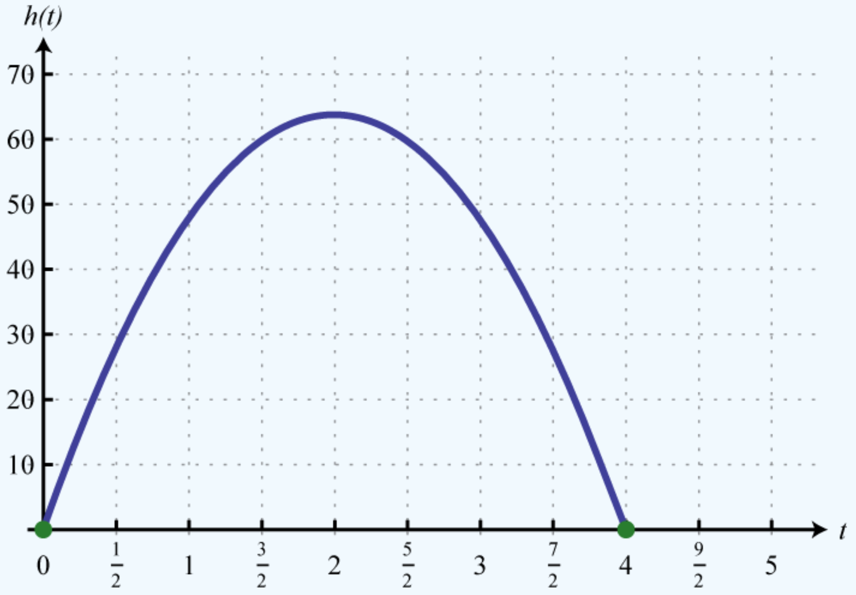

Figure 4.E.1

(a) Use the graph to determine the height of the projectile at 2.5 seconds.

(b) At what time does the projectile reach its maximum height?

(c) How long does it take the projectile to return to the ground?

6. Given the graph of the function f, find $f(-9), f(-3)$, and $f(12)$.

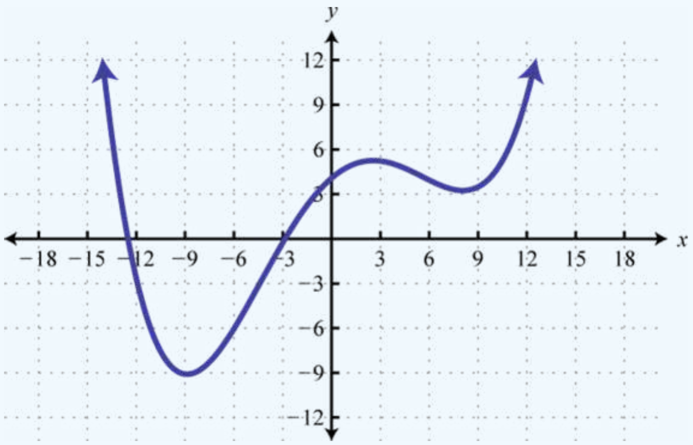

Figure 4.E.2

7. From the ground, a bullet is fired straight up into the air at 340 meters per second. Ignoring the effects of air friction, write a function that models the height of the bullet and use it to calculate the bullet's height after one-quarter of a second. (Round off to the nearest meter.)

8. An object is tossed into the air at an initial speed of 30 feet per second from a rooftop 10 feet high. Write a function that models the height of the object and use it to calculate the height of the object after 1 second.

Answer

1. $f(-3)=27; f(0)=6; f(10)=196$

3. $h(-3)=12; h(0)=3; h(2)=-13$

5. (a) 60 feet; (b) 2 seconds; (c) 4 seconds

7. $h(t)=-4.9t^2+340t$; at 0.25 second, the bullet's height is about 85 meters.

Exercise $4.E.2$

Perform the operations.

1. Given $f(x)=5x^2-3x+1$ and $g(x)=2x^2-x-1$, find $(f+g)(x)$.
2. Given $f(x)=x^2+3x-8$ and $g(x)=x^2-5x-7$, find $(f-g)(x)$.
3. Given $f(x)=3x^2-x+2$ and $g(x)=2x-3$, find $(f\cdot g)(x)$.
4. Given $f(x)=27x^5-15x^3-3x^2$ and $g(x)=3x^2$, find $(f/g)(x)$.
5. Given $g(x)=x^2-x+1$, find $g(-3u)$.
6. GIven $g(x)=x^3-1$, find $g(x-1)$.

Answer

1. $(f+g)(x)=7x^2-4x$

3. $(f\cdot g)(x)=6x^3-11x^2+7x-6$

5. $g(-3u)=9u^2+3u+1$

Exercise $4.E.3$

Given $f(x)=16x^3-12x^2+4x, g(x)=x^2-x+1$, and $h(x)=4x$, find the following:

1. $(g\cdot h)(x)$
2. $(f-g)(x)$
3. $(g+f)(x)$
4. $(f/h)(x)$
5. $(f\cdot h)(-1)$
6. $(g+h)(-3)$
7. $(g-f)(2)$
8. $(f/h)\left(\frac{3}{2}\right)$

Answer

1. $(g\cdot h)(x)=4x^3-4x^2+4x$

3. $(g+f)(x)=16x^3-11x^2+3x+1$

5. $(f\cdot h)(-1)=128$

7. $(g-f)(2)=-85$

Exercise $4.E.4$

Factor our the greatest common factor (GCF).

1. $2x^4-12x^3-2x^2$
2. $18a^3b-3a^2b^2+3ab^3$
3. $x^4y^3-3x^3y+x^2y$

4. $x^{3n} - x^{2n} - x^n$

Answer

1. $2x^2\left(x^2 - 6x - 1\right)$
3. $x^2y\left(x^2y^2 - 3x + 1\right)$

Exercise $4.E.5$

Factor by grouping.

1. $2x^3 - x^2 + 2x - 1$
2. $3x^3 - x^2 - 6x + 2$
3. $x^3 - 5x^2y + xy^2 - 5y^3$
4. $a^2b - a + ab^3 - b^2$
5. $2x^4 - 4xy^3 + 2x^2y^2 - 4x^3y$
6. $x^4y^2 - xy^5 + x^3y^4 - x^2y^3$

Answer

1. $\left(x^2 + 1\right)(2x - 1)$
3. $\left(x^2 + y^2\right)(x - 5y)$
5. $2x(x - 2y)\left(x^2 + y^2\right)$

Exercise $4.E.6$

Factor the special binomials.

1. $64x^2 - 1$
2. $9 - 100y^2$
3. $x^2 - 36y^2$
4. $4 - (2x - 1)^2$
5. $a^3b^3 + 125$
6. $64x^3 - y^3$
7. $81x^4 - y^4$
8. $x^8 - 1$
9. $x^6 - 64y^6$
10. $1 - a^6b^6$

Answer

1. $(8x + 1)(8x - 1)$
3. $(x + 6y)(x - 6y)$
5. $(ab + 5)\left(a^2b^2 - 5ab + 25\right)$
7. $\left(9x^2 + y^2\right)(3x + y)(3x - y)$
9. $(x + 2y)\left(x^2 - 2xy + 4y^2\right)(x - 2y)\left(x^2 + 2xy + 4y^2\right)$

Exercise $4.E.7$

Factor.

1. $x^2 - 8x - 48$
2. $x^2 - 15x + 54$
3. $x^2 - 4x - 6$
4. $x^2 - 12xy + 36y^2$

5. $x^2+20xy+75y^2$
6. $-x^2+5x+150$
7. $-2y^2+20y+48$
8. $28x^2+20x+3$
9. $150x^2-100x+6$
10. $24a^2-38ab+3b^2$
11. $27u^2-3uv-4v^2$
12. $16x^2y^2-78xy+27$
13. $16m^2+72mn+81n^2$
14. $4x^2-5x+20$
15. $25x^4-35x^2+6$
16. $2x^4+7x^2+3$
17. $x^6+3x^3y^3-10y^6$
18. $a^6-8a^3b^3+15b^6$
19. $x^{2n}-2x^n+1$
20. $6x^{2n}-x^n-2$

Answer

1. $(x-12)(x+4)$

3. Prime

5. $(x+5y)(x+15y)$

7. $-2(y-12)(y+2)$

9. $2(15x-1)(5x-3)$

11. $(3u+v)(9u-4v)$

13. $(4m+9n)^2$

15. $\left(5x^2-6\right)\left(5x^2-1\right)$

17. $\left(x^3+5y^3\right)\left(x^3-2y^3\right)$

19. $\left(x^n-1\right)^2$

Exercise $4.E.8$

Factor completely.

1. $45x^3-20x$
2. $12x^4-70x^3+50x^2$
3. $-20x^2+32x-3$
4. $-x^3y+9xy^3$
5. $24a^4b^2+3ab^5$
6. $64a^6b^6-1$
7. $64x^2+1$
8. $x^3+x^2y-xy^2-y^3$

Answer

1. $5x(3x+2)(3x-2)$

3. $-(10x-1)(2x-3)$

5. $3ab^2(2a+b)\left(4a^2-2ab+b^2\right)$

7. Prime

Exercise $4.E.9$

Solve by factoring.

1. $9x^2+8x=0$
2. $x^2-1=0$
3. $x^2-12x+20=0$
4. $x^2-2x-48=0$
5. $(2x+1)(x-2)=3$
6. $2-(x-4)^2=-7$
7. $(x-6)(x+3)=-18$
8. $(x+5)(2x-1)=3(2x-1)$
9. $\frac{1}{2}x^2+\frac{2}{3}x-\frac{1}{8}=0$
10. $\frac{1}{4}x^2-\frac{19}{12}x+\frac{1}{2}=0$
11. $x^3-2x^2-24x=0$
12. $x^4-5x^2+4=0$

Answer

1. $-\frac{8}{9}, 0$
3. $2, 10$
5. $-1, \frac{5}{2}$
7. $0, 3$
9. $-\frac{3}{2}, \frac{1}{6}$
11. $-4, 0, 6$

Exercise $4.E.10$

Find the roots of the given functions.

1. $f(x)=12x^2-8x$
2. $g(x)=2x^3-18x$
3. $h(t)=-16t^2+64$
4. $p(x)=5x^2-21x+4$
5.

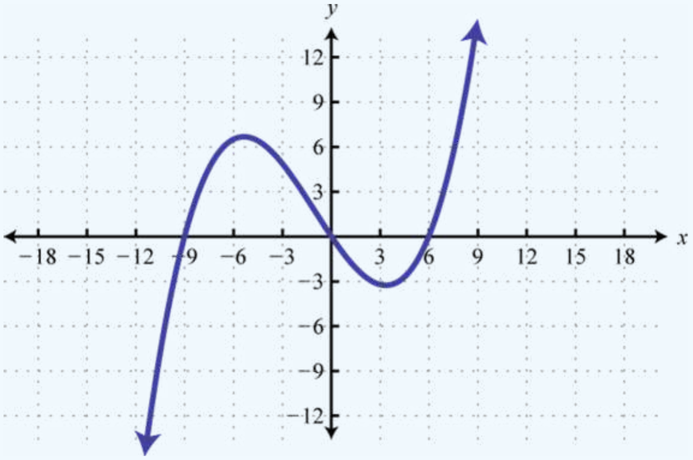

Figure 4.E.3

6.

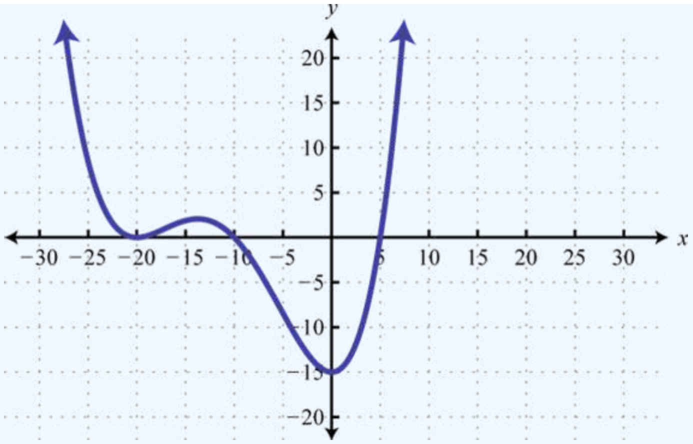

Figure 4.E.4

7. The height in feet of an object dropped from the top of a 16-foot ladder is given by $h(t)=-16t^2+16$, where t represents the time in seconds after the object has been dropped. How long will it take to hit the ground?

8. The length of a rectangle is 2 centimeters less than twice its width. If the area of the rectangle is 112 square centimeters, find its dimensions.

9. A triangle whose base is equal in measure to its height has an area of 72 square inches. Find the length of the base.

10. A box can be made by cutting out the corners and folding up the edges of a sheet of cardboard. A template for a rectangular cardboard box of height 2 inches is given.

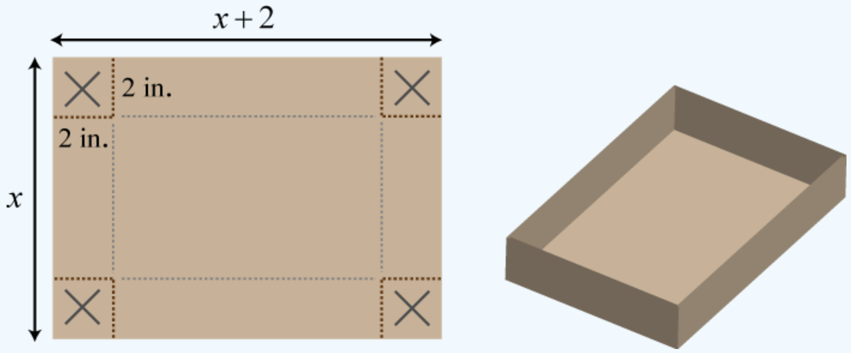

Figure 4.E.5

What are the dimensions of a cardboard sheet that will make a rectangular box with volume 240 cubic inches?

Answer

1. $0, \frac{2}{3}$

3. ± 2

5. $-9, 0, 6$

7. 1 second

9. 12 inches

Exercise $4.E.11$

Solve or factor.

1. x^2-25
2. $x^2-121=0$
3. $16x^2-22x-3=0$
4. $3x^2-14x-5$

5. $x^3 - x^2 - 2x - 2$
6. $3x^2 = -15x$

Answer

1. Factor; $(x+5)(x-5)$

3. Solve; $-\frac{1}{8}, \frac{3}{2}$

5. Factor; $(x-1)\left(x^2-2\right)$

Exercise $4.E.12$

Find a polynomial equation with integer coefficients, given the solutions.

1. $5, -2$
2. $\frac{2}{3}, -\frac{1}{2}$
3. $\pm\frac{4}{5}$
4. ± 10
5. $-4, 0, 3$
6. -8 double root

Answer

1. $x^2 - 3x - 10 = 0$

3. $25x^2 - 16 = 0$

5. $x^3 + x^2 - 12x = 0$

Exercise $4.E.13$

State the restrictions and simplify.

1. $\frac{108x^3}{12x^2}$
2. $\frac{56x^2(x-2)^2}{8x(x-2)^3}$
3. $\frac{64-x^2}{2x^2-15x-8}$
4. $\frac{3x^2+28x+9}{81-x^2}$
5. $\frac{x^2-25}{5x^2} \cdot \frac{10x^2-15x}{2x^2+7x-15}$
6. $\frac{7x^2-41x-6}{(x-7)^2} \cdot \frac{49-x^2}{x^2+x-42}$
7. $\frac{28x^2(2x-3)}{4x^2-9} \div \frac{7x}{4x^2-12x+9}$
8. $\frac{x^2-10x+24}{x^2-8x+16} \div \frac{2x^2-13x+6}{x^2+2x-24}$

Answer

1. $9x; x \neq 0$

3. $-\frac{x+8}{2x+1}; x \neq -\frac{1}{2}, 8$

5. $\frac{x-5}{x}; x \neq -5, 0, \frac{3}{2}$

7. $\frac{4x(2x-3)^2}{2x+3}; x \neq \pm\frac{3}{2}, 0$

Exercise $4.E.14$

Perform the operations and simplify. Assume all variable expressions in the denominator are nonzero.

1. $\frac{a^2-b^2}{4a^2b^2+4ab^3} \cdot \frac{2ab}{a^2-2ab+b^2}$
2. $\frac{a^2-5ab+6b^2}{a^2-4ab+4b^2} \div \frac{9b^2-a^2}{3a^3b-6a^2b^2}$
3. $\frac{x^2+xy+y^2}{4x^2+3xy-y^2} \cdot \frac{x^2-y^2}{x^3-y^3} \div \frac{x+y}{12x^2y-3xy^2}$
4. $\frac{x^4-y^4}{x^2-2xy+y^2} \div \frac{x^2-4xy-5y^2}{10x^3} \cdot \frac{2x^2-11xy+5y^2}{2x^3y+2xy^3}$

Answer

1. $\frac{1}{2b(a-b)}$
3. $\frac{3xy}{x+y}$

Exercise $4.E.15$

Perform the operations and state the restrictions.

1. Given $f(x)=\frac{4x^2+39x-10}{x^2+3x-10}$ and $g(x)=\frac{2x^2+7x-15}{x^2+13x+30}$, find $(f\cdot g)(x)$.
2. Given $f(x)=\frac{25-x^2}{3+x}$ and $g(x)=\frac{9-x^2}{5-x}$, find $(f\cdot g)(x)$.
3. Given $f(x)=\frac{42x^2}{2x^2+3x-2}$ and $g(x)=\frac{14x}{4x^2-4x+1}$, find $(f/g)(x)$.
4. Given $f(x)=\frac{x^2-20x+100}{x^2-1}$ and $g(x)=\frac{x^2-100}{x^2+2x+1}$, find $(f/g)(x)$.
5. The daily cost in dollars of running a small business is given by $C(x)=150+45x$ where x represents the number of hours the business is in operation. Determine the average cost per hour if the business is in operation for 8 hours in a day.
6. An electric bicycle manufacturer has determined that the cost of producing its product in dollars is given by the function $C(n)=2n^2+100n+2,500$ where n represents the number of electric bicycles produced in a day. Determine the average cost per bicycle if 10 and 20 are produced in a day.
7. Given $f(x)=3x-5$, simplify $\frac{f(x+h)-f(x)}{h}$.
8. Given $g(x)=2x^2-x+1$, simplify $\frac{g(x+h)-g(x)}{h}$.

Answer

1. $(f\cdot g)(x)=\frac{(4x-1)(2x-3)}{(x-2)(x+3)}; x\neq -10,-5,-3,2$
3. $(f/g)(x)=\frac{3x(2x-1)}{x+2}; x\neq -2,0,\frac{1}{2}$
5. $\$63.75$ per hour
7. 3

Exercise $4.E.16$

State the restrictions and simplify.

1. $\frac{5x-6}{x^2-36}-\frac{4x}{x^2-36}$
2. $\frac{2}{x}+5x$
3. $\frac{5}{x-5}+\frac{1}{2x}$
4. $\frac{x}{x-2}+\frac{3}{x+3}$
5. $\frac{7(x-1)}{4x^2-17x+15}-\frac{2}{x-3}$
6. $\frac{5}{x}-\frac{19x+25}{2x^2+5x}$
7. $\frac{x}{x-5}-\frac{2}{x-3}-\frac{5(x-3)}{x^2-8x+15}$
8. $\frac{3x}{2x-1}-\frac{x-4}{x+4}+\frac{12(2-x)}{2x^2+7x-4}$
9. $\frac{1}{t-1}+\frac{1}{(t-1)^2}-\frac{1}{t^2-1}$
10. $\frac{1}{t-1}-\frac{2t-5}{t^2-2t+1}-\frac{5t^2-3t-2}{(t-1)^3}$
11. $2x^{-1}+x^{-2}$
12. $(x-4)^{-1}-2x^{-2}$

Answer

1. $\frac{1}{x+6}; x\neq \pm 6$
3. $\frac{11x-5}{2x(x-5)}; x\neq 0,5$
5. $-\frac{1}{4x-5}; x\neq \frac{5}{4},3$
7. $\frac{x-5}{x-3}; x\neq 3,5$
9. $\frac{t^2+1}{(t+1)(t-1)^2}; t\neq \pm 1$
11. $\frac{2x+1}{x^2}; x\neq 0$

Exercise $4.E.17$

Simplify. Assume that all variable expressions used as denominators are nonzero.

1. $\frac{\frac{1}{7}+\frac{1}{x}}{\frac{1}{49}-\frac{1}{x^2}}$
2. $\frac{\frac{1}{100}-\frac{1}{x^2}}{\frac{1}{x}-\frac{1}{10}}$
3. $\frac{\frac{3}{x}-\frac{1}{x-5}}{\frac{5}{x+2}-\frac{2}{x}}$
4. $\frac{1-\frac{12}{x}+\frac{35}{x^2}}{1-\frac{25}{x^2}}$
5. $\frac{x-4x^{-1}}{2-5x^{-1}+2x^{-2}}$
6. $\frac{8x^{-1}+y^{-1}}{y^{-2}-64x^{-2}}$

Answer

1. $\frac{7x}{x-7}$

3. $\frac{(x+2)(2x-15)}{(x-5)(3x-4)}$

5. $\frac{x(x+2)}{2x-1}$

Exercise $4.E.18$

Perform the operations and state the restrictions.

1. Given $f(x)=\frac{3}{x-3}$ and $g(x)=\frac{x-2}{x+2}$, find $(f+g)(x)$.
2. Given $f(x)=\frac{1}{x^2+x}$ and $g(x)=\frac{2x}{x^2-1}$, find $(f+g)(x)$.
3. Given $f(x)=\frac{x-3}{x-5}$ and $g(x)=\frac{x^2-x}{x^2-25}$, find $(f-g)(x)$.
4. Given $f(x)=\frac{11x+4}{x^2-2x-8}$ and $g(x)=\frac{2x}{x-4}$, find $(f-g)(x)$.

Answer

1. $(f+g)(x)=\frac{x^2-2x+12}{(x-3)(x+2)}; x\neq -2,3$

3. $(f-g)(x)=\frac{3}{x+5}; x\neq \pm 5$

Exercise $4.E.19$

Solve.

1. $\frac{3}{x}=\frac{1}{2x+15}$
2. $\frac{x}{x-4}=\frac{x+8}{x-8}$
3. $\frac{x+5}{2(x+2)}+\frac{x-2}{x+4}=1$
4. $\frac{2x}{x-5}+\frac{1}{x+1}=0$
5. $\frac{x+1}{x-4}+\frac{4}{x+6}=-\frac{10}{x^2+2x-24}$
6. $\frac{2}{x}-\frac{12}{2x+3}=\frac{2-3x^2}{2x^2+3x}$
7. $\frac{x+7}{x-2}-\frac{9}{x+7}=\frac{81}{x^2+5x-14}$
8. $\frac{x}{x+5}+\frac{1}{x-4}=\frac{4x-7}{x^2+x-20}$
9. $\frac{2}{3x-1}+\frac{x}{2x+1}=\frac{2(3-4x)}{6x^2+x-1}$
10. $\frac{x}{x-1}+\frac{1}{x+1}=\frac{2x}{x^2-1}$
11. $\frac{2x}{x+5}-\frac{1}{2x-3}=\frac{4-7x}{2x^2+7x-15}$
12. $\frac{x}{x+4}+\frac{1}{2x+7}=\frac{x+8}{2x^2+15x+28}$
13. $\frac{1}{t-1}-\frac{2}{2t+1}=\frac{1}{t-2}-\frac{2}{2t-1}$
14. $\frac{t-1}{t-2}-\frac{t-2}{t-3}=\frac{t-3}{t-4}-\frac{t-4}{t-5}$
15. Solve for $a: \frac{1}{a}=\frac{1}{b}-\frac{1}{c}$
16. Solve for $y: x=\frac{3y-1}{y-5}$
17. A positive integer is 4 less than another. If the reciprocal of the larger integer is subtracted from twice the reciprocal of the smaller, the result is $\frac{1}{6}$. Find the two integers.

18. If 3 times the reciprocal of the larger of two consecutive odd integers is added to 7 times the reciprocal of the smaller, the result is $\frac{4}{3}$. Find the integers.
19. If the reciprocal of the smaller of two consecutive integers is subtracted from three times the reciprocal of the larger, the result is $\frac{3}{10}$. Find the integers.
20. A positive integer is twice that of another. The sum of the reciprocals of the two positive integers is $\frac{1}{4}$. Find the two integers.

Answer

1. -9
3. $-1, 4$
5. $-11, 0$
7. $\emptyset$
9. -4
11. $-\frac{3}{2}$
13. $\frac{3}{4}$
15. $a = \frac{bc}{c-b}$
17. $8, 12$
19. $5, 6$

Exercise $4.E.20$

Use algebra to solve the following applications.

1. Manuel traveled 8 miles on the bus and another 84 miles on a train. If the train was 16 miles per hour faster than the bus, and the total trip took 2 hours, what was the average speed of the train?
2. A boat can average 10 miles per hour in still water. On a trip downriver, the boat was able to travel 7.5 miles with the current. On the return trip, the boat was only able to travel 4.5 miles in the same amount of time against the current. What was the speed of the current?
3. Susan can jog, on average, $1\frac{1}{2}$ miles per hour faster than her husband Bill. Bill can jog 10 miles in the same amount of time it takes Susan to jog 13 miles. How fast, on average, can Susan jog?
4. In the morning, Raul drove 8 miles to visit his grandmother and then returned later that evening. Because of traffic, his average speed on the return trip was $\frac{1}{2}$ that of his average speed that morning. If the total driving time was $\frac{3}{4}$ of an hour, what was his average speed on the return trip?
5. One pipe can completely fill a water tank in 6 hours while another smaller pipe takes 8 hours to fill the same tank. How long will it take to fill the tank to $\frac{3}{4}$ capacity if both pipes are turned on?
6. It takes Bill 3 minutes longer than Jerry to fill an order. Working together they can fill 15 orders in 30 minutes. How long does it take Bill to fill an order by himself?
7. Manny takes twice as long as John to assemble a skateboard. If they work together, they can assemble a skateboard in 6 minutes. How long would it take Manny to assemble the skateboard without John's help?
8. Working alone, Joe can complete the yard work in 30 minutes. It takes Mike 45 minutes to complete work on the same yard. How long would it take them working together?

Answer

1. 48 miles per hour
3. 6.5 miles per hour
5. Approximately 2.6 hours
7. 18 minutes

Exercise $4.E.21$

Construct a mathematical model given the following:

1. y varies directly as x, where $y = 30$ when $x = 5$.
2. y varies inversely as x, where $y = 3$ when $x = -2$.
3. y is jointly proportional to x and z, where $y = -50$ when $x = -2$ and $z = 5$.
4. y is directly proportional to the square of x and inversely proportional to z, where $y = -6$ when $x = 2$ and $z = -8$.
5. The distance an object in free fall varies directly with the square of the time that it has been falling. It is observed that an object falls 36 feet in $1\frac{1}{2}$ seconds. Find an equation that models the distance an object will fall, and use it to determine how far it will fall in $2\frac{1}{2}$ seconds.
6. After the brakes are applied, the stopping distance d of an automobile varies directly with the square of the speed s of the car. If a car traveling 55 miles per hour takes 181.5 feet to stop, how many feet will it take to stop if it is moving 65 miles per hour?
7. The weight of an object varies inversely as the square of its distance from the center of the Earth. If an object weighs 180 lbs on the surface of the Earth (approximately $4,000$ miles from the center), then how much will it weigh at $2,000$ miles above the Earth's surface?
8. The cost per person of renting a limousine varies inversely with the number of people renting it. If 5 people go in on the rental, the limousine will cost $\$112$ per person. How much will the rental cost per person if 8 people go in on the rental?
9. To balance a seesaw, the distance from the fulcrum that a person must sit is inversely proportional to his weight. If a 52-pound boy is sitting 3 feet away from the fulcrum, then how far from the fulcrum must a 44-pound boy sit? Round to the nearest tenth of a foot.

Answer

1. $y = 6x$
3. $y = 5xz$
5. $d = 16t^2$; 100 feet
7. 80 lbs
9. Approximately 3.5 feet

Sample Exam

Exercise $4.E.22$

Given $f(x) = x^2 - x + 4, g(x) = 5x - 1$, and $h(x) = 2x^2 + x - 3$, find the following:

1. $(g \cdot h)(x)$
2. $(h - f)(x)$
3. $(f + g)(-1)$

Answer

1. $(g \cdot h)(x) = 10x^3 + 3x^2 - 16x + 3$
3. $(f + g)(-1) = 0$

Exercise $4.E.23$

Factor.

1. $x^3 + 16x - 2x^2 - 32$
2. $x^3 - 8y^3$
3. $x^4 - 81$
4. $25x^2y^2 - 40xy + 16$
5. $16x^3y + 12x^2y^2 - 18xy^3$

Answer

2. $(x-2y)\left(x^2+2xy+4y^2\right)$
4. $(5xy-4)^2$

Exercise $4.E.24$

Solve

1. $6x^2+24x=0$
2. $(2x+1)(3x+2)=12$
3. $(2x+1)^2=23x+6$
4. Find a quadratic equation with integer coefficients given the solutions $\left\{\frac{1}{2},-3\right\}$.
5. Given $f(x)=5x^2-x+4$, simplify $\frac{f(x+h)-f(x)}{h}$, where $h\neq 0$.

Answer

1. $-4, 0$
3. $-\frac{1}{4}, 5$
5. $10x+5h-1$

Exercise $4.E.25$

Simplify and state the restrictions.

1. $\frac{4x^2-33x+8}{x^2-10x+16} \div \frac{16x^2-1}{x^2-4x+4}$
2. $\frac{x-1}{x-7}+\frac{1}{1-x}-\frac{2(x+11)}{x^2-8x+7}$

Answer

2. $\frac{x+2}{x-1}; x\neq 1, 7$

Exercise $4.E.26$

Assume all variable expressions in the denominator are nonzero and simplify.

1. $\frac{\frac{3}{x}+\frac{1}{y}}{\frac{1}{y^2}-\frac{9}{x^2}}$

Exercise $4.E.27$

Solve.

1. $\frac{6x-5}{3x+2}=\frac{2x}{x+1}$
2. $\frac{2x}{x+5}-\frac{1}{5-x}=\frac{2x}{x^2-25}$
3. Find the root of the function defined by $f(x)=\frac{1}{x+3}-4$.
4. Solve for y: $x=\frac{4y}{3y-1}$

Answer

1. $-\frac{5}{3}$
3. $-\frac{11}{4}$

Exercise $4.E.28$

Use algebra to solve.

1. The height of an object dropped from a 64-foot building is given by the function $h(t)=-16t^2+64$, where t represents time in seconds after it was dropped.

   1. Determine the height of the object at $\frac{3}{4}$ of a second.
   2. How long will it take the object to hit the ground?
2. One positive integer is 3 units more than another. When the reciprocal of the larger is subtracted from twice the reciprocal of the smaller, the result is $\frac{2}{9}$. Find the two positive integers.
3. A light airplane can average 126 miles per hour in still air. On a trip, the airplane traveled 222 miles with a tailwind. On the return trip, against a headwind of the same speed, the plane was only able to travel 156 miles in the same amount of time. What was the speed of the wind?
4. On the production line, it takes John 2 minutes less time than Mark to assemble a watch. Working together they can assemble 5 watches in 12 minutes. How long does it take John to assemble a watch working alone?
5. Write an equation that relates x and y, given that y varies inversely with the square of x, where $y=-\frac{1}{3}$ when $x=3$. Use it to find y when $x=\frac{1}{2}$.

Answer

1. (1) 55 feet; (2) 2 seconds

3. 22 miles per hour

5. $y=-\frac{3}{x^2}; y=-12$

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CHAPTER OVERVIEW

5: Radical Functions and Equations

5.1: Roots and Radicals
5.2: Simplifying Radical Expressions
5.3: Adding and Subtracting Radical Expressions
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5.5: Rational Exponents
5.6: Solving Radical Equations
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5.E: Radical Functions and Equations (Exercises)

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5.1: Roots and Radicals

Learning Objectives

- Identify and evaluate square and cube roots.
- Determine the domain of functions involving square and cube roots.
- Evaluate nth roots.
- Simplify radicals using the product and quotient rules for radicals.

Square and Cube Roots

Recall that a **square root**[1] of a number is a number that when multiplied by itself yields the original number. For example, 5 is a square root of 25, because $5^2=25$. Since $(-5)^2=25$, we can say that -5 is a square root of 25 as well. Every positive real number has two square roots, one positive and one negative. For this reason, we use the radical sign $\sqrt{}$ to denote the **principal (nonnegative) square root**[2] and a negative sign in front of the radical $-\sqrt{}$ to denote the negative square root.

$$\begin{aligned} \sqrt{25}&=5 \quad \color{Cerulean}{Positive\ square\ root\ of\ 25} \\ -\sqrt{25}&=-5 \quad \color{Cerulean}{Negative\ square\ root\ of\ 25} \end{aligned}$$

Zero is the only real number with one square root.

$$\sqrt{0}=0 \text{ because } 0^2=0$$

Example 5.1.1:

Evaluate.

a. $\sqrt{121}$
b. $-\sqrt{81}$

Solution

a. $\sqrt{121}=\sqrt{11^2}=11$
b. $-\sqrt{81}=-\sqrt{9^2}=-9$

If the **radicand**[3], the number inside the radical sign, can be factored as the square of another number, then the square root of the number is apparent. In this case, we have the following property:

$$\sqrt{a^2}=a \quad \text{if} \quad a \geq 0$$

Or more generally,

$$\sqrt{a^2}=|a| \quad \text{if} \quad a \in R$$

The absolute value is important because a may be a negative number and the radical sign denotes the principal square root. For example,

$$\sqrt{(-8)^2}=|-8|=8$$

Make use of the absolute value to ensure a positive result.

Example 5.1.2:

Simplify: $\sqrt{(x-2)^2}$.

Solution

Here the variable expression $x-2$ could be negative, zero, or positive. Since the sign depends on the unknown quantity x, we must ensure that we obtain the principal square root by making use of the absolute value.

$$\sqrt{(x-2)^2}=|x-2|$$

Answer:

$|x-2|$

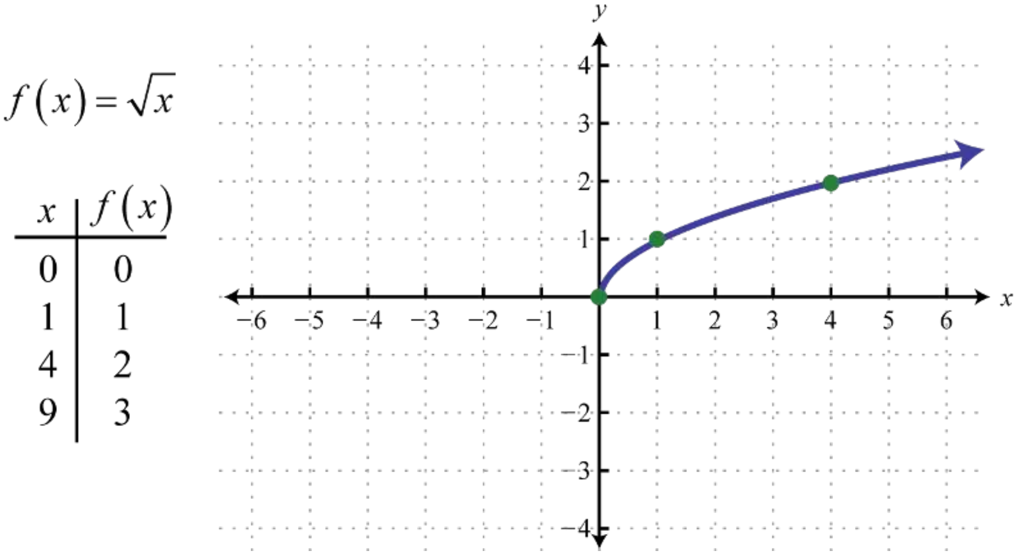
The importance of the use of the absolute value in the previous example is apparent when we evaluate using values that make the radicand negative. For example, when $x=1$,

$$\begin{aligned}\sqrt{(x-2)^2} &= |x-2| \\ &= |1-2| \\ &= |-1| \\ &= 1\end{aligned}$$

Next, consider the square root of a negative number. To determine the square root of -25, you must find a number that when squared results in -25:

$$\sqrt{-25} = ? \quad \text{or} \quad (?)^2 = -25$$

However, any real number squared always results in a positive number. The square root of a negative number is currently left undefined. For now, we will state that $\sqrt{-25}$ is not a real number. Therefore, the **square root function**[4] given by $f(x)=\sqrt{x}$ is not defined to be a real number if the x-values are negative. The smallest value in the domain is zero. For example,$f(0)=\sqrt{0}=0$ and $f(4)=\sqrt{4}=2$. Recall the graph of the square root function.



x	$f(x)$
0	0
1	1
4	2
9	3

Figure $5.1.1$

The domain and range both consist of real numbers greater than or equal to zero: $[0,\infty)$. To determine the domain of a function involving a square root we look at the radicand and find the values that produce nonnegative results.

Example $5.1.3$:

Determine the domain of the function defined by $f(x)=\sqrt{2x+3}$.

Solution

Here the radicand is $2x+3$. This expression must be zero or positive. In other words,

$$2x+3 \geq 0$$

Solve for x.

$$\begin{aligned}2x+3 &\geq 0 \\ 2x &\geq -3 \\ x &\geq -\frac{3}{2}\end{aligned}$$

Answer:

Domain: $[-\frac{3}{2},\infty)$

A **cube root**[5] of a number is a number that when multiplied by itself three times yields the original number. Furthermore, we denote a cube root using the symbol $\sqrt[3]{}$, where 3 is called the **index**[6]. For example,

$$\sqrt[3]{64} = 4, \text{ because } 4^3 = 64$$

The product of three equal factors will be positive if the factor is positive and negative if the factor is negative. For this reason, any real number will have only one real cube root. Hence the technicalities associated with the principal root do not apply. For example,

$$\sqrt[3]{-64} = -4, \text{ because } (-4)^3 = -64$$

In general, given any real number a, we have the following property:

$$\sqrt[3]{a^3} = a \quad \text{if} \quad a \in R$$

When simplifying cube roots, look for factors that are perfect cubes.

Example $5.1.4$:

Evaluate.

1. $\sqrt[3]{8}$
2. $\sqrt[3]{0}$
3. $\sqrt[3]{\frac{1}{27}}$
4. $\sqrt[3]{-1}$
5. $\sqrt[3]{-125}$

Solution

1. $\sqrt[3]{8} = \sqrt[3]{2^3} = 2$
2. $\sqrt[3]{0} = \sqrt[3]{0^3} = 0$
3. $\sqrt[3]{\frac{1}{27}} = \sqrt[3]{\left(\frac{1}{3}\right)^3} = \frac{1}{3}$
4. $\sqrt[3]{-1} = \sqrt[3]{(-1)^3} = -1$
5. $\sqrt[3]{-125} = \sqrt[3]{(-5)^3} = -5$

It may be the case that the radicand is not a perfect square or cube. If an integer is not a perfect power of the index, then its root will be irrational. For example, $\sqrt[3]{2}$ is an irrational number that can be approximated on most calculators using the root button $\sqrt[x]{}$.Depending on the calculator, we typically type in the index prior to pushing the button and then the radicand as follows:

$$3 \quad \sqrt[x]{y} \quad 2 \quad =$$

Therefore, we have

$$\sqrt[3]{2} \approx 1.260, \qquad \text{because} \qquad 1.260\text{^}3 \approx 2$$

Since cube roots can be negative, zero, or positive we do not make use of any absolute values.

Example $5.1.5$:

Simplify: $\sqrt[3]{(y-7)^3}$.

Solution

The cube root of a quantity cubed is that quantity.

$$\sqrt[3]{(y-7)^3} = y - 7$$

Answer:

$y - 7$

Exercise $5.1.1$

Evaluate: $\sqrt[3]{-1000}$.

Answer

$= 10$

www.youtube.com/v/B06NIs-3gig

Next, consider the **cube root function**[7]:

$$f(x)=\sqrt[3]{x} \quad \textit{Cube root function.}$$

Since the cube root could be either negative or positive, we conclude that the domain consists of all real numbers. Sketch the graph by plotting points. Choose some positive and negative values for x, as well as zero, and then calculate the corresponding y-values.

Table $5.1.1$

x	$f(x)$	$f(x)=\sqrt[3]{x}$	*Ordered Pairs*
-8	-2	$f(-8)=\sqrt[3]{-8}=-2$	$(-8,-2)$
-1	-1	$f(-1)=\sqrt[3]{-1}=-1$	$(-1,-1)$
0	0	$f(0)=\sqrt[3]{0}=0$	$(0,0)$
1	1	$f(1)=\sqrt[3]{1}=1$	$(1,1)$
8	2	$f(8)=\sqrt[3]{8}=2$	$(8,2)$

Plot the points and sketch the graph of the cube root function.

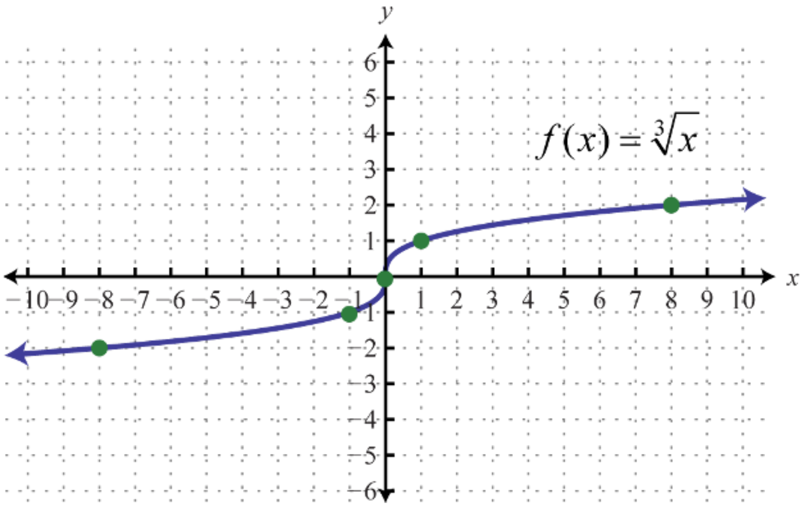


Figure $5.1.2$

The graph passes the vertical line test and is indeed a function. In addition, the range consists of all real numbers.

Example $5.1.6$:

Given $g(x)=\sqrt[3]{x+1}+2$, find $g(-9), g(-2), g(-1)$, and $g(0)$. Sketch the graph of g.

Solution

Replace x with the given values.

Table $5.1.2$

x	$g(x)$	$g(x)=\sqrt[3]{x+1}+2$	*Ordered Pairs*
-9	0	$g(-9)=\sqrt[3]{-9+1}+2=\sqrt[3]{-8}+2$ $=-2+2=0$	$(-9,0)$
-2	1	$g(-2)=\sqrt[3]{-2+1}+2=\sqrt[3]{-1}+2$ $=-1+2=1$	$(-2,1)$
-1	2	$g(-1)=\sqrt[3]{-1+1}+2=\sqrt[3]{0}+2$ $=0+2=2$	$(-1,2)$

x	$g(x)$	$g(x)=\sqrt[3]{x+1}+2$	*Ordered Pairs*
0	3	$g(0)=\sqrt[3]{0+1}+2=\sqrt[3]{1}+2=1+2=3$	$(0,3)$

We can also sketch the graph using the following translations:

$$\begin{aligned} &y=\sqrt[3]{x} && \textit{Basic cube root function} \\ &y=\sqrt[3]{x+1} && \textit{Horizontal shift left 1 unit} \\ &y=\sqrt[3]{x+1}+2 && \textit{Vertical shift up 2 units} \end{aligned}$$

Answer:

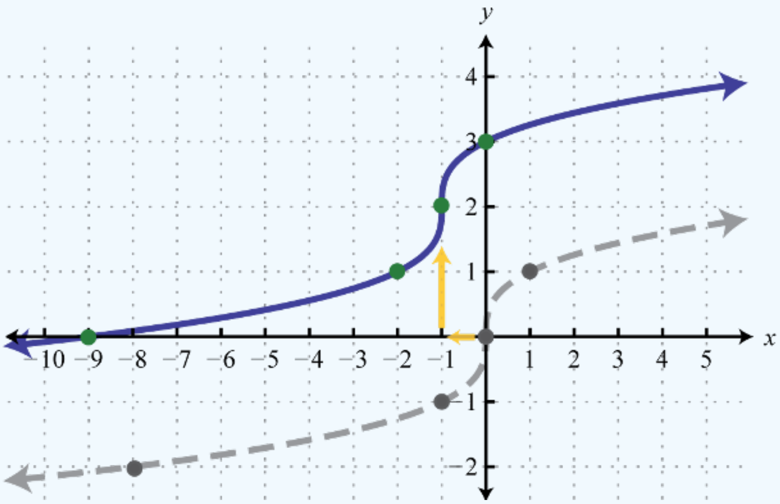


Figure $5.1.3$

nth Roots

For any integer $n\geq 2$, we define an **nth root**[8] of a positive real number as a number that when raised to the nth power yields the original number. Given any nonnegative real number a, we have the following property:

$$\sqrt[n]{a^n}=a, \quad \text{if} \quad a\geq 0$$

Here n is called the index and a^n is called the radicand. Furthermore, we can refer to the entire expression $\sqrt[n]{A}$ as a **radical**[9]. When the index is an integer greater than or equal to 4, we say "fourth root," "fifth root," and so on. The nth root of any number is apparent if we can write the radicand with an exponent equal to the index.

Example $5.1.7$:

Simplify:

1. $\sqrt[4]{81}$
2. $\sqrt[5]{32}$
3. $\sqrt[7]{1}$
4. $\sqrt[4]{\frac{1}{16}}$

Solution

1. $\sqrt[4]{81}=\sqrt[4]{3^4}=3$
2. $\sqrt[5]{32}=\sqrt[5]{2^5}=2$
3. $\sqrt[7]{1}=\sqrt[7]{1^7}=1$
4. $\sqrt[4]{\frac{1}{16}}=\sqrt[4]{\left(\frac{1}{2}\right)^4}=\frac{1}{2}$

Note

If the index is $n=2$, then the radical indicates a square root and it is customary to write the radical without the index; $\sqrt[2]{a}=\sqrt{a}$.

We have already taken care to define the principal square root of a real number. At this point, we extend this idea to nth roots when n is even. For example, 3 is a fourth root of 81, because $3^4=81$. And since $(-3)^4=81$, we can say that -3 is a fourth root of 81 as well. Hence we use the radical sign $\sqrt[n]{}$ to denote the **principal (nonnegative) nth root**[10] when n is even. In this case, for any real number a, we use the following property:

$$\sqrt[n]{a^n}=|a| \quad \color{#1a9fd6}{When\ n\ is\ even}$$

For example,

$$\begin{aligned}\sqrt[4]{81}=\sqrt[4]{3^4} &= |3| &&= 3\\ \sqrt[4]{81}=\sqrt[4]{(-3)^4} &= |-3| &&= 3\end{aligned}$$

The negative nth root, when n is even, will be denoted using a negative sign in front of the radical $-\sqrt[n]{}$.

$$-\sqrt[4]{81}=-\sqrt[4]{3^4}=-3$$

We have seen that the square root of a negative number is not real because any real number that is squared will result in a positive number. In fact, a similar problem arises for any even index:

$$\sqrt[4]{-81}=\color{#1a9fd6}{?} \quad \text{or} \quad (\color{#1a9fd6}{?})^4=-81$$

We can see that a fourth root of -81 is not a real number because the fourth power of any real number is always positive.

$$\left.\begin{array}{l}\sqrt{-4}\\ \sqrt[4]{-81}\\ \sqrt[6]{-64}\end{array}\right\} \quad \color{#1a9fd6}{These\ radicals\ are\ not\ real\ numbers.}$$

You are encouraged to try all of these on a calculator. What does it say?

Example 5.1.8:

Simplify.

1. $\sqrt[4]{(-10)^4}$
2. $\sqrt[4]{-10^4}$
3. $\sqrt[6]{(2y+1)^6}$

Solution

Since the indices are even, use absolute values to ensure nonnegative results.

1. $\sqrt[4]{(-10)^4}=|-10|=10$
2. $\sqrt[4]{-10^4}=\sqrt[4]{-10,000}$ is not a real number.
3. $\sqrt[6]{(2y+1)^6}=|2y+1|$

When the index n is odd, the same problems do not occur. The product of an odd number of positive factors is positive and the product of an odd number of negative factors is negative. Hence when the index n is odd, there is only one real nth root for any real number a. And we have the following property:

$$\sqrt[n]{a^n}=a \quad \color{#1a9fd6}{When\ n\ is\ odd}$$

Example 5.1.9:

Simplify.

1. $\sqrt[5]{(-10)^5}$
2. $\sqrt[5]{-32}$
3. $\sqrt[7]{(2y+1)^7}$

Solution

Since the indices are odd, the absolute value is not used.

1. $\sqrt[5]{(-10)^5} = -10$
2. $\sqrt[5]{-32} = \sqrt[5]{(-2)^5} = -2$
3. $\sqrt[7]{(2y+1)^7} = 2y+1$

In summary, for any real number a we have,

$$\begin{aligned}\sqrt[n]{a^n} &= |a| \quad \textit{When n is even}\\ \sqrt[n]{a^n} &= a \quad \textit{When n is odd}\end{aligned}$$

When n *is odd*, the nth root is *positive or negative* depending on the sign of the radicand.

$$\begin{aligned}\sqrt[3]{27} &= \sqrt[3]{3^3} = 3\\ \sqrt[3]{-27} &= \sqrt[3]{(-3)^3} = -3\end{aligned}$$

When n *is even*, the nth root is *positive or not real* depending on the sign of the radicand.

$$\begin{aligned}\sqrt[4]{16} &= \sqrt[4]{2^4} \qquad = 2\\ \sqrt[4]{16} &= \sqrt[4]{(-2)^4} = |-2| = 2\\ \sqrt[4]{-16} & \quad \textit{Not a real number}\end{aligned}$$

Exercise 5.1.2

Simplify: $-8\sqrt[5]{-32}$.

Answer

16

www.youtube.com/v/Ik1xXgq18f0

Simplifying Radicals

It will not always be the case that the radicand is a perfect power of the given index. If it is not, then we use the **product rule for radicals**[11] and the **quotient rule for radicals**[12] to simplify them. Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$,

Table 5.1.3

Product Rule for Radicals:	$\sqrt[n]{A \cdot B} = \sqrt[n]{A} \cdot \sqrt[n]{B}$
Quotient Rule for Radicals:	$\sqrt[n]{\frac{A}{B}} = \frac{\sqrt[n]{A}}{\sqrt[n]{B}}$

A **radical is simplified**[13] if it does not contain any factors that can be written as perfect powers of the index.

Example 5.1.10:

Simplify: $\sqrt{150}$.

Solution

Here 150 can be written as $2 \cdot 3 \cdot 5^2$.

$$\begin{aligned}\sqrt{150} &= \sqrt{2 \cdot 3 \cdot 5^2} \qquad \textit{Apply the product rule for radicals.}\\ &= \sqrt{2 \cdot 3} \cdot \sqrt{5^2} \qquad \textit{Simplify.}\\ &= \sqrt{6} \cdot 5\\ &= 5\sqrt{6}\end{aligned}$$

We can verify our answer on a calculator:

$$\sqrt{150} \approx 12.25 \quad \text{and} \quad 5\sqrt{6} \approx 12.25$$

Also, it is worth noting that

$$12.25^2 \approx 150$$

Answer:

$5\sqrt{6}$

Note

$5\sqrt{6}$ is the exact answer and 12.25 is an approximate answer. We present exact answers unless told otherwise.

Example $5.1.11$:

Simplify: $\sqrt[3]{160}$.

Solution

Use the prime factorization of 160 to find the largest perfect cube factor:

$$\begin{aligned} 160 &= 2^5 \cdot 5 \\ &= 2^3 \cdot 2^2 \cdot 5 \end{aligned}$$

Replace the radicand with this factorization and then apply the product rule for radicals.

$$\begin{aligned} \sqrt[3]{160} &= \sqrt[3]{2^3 \cdot 2^2 \cdot 5} && \textit{Apply the product rule for radicals.} \\ &= \sqrt[3]{2^3} \cdot \sqrt[3]{2^2 \cdot 5} && \textit{Simplify.} \\ &= 2 \cdot \sqrt[3]{20} \end{aligned}$$

We can verify our answer on a calculator.

$$\sqrt[3]{160} \approx 5.43 \text{ and } 2\sqrt[3]{20} \approx 5.43$$

Answer:

$2\sqrt[3]{20}$

Example $5.1.12$:

Simplify: $\sqrt[5]{-320}$.

Solution

Here we note that the index is odd and the radicand is negative; hence the result will be negative. We can factor the radicand as follows:

$$-320 = -1 \cdot 32 \cdot 10 = (-1)^5 \cdot (2)^5 \cdot 10$$

Then simplify:

$$\begin{aligned} \sqrt[5]{-320} &= \sqrt[5]{(-1)^5 \cdot (2)^5 \cdot 10} && \textit{Apply the product rule for radicals.} \\ &= \sqrt[5]{(-1)^5} \cdot \sqrt[5]{(2)^5} \cdot \sqrt[5]{10} && \textit{Simplify.} \\ &= -1 \cdot 2 \cdot \sqrt[5]{10} \\ &= -2 \cdot \sqrt[5]{10} \end{aligned}$$

Answer:

$-2\sqrt[5]{10}$

Example $5.1.13$:

Simplify: $\sqrt[3]{-\frac{8}{64}}$.

Solution

In this case, consider the equivalent fraction with $-8 = (-2)^3$ in the numerator and $64 = 4^3$ in the denominator and then simplify.

$$\begin{aligned}\sqrt[3]{-\frac{8}{64}} &= \sqrt[3]{\frac{-8}{64}} && \textit{Apply the quotient rule for radicals.}\\ &= \frac{\sqrt[3]{(-2)^3}}{\sqrt[3]{4^3}} && \textit{Simplify.}\\ &= \frac{-2}{4}\\ &= -\frac{1}{2}\end{aligned}$$

Answer:

$-\frac{1}{2}$

Exercise 5.1.3

Simplify: $\sqrt[4]{\frac{80}{81}}$

Answer

$\frac{2\sqrt[4]{5}}{3}$

www.youtube.com/v/8CwbDBFO2FQ

Key Takeaways

- To simplify a square root, look for the largest perfect square factor of the radicand and then apply the product or quotient rule for radicals.
- To simplify a cube root, look for the largest perfect cube factor of the radicand and then apply the product or quotient rule for radicals.
- When working with nth roots, n determines the definition that applies. We use $\sqrt[n]{a^n} = a_1$ when n is odd and $\sqrt[n]{a^n} = |a|$ when n is even.
- To simplify nth roots, look for the factors that have a power that is equal to the index n and then apply the product or quotient rule for radicals. Typically, the process is streamlined if you work with the prime factorization of the radicand.

Exercise 5.1.4

Simplify.

1. $\sqrt{36}$
2. $\sqrt{100}$
3. $\sqrt{\frac{4}{9}}$
4. $\sqrt{\frac{1}{64}}$
5. $-\sqrt{16}$
6. $-\sqrt{1}$
7. $\sqrt{(-5)^2}$
8. $\sqrt{(-1)^2}$
9. $\sqrt{-4}$
10. $\sqrt{-5^2}$
11. $-\sqrt{(-3)^2}$
12. $-\sqrt{(-4)^2}$
13. $\sqrt{x^2}$
14. $\sqrt{(-x)^2}$
15. $\sqrt{(x-5)^2}$
16. $\sqrt{(2x-1)^2}$
17. $\sqrt[3]{64}$
18. $\sqrt[3]{216}$
19. $\sqrt[3]{-216}$
20. $\sqrt[3]{-64}$

21. $\sqrt[3]{-8}$
22. $\sqrt[3]{1}$
23. $-\sqrt[3]{(-2)^3}$
24. $-\sqrt[3]{(-7)^3}$
25. $\sqrt[3]{\frac{1}{8}}$
26. $\sqrt[3]{\frac{8}{27}}$
27. $\sqrt[3]{(-y)^3}$
28. $-\sqrt[3]{y^3}$
29. $\sqrt[3]{(y-8)^3}$
30. $\sqrt[3]{(2x-3)^3}$

Answer

1. 6

3. $\frac{2}{3}$

5. -4

7. 5

9. Not a real number

11. -3

13. $|x|$

15. $|x-5|$

17. 4

19. -6

21. -2

23. 2

25. $\frac{1}{2}$

27. $-y$

29. $y-8$

Exercise 5.1.5

Determine the domain of the given function.

1. $g(x)=\sqrt{x+5}$
2. $g(x)=\sqrt{x-2}$
3. $f(x)=\sqrt{5x+1}$
4. $f(x)=\sqrt{3x+4}$
5. $g(x)=\sqrt{-x+1}$
6. $g(x)=\sqrt{-x-3}$
7. $h(x)=\sqrt{5-x}$
8. $h(x)=\sqrt{2-3x}$
9. $g(x)=\sqrt[3]{x+4}$
10. $g(x)=\sqrt[3]{x-3}$

Answer

1. $[-5,\infty)$

3. $[-\frac{1}{5},\infty)$

5. $(-\infty,1]$

7. $(-\infty, 5]$

9. $(-\infty, \infty)$

Exercise $5.1.6$

Evaluate given the function definition.

1. Given $f(x)=\sqrt{x-1}$, find $f(1), f(2)$, and $f(5)$
2. Given $f(x)=\sqrt{x+5}$, find $f(-5), f(-1)$, and $f(20)$
3. Given $f(x)=\sqrt{x}+3$, find $f(0), f(1)$, and $f(16)$
4. Given $f(x)=\sqrt{x}-5$, find $f(0), f(1)$, and $f(25)$
5. Given $g(x)=\sqrt[3]{x}$, find $g(-1), g(0)$, and $g(1)$
6. Given $g(x)=\sqrt[3]{x}-2$ find $g(-1), g(0)$, and $g(8)$
7. Given $g(x)=\sqrt[3]{x+7}$, find $g(-15), g(-7)$, and $g(20)$
8. Given $g(x)=\sqrt[3]{x-1}+2$, find $g(0), g(2)$, and $g(9)$

Answer

1. $f(1)=0; f(2)=1; f(5)=2$

3. $f(0)=3; f(1)=4; f(16)=7$

5. $g(-1)=-1; g(0)=0; g(1)=1$

7. $g(-15)=-2; g(-7)=0; g(20)=3$

Exercise $5.1.7$

Sketch the graph of the given function and give its domain and range.

1. $f(x)=\sqrt{x+9}$
2. $f(x)=\sqrt{x-3}$
3. $f(x)=\sqrt{x-1}+2$
4. $f(x)=\sqrt{x+1}+3$
5. $g(x)=\sqrt[3]{x-1}$
6. $g(x)=\sqrt[3]{x+1}$
7. $g(x)=\sqrt[3]{x}-4$
8. $g(x)=\sqrt[3]{x}+5$
9. $g(x)=\sqrt[3]{x+2}-1$
10. $g(x)=\sqrt[3]{x-2}+3$
11. $f(x)=-\sqrt[3]{x}$
12. $f(x)=-\sqrt[3]{x-1}$

Answer

1. Domain: $[-9, \infty)$; range: $[0, \infty)$

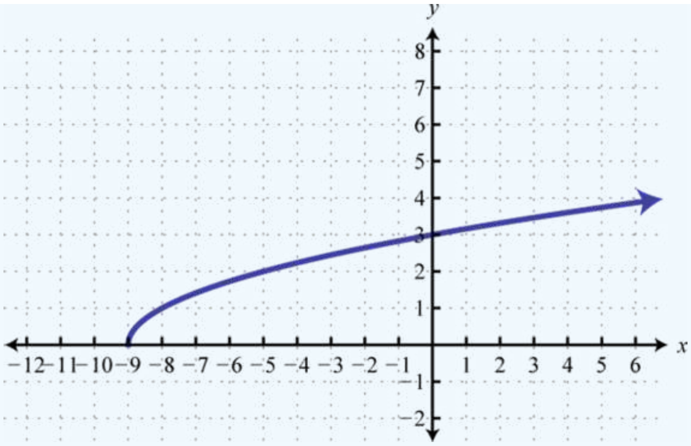


Figure 5.1.4

3. Domain: $[1, \infty)$; range: $[2, \infty)$

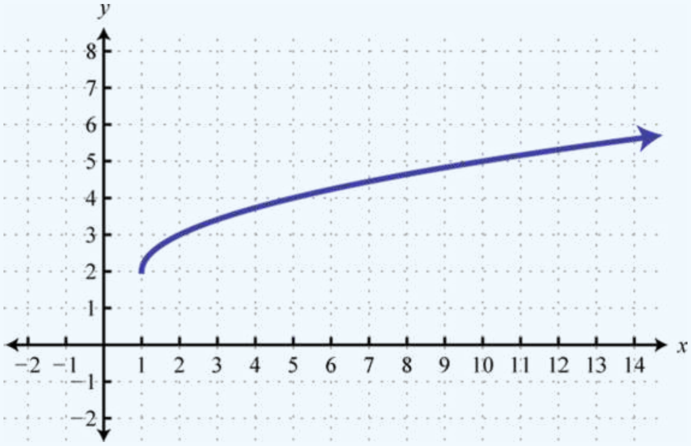


Figure 5.1.5

5. Domain: $\mathbb{R}$; range; $\mathbb{R}$

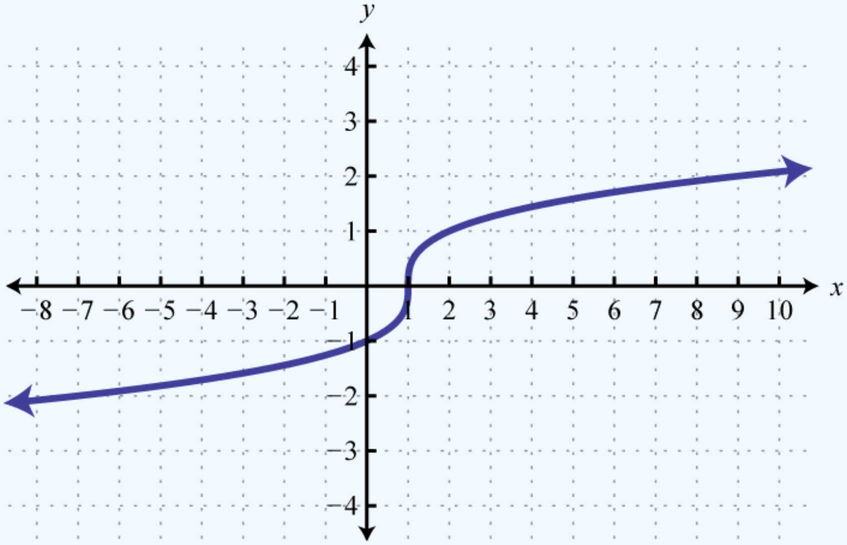


Figure 5.1.6

7. Domain: $\mathbb{R}$; range; $\mathbb{R}$

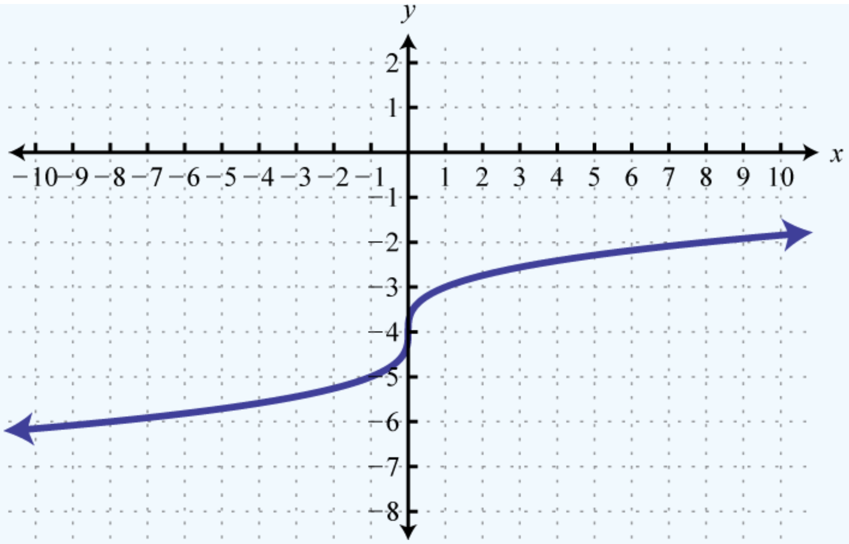


Figure 5.1.7

9. Domain: ℝ; range; ℝ

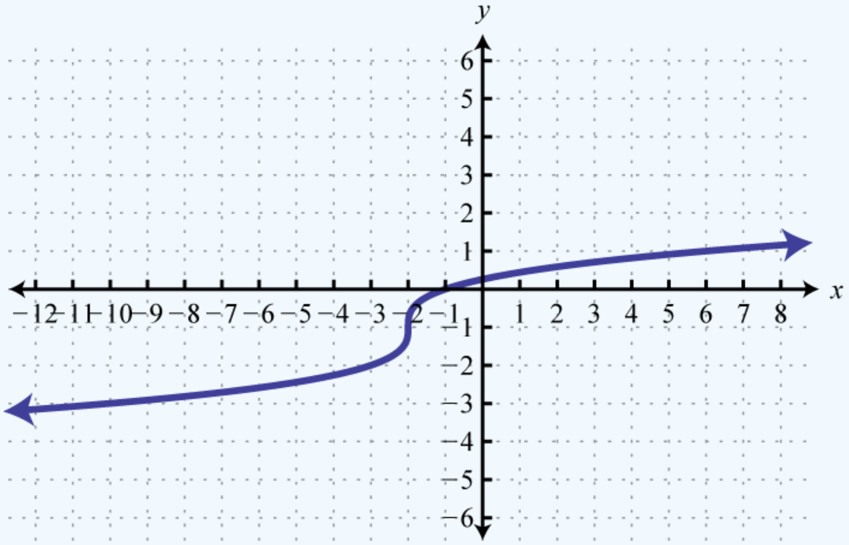


Figure 5.1.8

11. Domain: ℝ; range; ℝ

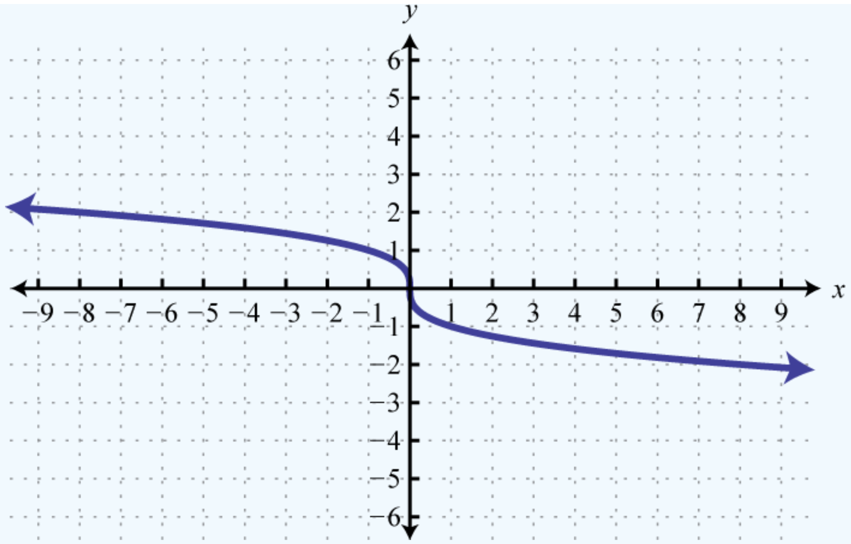


Figure 5.1.9

Exercise 5.1.8

Simplify.

1. $\sqrt[4]{64}$
2. $\sqrt[4]{16}$
3. $\sqrt[4]{625}$
4. $\sqrt[4]{1}$
5. $\sqrt[4]{256}$
6. $\sqrt[4]{10,000}$
7. $\sqrt[5]{243}$
8. $\sqrt[5]{100,000}$
9. $\sqrt[5]{\frac{1}{32}}$
10. $\sqrt[5]{\frac{1}{243}}$
11. $-\sqrt[4]{16}$
12. $-\sqrt[6]{1}$
13. $\sqrt[5]{-32}$
14. $\sqrt[5]{-1}$
15. $\sqrt{-1}$
16. $\sqrt[4]{-16}$
17. $-6\sqrt[3]{-27}$
18. $-5\sqrt[3]{-8}$
19. $2\sqrt[3]{-1,000}$
20. $7\sqrt[5]{-243}$
21. $6\sqrt[4]{-16}$
22. $12\sqrt[6]{-64}$
23. $\sqrt[3]{\frac{25}{16}}$
24. $6\sqrt{\frac{16}{9}}$
25. $5\sqrt[3]{\frac{27}{125}}$
26. $7\sqrt[5]{\frac{32}{7^5}}$
27. $-5\sqrt[3]{\frac{8}{27}}$
28. $-8\sqrt[4]{\frac{625}{16}}$
29. $2\sqrt[5]{100,000}$

30. $2\sqrt[7]{128}$

Answer

1. 4
3. 5
5. 4
7. 3
9. $\frac{1}{2}$
11. -2
13. -2
15. Not a real number
17. 18
19. -20
21. Not a real number
23. $\frac{15}{4}$
25. 3
27. $-\frac{10}{3}$
29. 20

Exercise 5.1.9

Simplify.

1. $\sqrt{96}$
2. $\sqrt{500}$
3. $\sqrt{480}$
4. $\sqrt{450}$
5. $\sqrt{320}$
6. $\sqrt{216}$
7. $5\sqrt{112}$
8. $10\sqrt{135}$
9. $-2\sqrt{240}$
10. $-3\sqrt{162}$
11. $\sqrt{\frac{150}{49}}$
12. $\sqrt{\frac{200}{9}}$
13. $\sqrt{\frac{675}{121}}$
14. $\sqrt{\frac{192}{81}}$
15. $\sqrt[3]{54}$
16. $\sqrt[3]{24}$
17. $\sqrt[3]{48}$
18. $\sqrt[3]{81}$
19. $\sqrt[3]{40}$
20. $\sqrt[3]{120}$
21. $\sqrt[3]{162}$
22. $\sqrt[3]{500}$
23. $\sqrt[3]{\frac{54}{125}}$

24. $\sqrt[3]{\frac{40}{343}}$
25. $5\sqrt[3]{-48}$
26. $2\sqrt[3]{-108}$
27. $8\sqrt[4]{96}$
28. $7\sqrt[4]{162}$
29. $\sqrt[5]{160}$
30. $\sqrt[5]{486}$
31. $\sqrt[5]{\frac{224}{243}}$
32. $\sqrt[5]{\frac{5}{32}}$
33. $\sqrt[5]{-\frac{1}{32}}$
34. $\sqrt[6]{-\frac{1}{64}}$

Answer

1. $4\sqrt{6}$
3. $4\sqrt{30}$
5. $8\sqrt{5}$
7. $20\sqrt{7}$
9. $-8\sqrt{15}$
11. $\frac{5\sqrt{6}}{7}$
13. $\frac{15\sqrt{3}}{11}$
15. $3\sqrt[3]{2}$
17. $2\sqrt[3]{6}$
19. $2\sqrt[3]{5}$
21. $3\sqrt[3]{6}$
23. $\frac{3\sqrt[3]{2}}{5}$
25. $-10\sqrt[3]{6}$
27. $16\sqrt[4]{6}$
29. $2\sqrt[5]{5}$
31. $\frac{2\sqrt[5]{7}}{3}$
33. $-\frac{1}{2}$

Exercise 5.1.10

Simplify. Give the exact answer and the approximate answer rounded to the nearest hundredth.

1. $\sqrt{60}$
2. $\sqrt{600}$
3. $\sqrt{\frac{96}{49}}$
4. $\sqrt{\frac{192}{25}}$
5. $\sqrt[3]{240}$
6. $\sqrt[3]{320}$
7. $\sqrt[3]{\frac{288}{125}}$
8. $\sqrt[3]{\frac{625}{8}}$
9. $\sqrt[4]{486}$
10. $\sqrt[5]{288}$

Answer

1. $2\sqrt{15}; 7.75$
3. $\frac{4\sqrt{6}}{7}; 1.40$
5. $2\sqrt[3]{30}; 6.21$
7. $\frac{2\sqrt[3]{36}}{5}; 1.32$
9. $3\sqrt[4]{6}; 4.70$

Exercise 5.1.11

Rewrite the following as a radical expression with coeffecient 1.

1. $2\sqrt{15}$
2. $3\sqrt{7}$
3. $5\sqrt{10}$
4. $10\sqrt{3}$
5. $2\sqrt[3]{7}$
6. $3\sqrt[3]{6}$
7. $2\sqrt[4]{5}$
8. $3\sqrt[4]{2}$
9. Each side of a square has a length that is equal to the square root of the square's area. If the area of a square is 72 square units, find the length of each of its sides.
10. Each edge of a cube has a length that is equal to the cube root of the cube's volume. If the volume of a cube is 375 cubic units, find the length of each of its edges.
11. The current I measured in amperes is given by the formula $I = \sqrt{\frac{P}{R}}$ where P is the power usage measured in watts and R is the resistance measured in ohms. If a 100 watt light bulb has 160 ohms of resistance, find the current needed. (Round to the nearest hundredth of an ampere.)
12. The time in seconds an object is in free fall is given by the formula $t = \frac{\sqrt{s}}{4}$ where s represents the distance in feet the object has fallen. How long will it take an object to fall to the ground from the top of an 8-foot stepladder? (Round to the nearest tenth of a second.)

Answer

1. $\sqrt{60}$
3. $\sqrt{250}$
5. $\sqrt[3]{56}$
7. $\sqrt[4]{80}$
9. $6\sqrt{2}$ units
11. 0.79 ampere

Exercise 5.1.12

1. Explain why there are two real square roots for any positive real number and one real cube root for any real number.
2. What is the square root of 1 and what is the cube root of 1? Explain why.
3. Explain why $\sqrt{-1}$ is not a real number and why $\sqrt[3]{-1}$ is a real number.
4. Research and discuss the methods used for calculating square roots before the common use of electronic calculators.

Answer

1. Answer may vary
3. Answer may vary

Footnotes

[1] A number that when multiplied by itself yields the original number.

[2] The positive square root of a positive real number, denoted with the symbol $\sqrt{\ }$.

[3] The expression A within a radical sign, $\sqrt[n]{A}$.

[4] The function defined by $f(x)=\sqrt{x}$.

[5] A number that when used as a factor with itself three times yields the original number, denoted with the symbol $\sqrt[3]{\ }$.

[6] The positive integer n in the notation $\sqrt[n]{\ }$ that is used to indicate an nth root.

[7] The function defined by $f(x)=\sqrt[3]{x}$.

[8] A number that when raised to the nth power $(n\geq 2)$ yields the original number.

[9] Used when referring to an expression of the form $\sqrt[n]{A}$.

[10] The positive nth root when n is even.

[11] Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$,$\sqrt[n]{A\cdot B}=\sqrt[n]{A}\cdot\sqrt[n]{B}$.

[12] Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$,$\sqrt[n]{\frac{A}{B}}=\frac{\sqrt[n]{A}}{\sqrt[n]{B}}$ where $B\neq 0$.

[13] A radical where the radicand does not consist of any factors that can be written as perfect powers of the index.

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5.2: Simplifying Radical Expressions

Learning Objectives

- Simplify radical expressions using the product and quotient rule for radicals.
- Use formulas involving radicals.

Simplifying Radical Expressions

An algebraic expression that contains radicals is called a **radical expression**[14]. We use the product and quotient rules to simplify them.

Example $5.2.1$:

Simplify: $\sqrt[3]{27x^3}$.

Solution

Use the fact that $\sqrt[n]{a^n} = a$ when n is odd.

$$\begin{aligned} \sqrt[3]{27x^3} &= \sqrt[3]{3^3 \cdot x^3} && \color{Cerulean}{Apply\: the\: product\: rule\: for\: radicals.} \\ &= \sqrt[3]{3^3} \cdot \sqrt[3]{x^3} && \color{Cerulean}{Simplify.} \\ &= 3 \cdot x \\ &= 3x \end{aligned}$$

Answer:

$3x$

Example $5.2.2$:

Simplify: $\sqrt[4]{16y^4}$.

Solution

Use the fact that $\sqrt[n]{a^n} = |a|$ when n is even.

$$\begin{aligned} \sqrt[4]{16y^4} &= \sqrt[4]{2^4y^4} && \color{Cerulean}{Apply\: the\: product\: rule\: for\: radicals.} \\ &= \sqrt[4]{2^4} \cdot \sqrt[4]{y^4} && \color{Cerulean}{Simplify.} \\ &= 2 \cdot |y| \\ &= 2|y| \end{aligned}$$

Since y is a variable, it may represent a negative number. Thus we need to ensure that the result is positive by including the absolute value.

Answer:

$2|y|$

Note

Typically, at this point in algebra we note that all variables are assumed to be positive. If this is the case, then y in the previous example is positive and the absolute value operator is not needed. The example can be simplified as follows.

$$\begin{aligned} \sqrt[4]{16y^4} &= \sqrt[4]{2^4y^4} \\ &= \sqrt[4]{2^4} \cdot \sqrt[4]{y^4} \\ &= 2y \end{aligned}$$

In this section, we will assume that all variables are positive. This allows us to focus on calculating nth roots without the technicalities associated with the principal nth root problem. For this reason, we will use the following property for the rest of the section,

$$\sqrt[n]{a^n} = a, \quad \text{if } a \geq 0 \quad \color{Cerulean}{nth\: root}$$

When simplifying radical expressions, look for factors with powers that match the index.

Example 5.2.3:

Simplify: $\sqrt{12x^6y^3}$.

Solution

Begin by determining the square factors of $12, x^6$, and y^3.

$$\left.\begin{aligned}12&=2^2\cdot 3\\x^6&=\left(x^3\right)^2\\y^3&=y^2\cdot y\end{aligned}\right\}\quad \textit{Square factors}$$

Make these substitutions, and then apply the product rule for radicals and simplify.

$$\begin{aligned}\sqrt{12x^6y^3}&=\sqrt{2^2\cdot 3\cdot\left(x^3\right)^2\cdot y^2\cdot y} && \textit{Apply the product rule for radicals.}\\&=\sqrt{2^2}\cdot\sqrt{\left(x^3\right)^2}\cdot\sqrt{y^2}\cdot\sqrt{3y} && \textit{Simplify.}\\&=2\cdot x^3\cdot y\cdot\sqrt{3y}\\&=2x^3y\sqrt{3y}\end{aligned}$$

Answer:

$2x^3y\sqrt{3y}$

Example 5.2.4:

Simplify: $\sqrt{\frac{18a^5}{b^8}}$.

Solution

Begin by determining the square factors of 18, a^5, and b^8.

$$\left.\begin{aligned}18&=2\cdot 3^2\\a^5&=a^2\cdot a^2\cdot a=\left(a^2\right)^2\cdot a\\b^8&=b^4\cdot b^4\qquad=\left(b^4\right)^2\end{aligned}\right\}\quad \textit{Square factors}$$

Make these substitutions, apply the product and quotient rules for radicals, and then simplify.

$$\begin{aligned}\sqrt{\frac{18a^5}{b^8}}&=\sqrt{\frac{2\cdot 3^2\cdot\left(a^2\right)^2\cdot a}{\left(b^4\right)^2}} && \textit{Apply the product and quotient rule for radicals.}\\&=\frac{\sqrt{3^2}\cdot\sqrt{\left(a^2\right)^2}\cdot\sqrt{2a}}{\sqrt{\left(b^4\right)^2}} && \textit{Simplify.}\\&=\frac{3a^2\sqrt{2a}}{b^4}\end{aligned}$$

Answer:

$\frac{3a^2\sqrt{2a}}{b^4}$

Example 5.2.5:

Simplify: $\sqrt[3]{80x^5y^7}$.

Solution

Begin by determining the cubic factors of $80, x^5$, and y^7.

$$\left.\begin{aligned}80&=2^4\cdot 5=2^3\cdot 2\cdot 5\\x^5&=x^3\cdot x^2\\y^7&=y^6\cdot y=\left(y^2\right)^3\cdot y\end{aligned}\right\}\quad \textit{Cubic factors}$$

Make these substitutions, and then apply the product rule for radicals and simplify.

$$\begin{aligned}
\sqrt[3]{80x^5y^7} &= \sqrt[3]{2^3 \cdot 2 \cdot 5 \cdot x^3 \cdot x^2 \cdot \left(y^2\right)^3 \cdot y} \\
&= \sqrt[3]{2^3} \cdot \sqrt[3]{x^3} \cdot \sqrt[3]{\left(y^2\right)^3} \cdot \sqrt[3]{2 \cdot 5 \cdot x^2 \cdot y} \\
&= 2 \cdot xy^2 \cdot \sqrt[3]{10x^2y} \\
&= 2xy^2\sqrt[3]{10x^2y}
\end{aligned}$$

Answer:

$2xy^2\sqrt[3]{10x^2y}$

Example 5.2.6:

Simplify: $\sqrt[3]{\frac{9x^6}{y^3z^9}}$.

Solution

The coefficient $9 = 3^2$, and thus does not have any perfect cube factors. It will be left as the only remaining radicand because all of the other factors are cubes, as illustrated below:

$$\left.\begin{aligned}
x^6 &= \left(x^2\right)^3 \\
y^3 &= (y)^3 \\
z^9 &= \left(z^3\right)^3
\end{aligned}\right\} \quad \textit{Cubic factors}$$

Replace the variables with these equivalents, apply the product and quotient rules for radicals, and then simplify.

$$\begin{aligned}
\sqrt[3]{\frac{9x^6}{y^3z^9}} &= \sqrt[3]{\frac{9 \cdot \left(x^2\right)^3}{y^3 \cdot \left(z^3\right)^3}} \\
&= \frac{\sqrt[3]{9} \cdot \sqrt[3]{\left(x^2\right)^3}}{\sqrt[3]{y^3} \cdot \sqrt[3]{\left(z^3\right)^3}} \\
&= \frac{\sqrt[3]{9} \cdot x^2}{y \cdot z^3} \\
&= \frac{x^2\sqrt[3]{9}}{yz^3}
\end{aligned}$$

Answer:

$\frac{x^2\sqrt[3]{9}}{yz^3}$

Example 5.2.7:

Simplify: $\sqrt[4]{81a^4b^5}$.

Solution

Determine all factors that can be written as perfect powers of 4. Here, it is important to see that $b^5 = b^4 \cdot b$. Hence the factor b will be left inside the radical.

$$\begin{aligned}
\sqrt[4]{81a^4b^5} &= \sqrt[4]{3^4 \cdot a^4 \cdot b^4 \cdot b} \\
&= \sqrt[4]{3^4} \cdot \sqrt[4]{a^4} \cdot \sqrt[4]{b^4} \cdot \sqrt[4]{b} \\
&= 3 \cdot a \cdot b \cdot \sqrt[4]{b} \\
&= 3ab\sqrt[4]{b}
\end{aligned}$$

Answer:

$3ab\sqrt[4]{b}$

Example 5.2.8:

Simplify: $\sqrt[5]{-32x^3y^6z^5}$.

Solution

Notice that the variable factor x cannot be written as a power of 5 and thus will be left inside the radical. In addition, $y^6 = y^5 \cdot y$; the factor y will be left inside the radical as well.

$$\begin{aligned} \sqrt[3]{-32x^3y^6z^5} &= \sqrt[5]{(-2)^5 \cdot x^3 \cdot y^5 \cdot y \cdot z^5} \\ &= \sqrt[5]{(-2)^5} \cdot \sqrt[5]{y^5} \cdot \sqrt[5]{z^5} \cdot \sqrt[5]{x^3 \cdot y} \\ &= -2 \cdot y \cdot z \cdot \sqrt[5]{x^3 \cdot y} \\ &= -2yz\sqrt[5]{x^3y} \end{aligned}$$

Answer:

$-2yz\sqrt[5]{x^3y}$

Tip: To simplify finding an nth root, divide the powers by the index.

$$\begin{aligned} \sqrt{a^6} &= a^3, \text{ which is } a^{6 \div 2} = a^3 \\ \sqrt[3]{b^6} &= b^2, \text{ which is } b^{6 \div 3} = b^2 \\ \sqrt[6]{c^6} &= c\ , \text{which is } c^{6 \div 6} = c^1 \end{aligned}$$

If the index does not divide into the power evenly, then we can use the quotient and remainder to simplify. For example,

$$\begin{aligned} \sqrt{a^5} &= a^2 \cdot \sqrt{a}, \text{ which is } a^{5 \div 2} = a^{2r1} \\ \sqrt[3]{b^5} &= b \cdot \sqrt[3]{b^2}, \text{ which is } b^{5 \div 3} = b^{1r2} \\ \sqrt[5]{c^{14}} &= c^2 \cdot \sqrt[5]{c^4}, \text{which is } c^{14 \div 5} = c^{2r4} \end{aligned}$$

The quotient is the exponent of the factor outside of the radical, and the remainder is the exponent of the factor left inside the radical.

Exercise 5.2.1

Simplify: $\sqrt[3]{162a^7b^5c^4}$.

Answer

$3a^2bc\sqrt[3]{6ab^2c}$

www.youtube.com/v/kT9Qa1nfmlk

Formulas Involving Radicals

Formulas often consist of radical expressions. For example, the period of a pendulum, or the time it takes a pendulum to swing from one side to the other and back, depends on its length according to the following formula.

$$T = 2\pi\sqrt{\frac{L}{32}}$$

Here T represents the period in seconds and L represents the length in feet of the pendulum.

Example 5.2.9:

If the length of a pendulum measures $1\frac{1}{2}$ feet, then calculate the period rounded to the nearest tenth of a second.

Solution

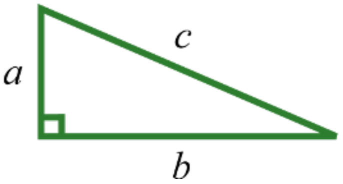
Substitute $1\frac{1}{2} = \frac{3}{2}$ for L and then simplify.

$$\begin{aligned}T&=2\pi\sqrt{\frac{L}{32}}\\&=2\pi\sqrt{\frac{\frac{3}{2}}{32}}\\&=2\pi\sqrt{\frac{3}{2}\cdot\frac{1}{32}}&&\color{blue}{Apply\ the\ quotient\ rule\ for\ radicals.}\\&=2\pi\frac{\sqrt{3}}{\sqrt{64}}&&\color{blue}{Simplify.}\\&=\frac{\pi\sqrt{3}}{4}\approx 1.36\end{aligned}$$

Answer:

The period is approximately 1.36 seconds.

Frequently you need to calculate the distance between two points in a plane. To do this, form a right triangle using the two points as vertices of the triangle and then apply the Pythagorean theorem. Recall that the Pythagorean theorem states that if given any right triangle with legs measuring a and b units, then the square of the measure of the hypotenuse c is equal to the sum of the squares of the legs: $a^2+b^2=c^2$. In other words, the hypotenuse of any right triangle is equal to the square root of the sum of the squares of its legs.



$$a^2+b^2=c^2$$
or
$$c=\sqrt{a^2+b^2}$$

Figure $5.2.1$

Example $5.2.10$:

Find the distance between $(-5,3)$ and $(1,1)$.

Solution

Form a right triangle by drawing horizontal and vertical lines though the two points. This creates a right triangle as shown below:

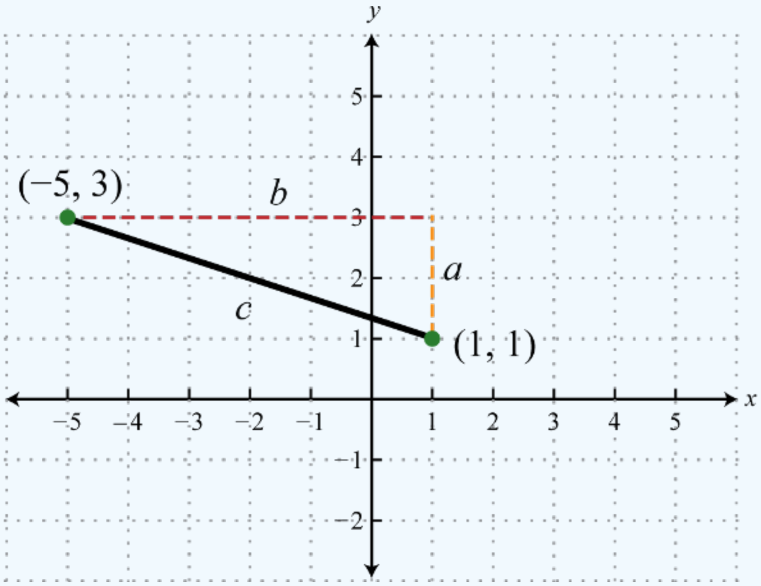


Figure $5.2.2$

The length of leg b is calculated by finding the distance between the x-values of the given points, and the length of leg a is calculated by finding the distance between the given y-values.

$$a = 3 - 1 = 2 \text{ units}$$
$$b = 1 - (-5) = 1 + 5 = 6 \text{ units}$$

Next, use the Pythagorean theorem to find the length of the hypotenuse.

$$\begin{aligned} c &= \sqrt{2^2 + 6^2} \\ &= \sqrt{4 + 36} \\ &= \sqrt{40} \\ &= \sqrt{4 \cdot 10} \\ &= 2\sqrt{10} \text{ units} \end{aligned}$$

Answer:

The distance between the two points is $2\sqrt{10}$ units.

Generalize this process to produce a formula that can be used to algebraically calculate the distance between any two given points.

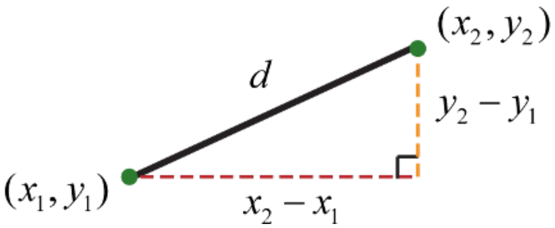


Figure 5.2.3

Given two points, (x_1, y_1) and (x_2, y_2) the distance, \(d|), between them is given by the **distance formula**[15], $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Example 5.2.11:

Calculate the distance between $(-4, 7)$ and $(2, 1)$.

Solution

Use the distance formula with the following points.

$$\begin{array}{cc} (x_1, y_1) & (x_2, y_2) \\ (-4,7) & (2,1) \end{array}$$

It is a good practice to include the formula in its general form before substituting values for the variables; this improves readability and reduces the probability of making errors.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(2-(-4))^2 + (1-7)^2} \\ &= \sqrt{(2 + 4)^2 + (1 - 7)^2} \\ &= \sqrt{(6)^2 + (-6)^2} \\ &= \sqrt{72} \\ &= \sqrt{36 \cdot 2} \\ &= 6\sqrt{2} \end{aligned}$$

Answer:

The distance between the two points is $6\sqrt{2}$ units.

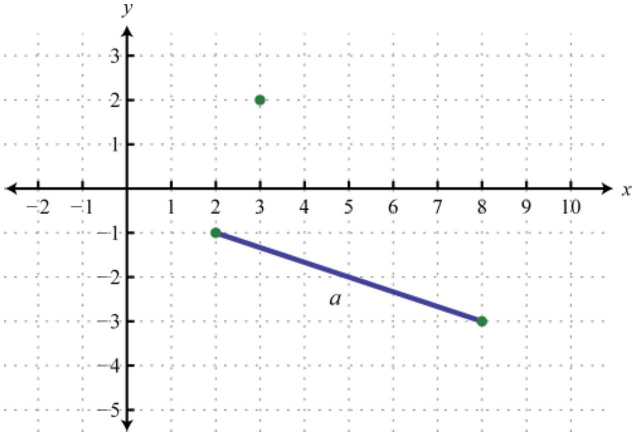
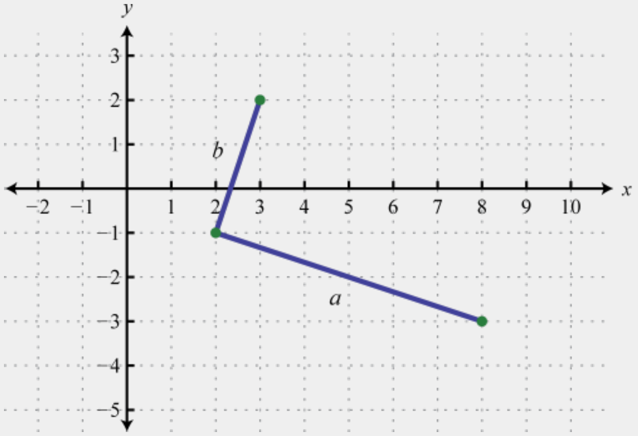
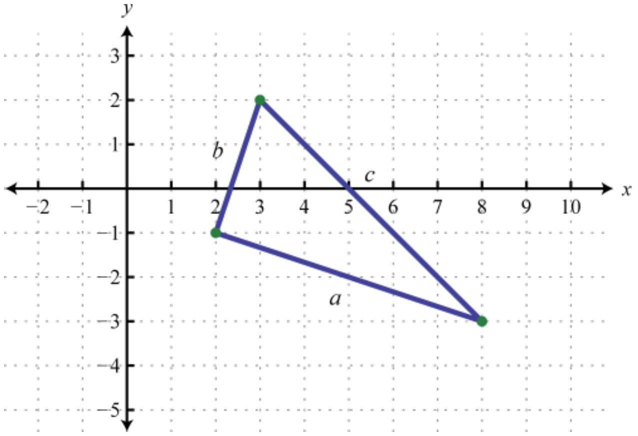
Example 5.2.12:

Do the three points $(2, -1)$, $(3, 2)$, and $(8, -3)$ form a right triangle?

Solution

The Pythagorean theorem states that having side lengths that satisfy the property $a^2+b^2=c^2$ is a necessary and sufficient condition of right triangles. In other words, if you can show that the sum of the squares of the leg lengths of the triangle is equal to the square of the length of the hypotenuse, then the triangle must be a right triangle. First, calculate the length of each side using the distance formula.

Table 5.2.1

Geometry	Calculation
 ***Figure*** 5.2.4	Points: $(2,-1)$ and $(8,-3)$ $a=\sqrt{(8-2)^2+[-3-(-1)]^2}$ $=\sqrt{(6)^2+(-3+1)^2}$ $=\sqrt{36+(-2)^2}$ $=\sqrt{36+4}$ $=\sqrt{40}$ $=2\sqrt{10}$
 Figure 5.2.5	Points: $(2,-1)$ and $(3,2)$ $b=\sqrt{(3-2)^2+[2-(-1)]^2}$ $=\sqrt{(1)^2+(2+1)^2}$ $=\sqrt{1+(3)^2}$ $=\sqrt{1+9}$ $=\sqrt{10}$
 ***Figure*** 5.2.6	Points: $(3,2)$ and $(8,-3)$ $c=\sqrt{(8-3)^2+(-3-2)^2}$ $=\sqrt{(5)^2+(-5)^2}$ $=\sqrt{25+25}$ $=\sqrt{50}$ $=5\sqrt{2}$

Now we check to see if $a^2+b^2=c^2$.

$$\begin{aligned} a^2+b^2 &= c^2 \\ (2\sqrt{10})^2+(\sqrt{10})^2 &= (5\sqrt{2})^2 \\ 4(\sqrt{10})^2+(\sqrt{10})^2 &= 25(\sqrt{2})^2 \\ 4\cdot 10+10 &= 25\cdot 2 \\ 50 &= 50\ \checkmark \end{aligned}$$

Answer:

Yes, the three points form a right triangle.

Exercise 5.2.2

The speed of a vehicle before the brakes were applied can be estimated by the length of the skid marks left on the road. On wet concrete, the speed v in miles per hour can be estimated by the formula $v=2\sqrt{3d}$, where d represents the length of the skid marks in feet. Estimate the speed of a vehicle before applying the brakes if the skid marks left behind measure 27 feet. Round to the nearest mile per hour.

Answer

18 miles per hour

www.youtube.com/v/8IscYu3YWqw

Key Takeaways

- To simplify a radical expression, look for factors of the radicand with powers that match the index. If found, they can be simplified by applying the product and quotient rules for radicals, as well as the property $\sqrt[n]{a^n}=a$, where a is nonnegative.
- A radical expression is simplified if its radicand does not contain any factors that can be written as perfect powers of the index.
- We typically assume that all variable expressions within the radical are nonnegative. This allows us to focus on simplifying radicals without the technical issues associated with the principal nth root. If this assumption is not made, we will ensure a positive result by using absolute values when simplifying radicals with even indices.

Exercise 5.2.3

Assume that the variable could represent any real number and then simplify.

1. $\sqrt{9x^2}$
2. $\sqrt{16y^2}$
3. $\sqrt[3]{8y^3}$
4. $\sqrt[3]{125a^3}$
5. $\sqrt[4]{64x^4}$
6. $\sqrt[4]{81y^4}$
7. $\sqrt{36a^4}$
8. $\sqrt{100a^8}$
9. $\sqrt{4a^6}$
10. $\sqrt{a^{10}}$
11. $\sqrt{18a^4b^5}$
12. $\sqrt{48a^5b^3}$
13. $\sqrt[6]{128x^6y^8}$
14. $\sqrt[6]{a^6b^7c^8}$
15. $\sqrt{(5x-4)^2}$
16. $\sqrt{(3x-5)^4}$
17. $\sqrt{x^2-6x+9}$
18. $\sqrt{x^2-10x+25}$
19. $\sqrt{4x^2+12x+9}$
20. $\sqrt{9x^2+6x+1}$

Answer

1. $3|x|$
3. $2y$
5. $2|x|$
7. $6a^2$
9. $2\left|a^3\right|$
11. $3a^2b^2\sqrt{2b}$
13. $2|xy|\sqrt[6]{2y^2}$
15. $|5x-4|$
17. $|x-3|$
19. $|2x+3|$

Exercise 5.2.4

Simplify. (Assume all variable expressions represent positive numbers.)

1. $\sqrt{49a^2}$
2. $\sqrt{64b^2}$
3. $\sqrt{x^2y^2}$
4. $\sqrt{25x^2y^2z^2}$
5. $\sqrt{180x^3}$
6. $\sqrt{150y^3}$
7. $\sqrt{49a^3b^2}$
8. $\sqrt{4a^4b^3c}$
9. $\sqrt{45x^5y^3}$
10. $\sqrt{50x^6y^4}$
11. $\sqrt{64r^2s^6t^5}$
12. $\sqrt{144r^8s^6t^2}$
13. $\sqrt{(x+1)^2}$
14. $\sqrt{(2x+3)^2}$
15. $\sqrt{4(3x-1)^2}$
16. $\sqrt{9(2x+3)^2}$
17. $\sqrt{\frac{9x^3}{25y^2}}$
18. $\sqrt{\frac{4x^5}{9y^4}}$
19. $\sqrt{\frac{m^7}{36n^4}}$
20. $\sqrt{\frac{147m^9}{n^6}}$
21. $\sqrt{\frac{2r^2s^5}{25t^4}}$
22. $\sqrt{\frac{36r^5}{s^2t^6}}$
23. $\sqrt[3]{27a^3}$
24. $\sqrt[3]{125b^3}$
25. $\sqrt[3]{250x^4y^3}$
26. $\sqrt[3]{162a^3b^5}$
27. $\sqrt[3]{64x^3y^6z^9}$
28. $\sqrt[3]{216x^{12}y^3}$
29. $\sqrt[3]{8x^3y^4}$
30. $\sqrt[3]{27x^5y^3}$
31. $\sqrt[3]{a^4b^5c^6}$
32. $\sqrt[3]{a^7b^5c^3}$
33. $\sqrt[3]{\frac{8x^4}{27y^3}}$

34. $\sqrt[3]{\frac{x^5}{125y^6}}$
35. $\sqrt[3]{360r^5s^{12}t^{13}}$
36. $\sqrt[3]{540r^3s^2t^9}$
37. $\sqrt[4]{81x^4}$
38. $\sqrt[4]{x^4y^4}$
39. $\sqrt[4]{16x^4y^8}$
40. $\sqrt[4]{81x^{12}y^4}$
41. $\sqrt[4]{a^4b^5c^6}$
42. $\sqrt[4]{5^4a^6c^8}$
43. $\sqrt[4]{128x^6}$
44. $\sqrt[4]{243y^7}$
45. $\sqrt[5]{\frac{32m^{10}}{n^5}}$
46. $\sqrt[5]{\frac{3^7m^9}{n^{10}}}$
47. $-3\sqrt{4x^2}$
48. $7\sqrt{9y^2}$
49. $-5x\sqrt{4x^2y}$
50. $-3y\sqrt{16x^3y^2}$
51. $12ab\sqrt{a^5b^3}$
52. $6a^2b\sqrt{9a^7b^2}$
53. $2x\sqrt[3]{8x^6}$
54. $-5x^2\sqrt[3]{27x^3}$
55. $2ab\sqrt[3]{-8a^4b^5}$
56. $5a^2b\sqrt[3]{-27a^3b^3}$

Answer

1. $7a$

3. xy

5. $6x\sqrt{5x}$

7. $7ab\sqrt{a}$

9. $3x^2y\sqrt{5xy}$

11. $8rs^3t^2\sqrt{t}$

13. $x+1$

15. $2(3x-1)$

17. $\frac{3x\sqrt{x}}{5y}$

19. $\frac{m^3\sqrt{m}}{6n^2}$

21. $\frac{rs^2\sqrt{2s}}{5t^2}$

23. $3a$

25. $5xy\sqrt[3]{2x}$

27. $4xy^2z^3$

29. $2xy\sqrt[3]{y}$

31. $abc^2\sqrt[3]{ab^2}$

33. $\frac{2x\sqrt[3]{x}}{3y}$

35. $2rs^4t^4\sqrt[3]{45r^2t}$

37. $3x$

39. $2xy^2$

41. $abc\sqrt[4]{bc^2}$

43. $2x\sqrt[4]{8x^2}$

45. $\frac{2m^2}{n}$

47. $-6x$

49. $-10x^2\sqrt{y}$

51. $12a^3b^2\sqrt{ab}$

53. $4x^3$

55. $-4a^2b^2\sqrt[3]{ab^2}$

Exercise 5.2.5

Rewrite the following as a radical expression with coefficient 1.

1. $3x\sqrt{6x}$
2. $5y\sqrt{5y}$
3. $ab\sqrt{10a}$
4. $2ab^2\sqrt{a}$
5. $m^2n\sqrt{mn}$
6. $2m^2n^3\sqrt{3n}$
7. $2x\sqrt[3]{3x}$
8. $3y\sqrt[3]{y^2}$
9. $2y^2\sqrt[4]{4y}$
10. $x^2y\sqrt[5]{9xy^2}$

Answer

1. $\sqrt{54x^3}$

3. $\sqrt{10a^3b^2}$

5. $\sqrt{m^5n^3}$

7. $\sqrt[3]{24x^4}$

9. $\sqrt[4]{64y^9}$

Exercise 5.2.6

The period T in seconds of a pendulum is given by the formula

$$T = 2\pi\sqrt{\frac{L}{32}}$$

where L represents the length in feet of the pendulum. Calculate the period, given each of the following lengths. Give the exact value and the approximate value rounded to the nearest tenth of a second.

1. 8 feet
2. 32 feet
3. $\frac{1}{2}$ foot
4. $\frac{1}{8}$ foot

Answer

1. π seconds; 3.1 seconds

3. $\frac{\pi}{4}$ seconds; 0.8 seconds

Exercise 5.2.7

The time t in seconds an object is in free fall is given by the formula

$$t = \frac{\sqrt{s}}{4}$$

where s represents the distance in feet the object has fallen. Calculate the time it takes an object to fall, given each of the following distances. Give the exact value and the approximate value rounded to the nearest tenth of a second.

1. 48 feet
2. 80 feet
3. 192 feet
4. 288 feet
5. The speed of a vehicle before the brakes were applied can be estimated by the length of the skid marks left on the road. On dry pavement, the speed v in miles per hour can be estimated by the formula $v = 2\sqrt{6d}$, where d represents the length of the skid marks in feet. Estimate the speed of a vehicle before applying the brakes on dry pavement if the skid marks left behind measure 27 feet. Round to the nearest mile per hour.
6. The radius r of a sphere can be calculated using the formula $r = \frac{\sqrt[3]{6\pi^2 V}}{2\pi}$, where V represents the sphere's volume. What is the radius of a sphere if the volume is 36π cubic centimeters?

Answer

1. $\sqrt{3}$ seconds; 1.7 seconds

3. $2\sqrt{3}$ seconds; 3.5 seconds

5. 25 miles per hour

Exercise 5.2.8

Given the function find the y-intercept

1. $f(x) = \sqrt{x+12}$
2. $f(x) = \sqrt{x+8} - 3$
3. $f(x) = \sqrt[3]{x-8}$
4. $f(x) = \sqrt[3]{x+27}$
5. $f(x) = \sqrt[3]{x+16}$
6. $f(x) = \sqrt[3]{x+3} - 1$

Answer

1. $(0, 2\sqrt{3})$

3. $(0, -2)$

5. $(0, 2\sqrt[3]{2})$

Exercise 5.2.9

Use the distance formula to calculate the distance between the given two points.

1. $(5, -7)$ and $(3, -8)$
2. $(-9, 7)$ and $(-8, 4)$
3. $(-3, -4)$ and $(3, -6)$
4. $(-5, -2)$ and $(1, -6)$
5. $(-1, 1)$ and $(-4, 10)$
6. $(8, -3)$ and $(2, -12)$
7. $(0, -6)$ and $(-3, 0)$
8. $(0, 0)$ and $(8, -4)$
9. $(\frac{1}{2}, -\frac{1}{2})$ and $(-1, \frac{3}{2})$
10. $(-\frac{1}{3}, 2)$ and $(\frac{5}{3}, -\frac{2}{3})$

Answer

1. $\sqrt{5}$ units
3. $2\sqrt{10}$ units
5. $3\sqrt{10}$ units
7. $3\sqrt{5}$ units
9. $\frac{5}{2}$ units

Exercise 5.2.10

Determine whether or not the three points form a right triangle. Use the Pythagorean theorem to justify your answer.

1. $(2,-1),(-1,2),\text{ and }(6,3)$
2. $(-5,2),(-1,-2),\text{ and }(-2,5)$
3. $(-5,0),(0,3),\text{ and }(6,-1)$
4. $(-4,-1),(-2,5),\text{ and }(7,2)$
5. $(1,-2),(2,3),\text{ and }(-3,4)$
6. $(-2,1),(-1,-1),\text{ and }(1,3)$
7. $(-4,0),(-2,-10),\text{ and }(3,-9)$
8. $(0,0),(2,4),\text{ and }(-2,6)$

Answer

1. Right triangle
3. Not a right triangle
5. Right triangle
7. Right triangle

Exercise 5.2.11

1. Give a value for x such that $\sqrt{x^2} \neq x$. Explain why it is important to assume that the variables represent nonnegative numbers.
2. Research and discuss the accomplishments of Christoph Rudolff. What is he credited for?
3. What is a surd, and where does the word come from?
4. Research ways in which police investigators can determine the speed of a vehicle after an accident has occurred. Share your findings on the discussion board.

Answer

1. Answer may vary
3. Answer may vary

Footnotes

[14] An algebraic expression that contains radicals.

[15] Given two points (x_1,y_1) and (x_2,y_2), calculate the distance d between them using the formula $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

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5.3: Adding and Subtracting Radical Expressions

Learning Objectives

- Add and subtract like radicals.
- Simplify radical expressions involving like radicals.

Adding and Subtracting Like Radicals

Adding and subtracting radical expressions is similar to adding and subtracting like terms. Radicals are considered to be **like radicals**[16], or **similar radicals**[17], when they share the same index and radicand. For example, the terms $2\sqrt{6}$ and $5\sqrt{6}$ contain like radicals and can be added using the distributive property as follows:

$$\begin{aligned} 2\sqrt{6}+5\sqrt{6} &= (2+5)\sqrt{6} \\ &= 7\sqrt{6} \end{aligned}$$

Typically, we do not show the step involving the distributive property and simply write,

$$2\sqrt{6}+5\sqrt{6}=7\sqrt{6}$$

When adding terms with like radicals, add only the coefficients; the radical part remains the same.

Example 5.3.1:

Add: $7\sqrt[3]{5}+3\sqrt[3]{5}$.

Solution

The terms are like radicals; therefore, add the coefficients.

$$7\sqrt[3]{5}+3\sqrt[3]{5}=10\sqrt[3]{5}$$

Answer:

$10\sqrt[3]{5}$

Subtraction is performed in a similar manner.

Example 5.3.2:

Subtract: $4\sqrt{10}-5\sqrt{10}$.

Solution

$$\begin{aligned} 4\sqrt{10}-5\sqrt{10} &= (4-5)\sqrt{10} \\ &= -1\sqrt{10} \\ &= -\sqrt{10} \end{aligned}$$

Answer:

$-\sqrt{10}$

If the radicand and the index are not exactly the same, then the radicals are not similar and we cannot combine them.

Example 5.3.3:

Simplify: $10\sqrt{5}+6\sqrt{2}-9\sqrt{5}-7\sqrt{2}$.

Solution

$$\begin{aligned} 10\sqrt{5}+6\sqrt{2}-9\sqrt{5}-7\sqrt{2} &= 10\sqrt{5}-9\sqrt{5}+6\sqrt{2}-7\sqrt{2} \\ &= \sqrt{5}-\sqrt{2} \end{aligned}$$

We cannot simplify any further because $\sqrt{5}$ and $\sqrt{2}$ are not like radicals; the radicands are not the same.

Answer:

$\sqrt{5}-\sqrt{2}$

Caution: It is important to point out that $\sqrt{5}-\sqrt{2}\neq\sqrt{5-2}$. We can verify this by calculating the value of each side with a calculator.

$$\sqrt{5}-\sqrt{2}\approx 0.82$$
$$\sqrt{5-2}=\sqrt{3}\approx 1.73$$

In general, note that $\sqrt[n]{a}\pm\sqrt[n]{b}\neq\sqrt[n]{a\pm b}$.

Example 5.3.4:

Simplify: $5\sqrt[3]{10}+3\sqrt{10}-\sqrt[3]{10}-2\sqrt{10}$.

Solution

$$\begin{aligned}5\sqrt[3]{10}+3\sqrt{10}-\sqrt[3]{10}-2\sqrt{10}&=5\sqrt[3]{10}-\sqrt[3]{10}+3\sqrt{10}-2\sqrt{10}\\&=4\sqrt[3]{10}+\sqrt{10}\end{aligned}$$

We cannot simplify any further, because $\sqrt[3]{10}$ and $\sqrt{10}$ are not like radicals; the indices are not the same.

Answer:

$4\sqrt[3]{10}+\sqrt{10}$

Adding and Subtracting Radical Expressions

Often, we will have to simplify before we can identify the like radicals within the terms.

Example 5.3.5:

Subtract: $\sqrt{32}-\sqrt{18}+\sqrt{50}$.

Solution

At first glance, the radicals do not appear to be similar. However, after simplifying completely, we will see that we can combine them.

$$\begin{aligned}\sqrt{32}-\sqrt{18}+\sqrt{50}&=\sqrt{16\cdot 2}-\sqrt{9\cdot 2}+\sqrt{25\cdot 2}\\&=4\sqrt{2}-3\sqrt{2}+5\sqrt{2}\\&=6\sqrt{2}\end{aligned}$$

Answer:

$6\sqrt{2}$

Example 5.3.6:

Simplify: $\sqrt[3]{108}+\sqrt[3]{24}-\sqrt[3]{32}-\sqrt[3]{81}$.

Solution:

Begin by looking for perfect cube factors of each radicand.

$$\begin{aligned}\sqrt[3]{108}+\sqrt[3]{24}-\sqrt[3]{32}-\sqrt[3]{81}&=\sqrt[3]{27\cdot 4}+\sqrt[3]{8\cdot 3}-\sqrt[3]{8\cdot 4}-\sqrt[3]{27\cdot 3} && \textit{Simplify.}\\&=3\sqrt[3]{4}+2\sqrt[3]{3}-2\sqrt[3]{4}-3\sqrt[3]{3} && \textit{Combine like terms.}\\&=\sqrt[3]{4}-\sqrt[3]{3}\end{aligned}$$

Answer:

$\sqrt[3]{4}-\sqrt[3]{3}$

Exercise 5.3.1

Simplify: $\sqrt{20}+\sqrt{27}-3\sqrt{5}-2\sqrt{12}$.

Answer

$-\sqrt{5}-\sqrt{3}$

www.youtube.com/v/QtaPdVybP1g

Next, we work with radical expressions involving variables. In this section, assume all radicands containing variable expressions are nonnegative.

Example 5.3.7:

Simplify: $-9\sqrt[3]{5x}-\sqrt[3]{2x}+10\sqrt[3]{5x}$.

Solution

Combine like radicals.

$$\begin{aligned}-9\sqrt[3]{5x}-\sqrt[3]{2x}+10\sqrt[3]{5x} &= -9\sqrt[3]{5x}+10\sqrt[3]{5x}-\sqrt[3]{2x}\\ &= \sqrt[3]{5x}-\sqrt[3]{2x}\end{aligned}$$

We cannot combine any further because the remaining radical expressions do not share the same radicand; they are not like radicals. Note: $\sqrt[3]{5x}-\sqrt[3]{2x}\neq\sqrt[3]{5x-2x}$.

Answer:

$\sqrt[3]{5x}-\sqrt[3]{2x}$

We will often find the need to subtract a radical expression with multiple terms. If this is the case, remember to apply the distributive property before combining like terms.

Example 5.3.8:

Simplify: $(5\sqrt{x}-4\sqrt{y})-(4\sqrt{x}-7\sqrt{y})$.

Solution

$$\begin{aligned}(5\sqrt{x}-4\sqrt{y})-(4\sqrt{x}-7\sqrt{y}) &= 5\sqrt{x}-4\sqrt{y}-4\sqrt{x}+7\sqrt{y} \quad \textit{Distribute.}\\ &= 5\sqrt{x}-4\sqrt{x}-4\sqrt{y}+7\sqrt{y}\\ &= \sqrt{x}+3\sqrt{y}\end{aligned}$$

Answer:

$\sqrt{x}+3\sqrt{y}$

Until we simplify, it is often unclear which terms involving radicals are similar. The general steps for simplifying radical expressions are outlined in the following example.

Example 5.3.9:

Simplify: $5\sqrt[3]{3x^4}+\sqrt[3]{24x^3}-\left(x\sqrt[3]{24x}+4\sqrt[3]{3x^3}\right)$.

Solution

Step 1: Simplify the radical expression. In this case, distribute and then simplify each term that involves a radical.

$$\begin{aligned}&5\sqrt[3]{3x^4}+\sqrt[3]{24x^3}-\left(x\sqrt[3]{24x}+4\sqrt[3]{3x^3}\right)\\ &=5\sqrt[3]{3x^4}+\sqrt[3]{24x^3}-x\sqrt[3]{24x}-4\sqrt[3]{3x^3}\\ &=5\sqrt[3]{3\cdot x\cdot x^3}+\sqrt[3]{8\cdot 3\cdot x^3}-x\sqrt[3]{8\cdot 3x}-4\sqrt[3]{3x^3}\\ &=5x\sqrt[3]{3x}+2x\sqrt[3]{3}-2x\sqrt[3]{3x}-4x\sqrt[3]{3}\end{aligned}$$

Step 2: Combine all like radicals. Remember to add only the coefficients; the variable parts remain the same.

$$\begin{aligned}&= 5x\sqrt[3]{3x}+2x\sqrt[3]{3}-2x\sqrt[3]{3x}-4x\sqrt[3]{3}\\&= 3x\sqrt[3]{3x}-2x\sqrt[3]{3}\end{aligned}$$

Answer: $3x\sqrt[3]{3x}-2x\sqrt[3]{3}$

Example 5.3.10:

Simplify: $2a\sqrt{125a^2b}-a^2\sqrt{80b}+4\sqrt{20a^4b}$.

Solution

$$\begin{aligned}&2a\sqrt{125a^2b}-a^2\sqrt{80b}+4\sqrt{20a^4b}\\&=2a\sqrt{25\cdot 5\cdot a^2\cdot b}-a^2\sqrt{16\cdot 5\cdot b}+4\sqrt{4\cdot 5\cdot (a^2)^2\, b} && \textit{Factor.}\\&=2a\cdot 5\cdot a\sqrt{5b}-a^2\cdot 4\sqrt{5b}+4\cdot 2\cdot a^2\sqrt{5b} && \textit{Simplify.}\\&=10a^2\sqrt{5b}-4a^2\sqrt{5b}+8a^2\sqrt{5b} && \textit{Combine like terms.}\\&=14a^2\sqrt{5b}\end{aligned}$$

Answer:

$14a^2\sqrt{5b}$

Exercise 5.3.2

$\sqrt[3]{2x^6y}+\sqrt[3]{xy^3}-\left(y\sqrt[3]{27x}-2x\sqrt[3]{2x^3y}\right)$

Answer

$3x^2\sqrt[3]{2y}-2y\sqrt[3]{x}$

www.youtube.com/v/oYeyj7ubc-Q

note

Take careful note of the differences between products and sums within a radical. Assume both x and y are nonnegative.

$$\begin{array}{cc}\textit{Products} & \textit{Sums}\\ \hline \sqrt{x^2y^2}=xy & \sqrt{x^2+y^2}\neq x+y\\ \sqrt[3]{x^3y^3}=xy & \sqrt[3]{x^3+y^3}\neq x+y\end{array}$$

The property $\sqrt[n]{a\cdot b}=\sqrt[n]{a}\cdot\sqrt[n]{b}$ says that we can simplify radicals when the operation in the radicand is multiplication. There is no corresponding property for addition.

Example 5.3.11:

Calculate the perimeter of the triangle formed by the points $(-2,-1)$, $(-3,6)$, and $(2,1)$.

Solution

The formula for the perimeter of a triangle is $P=a+b+c$ where a, b, and c represent the lengths of each side. Plotting the points we have,

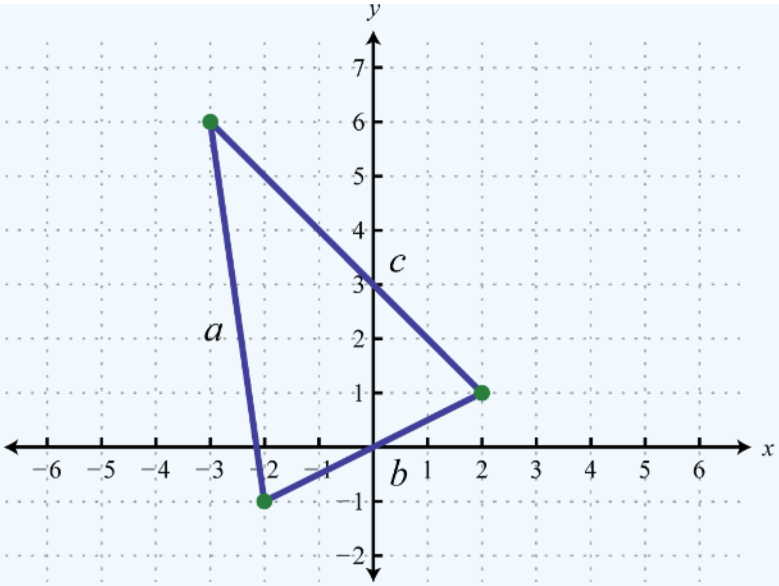

Figure 5.3.1

Use the distance formula to calculate the length of each side.

$$\begin{aligned} a &= \sqrt{[-3-(-2)]^2+[6-(-1)]^2} \\ &= \sqrt{(-3+2)^2+(6+1)^2} \\ &= \sqrt{(-1)^2+(7)^2} \\ &= \sqrt{1+49} \\ &= \sqrt{50} \\ &= 5\sqrt{2} \end{aligned} \qquad \begin{aligned} b &= \sqrt{[2-(-2)]^2+[1-(-1)]^2} \\ &= \sqrt{(2+2)^2+(1+1)^2} \\ &= \sqrt{(4)^2+(2)^2} \\ &= \sqrt{16+4} \\ &= \sqrt{20} \\ &= 2\sqrt{5} \end{aligned}$$

Similarly we can calculate the distance between $(-3, 6)$ and $(2, 1)$ and find that $c = 5\sqrt{2}$ units. Therefore, we can calculate the perimeter as follows:

$$\begin{aligned} P &= a+b+c \\ &= 5\sqrt{2}+2\sqrt{5}+5\sqrt{2} \\ &= 10\sqrt{2}+2\sqrt{5} \end{aligned}$$

Answer:

$10\sqrt{2}+2\sqrt{5}$ units

Key Takeaways

- Add and subtract terms that contain like radicals just as you do like terms. If the index and radicand are exactly the same, then the radicals are similar and can be combined. This involves adding or subtracting only the coefficients; the radical part remains the same.
- Simplify each radical completely before combining like terms.

Exercise 5.3.3

Simplify

1. $10\sqrt{3}-5\sqrt{3}$
2. $15\sqrt{6}-8\sqrt{6}$
3. $9\sqrt{3}+5\sqrt{3}$
4. $12\sqrt{6}+3\sqrt{6}$
5. $4\sqrt{5}-7\sqrt{5}-2\sqrt{5}$
6. $3\sqrt{10}-8\sqrt{10}-2\sqrt{10}$
7. $\sqrt{6}-4\sqrt{6}+2\sqrt{6}$

8. $5\sqrt{10}-15\sqrt{10}-2\sqrt{10}$
9. $13\sqrt{7}-6\sqrt{2}-5\sqrt{7}+5\sqrt{2}$
10. $10\sqrt{13}-12\sqrt{15}+5\sqrt{13}-18\sqrt{15}$
11. $6\sqrt{5}-(4\sqrt{3}-3\sqrt{5})$
12. $-12\sqrt{2}-(6\sqrt{6}+\sqrt{2})$
13. $(2\sqrt{5}-3\sqrt{10})-(\sqrt{10}+3\sqrt{5})$
14. $(-8\sqrt{3}+6\sqrt{15})-(\sqrt{3}-\sqrt{15})$
15. $4\sqrt[3]{6}-3\sqrt[3]{5}+6\sqrt[3]{6}$
16. $\sqrt[3]{10}+5\sqrt[3]{10}-4\sqrt[3]{10}$
17. $(7\sqrt[3]{9}-4\sqrt[3]{3})-(\sqrt[3]{9}-3\sqrt[3]{3})$
18. $(-8\sqrt[3]{5}+\sqrt[3]{25})-(2\sqrt[3]{5}+6\sqrt[3]{25})$

Answer

1. $5\sqrt{3}$

3. $14\sqrt{3}$

5. $-5\sqrt{5}$

7. $-\sqrt{6}$

9. $8\sqrt{7}-\sqrt{2}$

11. $9\sqrt{5}-4\sqrt{3}$

13. $-\sqrt{5}-4\sqrt{10}$

15. $10\sqrt[3]{6}-3\sqrt[3]{5}$

17. $6\sqrt[3]{9}-\sqrt[3]{3}$

Exercise 5.3.4

Simplify. (Assume all radicands containing variable expressions are positive.)

1. $\sqrt{2x}-4\sqrt{2x}$
2. $5\sqrt{3y}-6\sqrt{3y}$
3. $9\sqrt{x}+7\sqrt{x}$
4. $-8\sqrt{y}+4\sqrt{y}$
5. $7x\sqrt{y}-3x\sqrt{y}+x\sqrt{y}$
6. $10y^2\sqrt{x}-12y^2\sqrt{x}-2y^2\sqrt{x}$
7. $2\sqrt{ab}-5\sqrt{a}+6\sqrt{ab}-10\sqrt{a}$
8. $-3x\sqrt{y}+6\sqrt{y}-4x\sqrt{y}-7\sqrt{y}$
9. $5\sqrt{xy}-(3\sqrt{xy}-7\sqrt{xy})$
10. $-8a\sqrt{b}-(2a\sqrt{b}-4\sqrt{ab})$
11. $(3\sqrt{2x}-\sqrt{3x})-(\sqrt{2x}-7\sqrt{3x})$
12. $(\sqrt{y}-4\sqrt{2y})-(\sqrt{y}-5\sqrt{2y})$
13. $5\sqrt[3]{x}-12\sqrt[3]{x}$
14. $-2\sqrt[3]{y}-3\sqrt[3]{y}$
15. $a\sqrt[5]{3b}+4a\sqrt[5]{3b}-a\sqrt[5]{3b}$
16. $-8\sqrt[4]{ab}+3\sqrt[4]{ab}-2\sqrt[4]{ab}$
17. $6\sqrt{2a}-4\sqrt[3]{2a}+7\sqrt{2a}-\sqrt[3]{2a}$
18. $4\sqrt[5]{3a}+\sqrt[3]{3a}-9\sqrt[5]{3a}+\sqrt[3]{3a}$
19. $(\sqrt[4]{4xy}-\sqrt[3]{xy})-(2\sqrt[4]{4xy}-\sqrt[3]{xy})$
20. $(5\sqrt[5]{6y}-5\sqrt{y})-(2\sqrt[6]{6y}+3\sqrt{y})$
21. $2x^2\sqrt[3]{3x}-\left(x^2\sqrt[3]{3x}-x\sqrt[3]{3x}\right)$
22. $5y^3\sqrt{6y}-\left(\sqrt{6y}-4y^3\sqrt{6y}\right)$

Answer

1. $-3\sqrt{2x}$
3. $16\sqrt{x}$
5. $5x\sqrt{y}$
7. $8\sqrt{ab}-15\sqrt{a}$
9. $9\sqrt{xy}$
11. $2\sqrt{2x}+6\sqrt{3x}$
13. $-7\sqrt[3]{x}$
15. $4a\sqrt[5]{3b}$
17. $13\sqrt{2a}-5\sqrt[3]{2a}$
19. $-\sqrt[4]{4xy}$
21. $x^2\sqrt[3]{3x}+x\sqrt[3]{3x}$

Exercise $5.3.5$

Simplify.

1. $\sqrt{75}-\sqrt{12}$
2. $\sqrt{24}-\sqrt{54}$
3. $\sqrt{32}+\sqrt{27}-\sqrt{8}$
4. $\sqrt{20}+\sqrt{48}-\sqrt{45}$
5. $\sqrt{28}-\sqrt{27}+\sqrt{63}-\sqrt{12}$
6. $\sqrt{90}+\sqrt{24}-\sqrt{40}-\sqrt{54}$
7. $\sqrt{45}-\sqrt{80}+\sqrt{245}-\sqrt{5}$
8. $\sqrt{108}+\sqrt{48}-\sqrt{75}-\sqrt{3}$
9. $4\sqrt{2}-(\sqrt{27}-\sqrt{72})$
10. $-3\sqrt{5}-(\sqrt{20}-\sqrt{50})$
11. $\sqrt[3]{16}-\sqrt[3]{54}$
12. $\sqrt[3]{81}-\sqrt[3]{24}$
13. $\sqrt[3]{135}+\sqrt[3]{40}-\sqrt[3]{5}$
14. $\sqrt[3]{108}-\sqrt[3]{32}-\sqrt[3]{4}$
15. $2\sqrt{27}-2\sqrt{12}$
16. $3\sqrt{50}-4\sqrt{32}$
17. $3\sqrt{243}-2\sqrt{18}-\sqrt{48}$
18. $6\sqrt{216}-2\sqrt{24}-2\sqrt{96}$
19. $2\sqrt{18}-3\sqrt{75}-2\sqrt{98}+4\sqrt{48}$
20. $2\sqrt{45}-\sqrt{12}+2\sqrt{20}-\sqrt{108}$
21. $(2\sqrt{363}-3\sqrt{96})-(7\sqrt{12}-2\sqrt{54})$
22. $(2\sqrt{288}+3\sqrt{360})-(2\sqrt{72}-7\sqrt{40})$
23. $3\sqrt[3]{54}+5\sqrt[3]{250}-4\sqrt[3]{16}$
24. $4\sqrt[3]{162}-2\sqrt[3]{384}-3\sqrt[3]{750}$

Answer

1. $3\sqrt{3}$
3. $2\sqrt{2}+3\sqrt{3}$
5. $5\sqrt{7}-5\sqrt{3}$
7. $5\sqrt{5}$
9. $10\sqrt{2}-3\sqrt{3}$

11. $-\sqrt[3]{2}$

13. $4\sqrt[3]{5}$

15. $2\sqrt{3}$

17. $23\sqrt{3}-6\sqrt{2}$

19. $-8\sqrt{2}+\sqrt{3}$

21. $8\sqrt{3}-6\sqrt{6}$

23. $26\sqrt[3]{2}$

Exercise 5.3.6

Simplify. (Assume all radicands containing variable expressions are positive.)

1. $\sqrt{81b}+\sqrt{4b}$
2. $\sqrt{100a}+\sqrt{a}$
3. $\sqrt{9a^2b}-\sqrt{36a^2b}$
4. $\sqrt{50a^2}-\sqrt{18a^2}$
5. $\sqrt{49x}-\sqrt{9y}+\sqrt{x}-\sqrt{4y}$
6. $\sqrt{9x}+\sqrt{64y}-\sqrt{25x}-\sqrt{y}$
7. $7\sqrt{8x}-(3\sqrt{16y}-2\sqrt{18x})$
8. $2\sqrt{64y}-(3\sqrt{32y}-\sqrt{81y})$
9. $2\sqrt{9m^2n}-5m\sqrt{9n}+\sqrt{m^2n}$
10. $4\sqrt{18n^2m}-2n\sqrt{8m}+n\sqrt{2m}$
11. $\sqrt{4x^2y}-\sqrt{9xy^2}-\sqrt{16x^2y}+\sqrt{y^2x}$
12. $\sqrt{32x^2y^2}+\sqrt{12x^2y}-\sqrt{18x^2y^2}-\sqrt{27x^2y}$
13. $\left(\sqrt{9x^2y}-\sqrt{16y}\right)-\left(\sqrt{49x^2y}-4\sqrt{y}\right)$
14. $\left(\sqrt{72x^2y^2}-\sqrt{18x^2y}\right)-\left(\sqrt{50x^2y^2}+x\sqrt{2y}\right)$
15. $\sqrt{12m^4n}-m\sqrt{75m^2n}+2\sqrt{27m^4n}$
16. $5n\sqrt{27mn^2}+2\sqrt{12mn^4}-n\sqrt{3mn^2}$
17. $2\sqrt{27a^3b}-a\sqrt{48ab}-a\sqrt{144a^3b}$
18. $2\sqrt{98a^4b}-2a\sqrt{162a^2b}+a\sqrt{200b}$
19. $\sqrt[3]{125a}-\sqrt[3]{27a}$
20. $\sqrt[3]{1000a^2}-\sqrt[3]{64a^2}$
21. $2x\sqrt[3]{54x}-2\sqrt[3]{16x^4}+5\sqrt[3]{2x^4}$
22. $x\sqrt[3]{54x^3}-\sqrt[3]{250x^6}+x^2\sqrt[3]{2}$
23. $\sqrt[4]{16y^2}+\sqrt[4]{81y^2}$
24. $\sqrt[5]{32y^4}-\sqrt[5]{y^4}$
25. $\sqrt[4]{32a^3}-\sqrt[4]{162a^3}+5\sqrt[4]{2a^3}$
26. $\sqrt[4]{80a^4b}+\sqrt[4]{5a^4b}-a\sqrt[4]{5b}$
27. $\sqrt[3]{27x^3}+\sqrt[3]{8x}-\sqrt[3]{125x^3}$
28. $\sqrt[3]{24x}-\sqrt[3]{128x}-\sqrt[3]{81x}$
29. $\sqrt[3]{27x^4y}-\sqrt[3]{8xy^3}+x\sqrt[3]{64xy}-y\sqrt[3]{x}$
30. $\sqrt[3]{125xy^3}+\sqrt[3]{8x^3y}-\sqrt[3]{216xy^3}+10x^3\sqrt{y}$
31. $\left(\sqrt[3]{162x^4y}-\sqrt[3]{250x^4y^2}\right)-\left(\sqrt[3]{2x^4y^2}-\sqrt[3]{384x^4y}\right)$
32. $\left(\sqrt[5]{32x^2y^6}-\sqrt[5]{243x^6y^2}\right)-\left(\sqrt[5]{x^2y^6}-x\sqrt[5]{xy^2}\right)$

Answer

1. $11\sqrt{b}$

3. $-3a\sqrt{b}$

5. $8\sqrt{x}-5\sqrt{y}$

7. $20\sqrt{2x}-12\sqrt{y}$

9. $-8m\sqrt{n}$

11. $-2x\sqrt{y}-2y\sqrt{x}$

13. $-4x\sqrt{y}$

15. $3m^2\sqrt{3n}$

17. $2a\sqrt{3ab}-12a^2\sqrt{ab}$

19. $2\sqrt[3]{a}$

21. $7x\sqrt[3]{2x}$

23. $5\sqrt[4]{y^2}$

25. $4\sqrt[4]{2a^3}$

27. $-2x+2\sqrt[3]{x}$

29. $7x\sqrt[3]{xy}-3y\sqrt[3]{x}$

31. $7x\sqrt[3]{6xy}-6x\sqrt[3]{2xy^2}$

Exercise $5.3.7$

Calculate the perimeters of the triangles formed by the following set of vertices.

1. $\{(-4,-5),(-4,3),(2,3)\}$
2. $\{(-1,1),(3,1),(3,-2)\}$
3. $\{(-3,1),(-3,5),(1,5)\}$
4. $\{(-3,-1),(-3,7),(1,-1)\}$
5. $\{(0,0),(2,4),(-2,6)\}$
6. $\{(-5,-2),(-3,0),(1,-6)\}$
7. A square garden that is 10 feet on each side is to be fenced in. In addition, the space is to be partitioned in half using a fence along its diagonal. How much fencing is needed to do this? (Round to the nearest tenth of a foot.)
8. A garden in the shape of a square has an area of 150 square feet. How much fencing is needed to fence it in? (Hint: The length of each side of a square is equal to the square root of the area. Round to the nearest tenth of a foot.)

Answer

1. 24 units

3. $8+4\sqrt{2}$ units

5. $4\sqrt{5}+2\sqrt{10}$ units

7. 54.1 feet

Exercise $5.3.8$

1. Choose values for x and y and use a calculator to show that $\sqrt{x+y}\neq\sqrt{x}+\sqrt{y}$.
2. Choose values for x and y and use a calculator to show that $\sqrt{x^2+y^2}\neq x+y$.

Answer

1. Answer may vary

Footnotes

[16] Radicals that share the same index and radicand.

[17] Term used when referring to like radicals.

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5.4: Multiplying and Dividing Radical Expressions

Learning Objectives

- Multiply radical expressions.
- Divide radical expressions.
- Rationalize the denominator.

Multiplying Radical Expressions

When multiplying radical expressions with the same index, we use the product rule for radicals. Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$,

$$\sqrt[n]{A} \cdot \sqrt[n]{B} = \sqrt[n]{A \cdot B}\backslash$$

Example 5.4.1:

Multiply: $\sqrt[3]{12} \cdot \sqrt[3]{6}$.

Solution:

Apply the product rule for radicals, and then simplify.

$$\begin{aligned} \sqrt[3]{12} \cdot \sqrt[3]{6} &= \sqrt[3]{12 \cdot 6} && \color{Cerulean}{Multiply\ the\ radicands.} \\ &= \sqrt[3]{72} && \color{Cerulean}{Simplify.} \\ &= \sqrt[3]{2^3 \cdot 3^2} \\ &= 2\sqrt[3]{3^2} \\ &= 2\sqrt[3]{9} \end{aligned}$$

Answer:

$2\sqrt[3]{9}$

Often, there will be coefficients in front of the radicals.

Example 5.4.2:

Multiply: $3\sqrt{6} \cdot 5\sqrt{2}$

Solution

Using the product rule for radicals and the fact that multiplication is commutative, we can multiply the coefficients and the radicands as follows.

$$\begin{aligned} 3\sqrt{6} \cdot 5\sqrt{2} &= \color{Cerulean}{3 \cdot 5} \cdot \color{OliveGreen}{\sqrt{6} \cdot \sqrt{2}} && \color{Cerulean}{Multiplication\ is\ commutative.} \\ &= 15 \cdot \sqrt{12} && \color{Cerulean}{Multiply\ the\ coefficients\ and\ the\ radicands.} \\ &= 15\sqrt{4 \cdot 3} && \color{Cerulean}{Simplify.} \\ &= 15 \cdot 2 \cdot \sqrt{3} \\ &= 30\sqrt{3} \end{aligned}$$

Typically, the first step involving the application of the commutative property is not shown.

Answer:

$30\sqrt{3}$

Example 5.4.3:

Multiply: $-3\sqrt[3]{4y^2} \cdot 5\sqrt[3]{16y}$.

Solution

$$\begin{aligned} -3\sqrt[3]{4y^2}\cdot 5\sqrt[3]{16y} &= -15\sqrt[3]{64y^3} && \textit{Multiply the coefficients and then multipy the rest.} \\ &= -15\sqrt[3]{4^3y^3} && \textit{Simplify.} \\ &= -15\cdot 4y \\ &= -60y \end{aligned}$$

Answer:

$-60y$

Use the distributive property when multiplying rational expressions with more than one term.

Example 5.4.4:

Multiply: $5\sqrt{2x}(3\sqrt{x}-\sqrt{2x})$.

Solution:

Apply the distributive property and multiply each term by $5\sqrt{2x}$.

$$\begin{aligned} 5\sqrt{2x}(3\sqrt{x}-\sqrt{2x}) &= 5\sqrt{2x}\cdot 3\sqrt{x} - 5\sqrt{2x}\cdot\sqrt{2x} && \textit{Distribute.} \\ &= 15\sqrt{2x^2} - 5\sqrt{4x^2} && \textit{Simplify.} \\ &= 15x\sqrt{2} - 5\cdot 2x \\ &= 15x\sqrt{2} - 10x \end{aligned}$$

Answer:

$15x\sqrt{2}-10x$

Example 5.4.5:

Multiply: $\sqrt[3]{6x^2y}\left(\sqrt[3]{9x^2y^2}-5\cdot\sqrt[3]{4xy}\right)$.

Solution

Apply the distributive property, and then simplify the result.

$$\begin{aligned} \sqrt[3]{6x^2y}\left(\sqrt[3]{9x^2y^2}-5\cdot\sqrt[3]{4xy}\right) &= \sqrt[3]{6x^2y}\cdot\sqrt[3]{9x^2y^2} - \sqrt[3]{6x^2y}\cdot 5\sqrt[3]{4xy} \\ &= \sqrt[3]{54x^4y^3} - 5\sqrt[3]{24x^3y^2} \\ &= \sqrt[3]{27\cdot 2\cdot x\cdot x^3\cdot y^3} - 5\sqrt[3]{8\cdot 3\cdot x^3\cdot y^2} \\ &= 3xy\sqrt[3]{2x} - 10x\sqrt[3]{3y^2} \\ &= 3xy\sqrt[3]{2x} - 10x\sqrt[3]{3y^2} \end{aligned}$$

Answer:

$3xy\sqrt[3]{2x}-10x\sqrt[3]{3y^2}$

The process for multiplying radical expressions with multiple terms is the same process used when multiplying polynomials. Apply the distributive property, simplify each radical, and then combine like terms.

Example 5.4.6:

Multiply: $(\sqrt{x}-5\sqrt{y})^2$.

Solution

$$(\sqrt{x}-5\sqrt{y})^2 = (\sqrt{x}-5\sqrt{y})(\sqrt{x}-5\sqrt{y})$$

Begin by applying the distributive property.

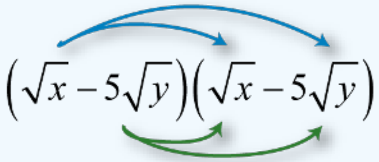

Figure 5.4.1

$$\begin{aligned} &= \sqrt{x}\cdot\sqrt{x} + \sqrt{x}(-5\sqrt{y}) + (-5\sqrt{y})\sqrt{x} + (-5\sqrt{y})(-5\sqrt{y}) \\ &= \sqrt{x^2} - 5\sqrt{xy} - 5\sqrt{xy} + 25\sqrt{y^2} \\ &= x - 10\sqrt{xy} + 25y \end{aligned}$$

Answer:

$x - 10\sqrt{xy} + 25y$

The binomials $(a+b)$ and $(a-b)$ are called **conjugates**[18]. When multiplying conjugate binomials the middle terms are opposites and their sum is zero.

Example 5.4.7:

Multiply: $(\sqrt{10}+\sqrt{3})(\sqrt{10}-\sqrt{3})$.

Solution

Apply the distributive property, and then combine like terms.

$$\begin{aligned} (\sqrt{10}+\sqrt{3})(\sqrt{10}-\sqrt{3}) &= \sqrt{10}\cdot\sqrt{10} + \sqrt{10}(-\sqrt{3}) + \sqrt{3}(\sqrt{10}) + \sqrt{3}(-\sqrt{3}) \\ &= \sqrt{100} - \sqrt{30} + \sqrt{30} - \sqrt{9} \\ &= 10 - \sqrt{30} + \sqrt{30} - 3 \\ &= 10 - 3 \\ &= 7 \end{aligned}$$

Answer:

7

It is important to note that when multiplying conjugate radical expressions, we obtain a rational expression. This is true in general

$$\begin{aligned} (\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y}) &= \sqrt{x^2} - \sqrt{xy} + \sqrt{xy} - \sqrt{y^2} \\ &= x - y \end{aligned}$$

Alternatively, using the formula for the difference of squares we have,

$$\begin{aligned} (a+b)(a-b) &= a^2 - b^2 \qquad \textit{Difference of squares.} \\ (\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y}) &= (\sqrt{x})^2 - (\sqrt{y})^2 \\ &= x - y \end{aligned}$$

Exercise 5.4.1

Multiply: $(3-2\sqrt{y})(3+2\sqrt{y})$. (Assume y is positive.)

Answer

$9 - 4y$

www.youtube.com/v/HPggvr8g68U

Dividing Radical Expressions

To divide radical expressions with the same index, we use the quotient rule for radicals. Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$,

$$\frac{\sqrt[n]{A}}{\sqrt[n]{B}} = \sqrt[n]{\frac{A}{B}}$$

Example 5.4.8:

Divide: $\frac{\sqrt[3]{96}}{\sqrt[3]{6}}$.

Solution

In this case, we can see that 6 and 96 have common factors. If we apply the quotient rule for radicals and write it as a single cube root, we will be able to reduce the fractional radicand.

$$\begin{aligned}\frac{\sqrt[3]{96}}{\sqrt[3]{6}} &= \sqrt[3]{\frac{96}{6}} \quad \textit{Apply the quotient rule for radicals and reduce the radicand.}\\ &= \sqrt[3]{16}\\ &= \sqrt[3]{8\cdot 2} \textit{Simplify.}\\ &= 2\sqrt[3]{2}\end{aligned}$$

Answer:

$2\sqrt[3]{2}$

Example 5.4.9:

Divide: $\frac{\sqrt{50x^6y^4}}{\sqrt{8x^3y}}$.

Solution

Write as a single square root and cancel common factors before simplifying.

$$\begin{aligned}\frac{\sqrt{50x^6y^4}}{\sqrt{8x^3y}} &= \sqrt{\frac{50x^6y^4}{8x^3y}} \quad \textit{Apply the quotient rule for radicals and cancel.}\\ &= \sqrt{\frac{25x^3y^3}{4}} \quad \textit{Simplify.}\\ &= \frac{\sqrt{25x^3y^3}}{\sqrt{4}}\\ &= \frac{5xy\sqrt{xy}}{2}\end{aligned}$$

Answer:

$\frac{5xy\sqrt{xy}}{2}$

Rationalizing the Denominator

When the denominator (divisor) of a radical expression contains a radical, it is a common practice to find an equivalent expression where the denominator is a rational number. Finding such an equivalent expression is called **rationalizing the denominator**[19].

$$\begin{array}{ccc}\textit{Radical expression} & & \textit{Rational denominator}\\ \frac{1}{\sqrt{2}} & = & \frac{\sqrt{2}}{2}\end{array}$$

To do this, multiply the fraction by a special form of 1 so that the radicand in the denominator can be written with a power that matches the index. After doing this, simplify and eliminate the radical in the denominator. For example:

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2}$$

Remember, to obtain an equivalent expression, you must multiply the numerator and denominator by the exact same nonzero factor.

Example 5.4.10:

Rationalize the denominator: $\frac{\sqrt{2}}{\sqrt{5x}}$.

Solution

The goal is to find an equivalent expression without a radical in the denominator. The radicand in the denominator determines the factors that you need to use to rationalize it. In this example, multiply by 1 in the form $\frac{\sqrt{5x}}{\sqrt{5x}}$.

$$\begin{aligned}\frac{\sqrt{2}}{\sqrt{5x}} &= \frac{\sqrt{2}}{\sqrt{5x}} \cdot \frac{\sqrt{5x}}{\sqrt{5x}} \quad Multiply\ by\ \frac{\sqrt{5x}}{\sqrt{5x}}.\\ &= \frac{\sqrt{10x}}{\sqrt{25x^2}} \quad Simplify.\\ &= \frac{\sqrt{10x}}{5x}\end{aligned}$$

Answer:

$\frac{\sqrt{10x}}{5x}$

Sometimes, we will find the need to reduce, or cancel, after rationalizing the denominator.

Example 5.4.11:

Rationalize the denominator: $\frac{3a\sqrt{2}}{\sqrt{6ab}}$.

Solution

In this example, we will multiply by 1 in the form $\frac{\sqrt{6ab}}{\sqrt{6ab}}$.

$$\begin{aligned}\frac{3a\sqrt{2}}{\sqrt{6ab}} &= \frac{3a\sqrt{2}}{\sqrt{6ab}} \cdot \frac{\sqrt{6ab}}{\sqrt{6ab}}\\ &= \frac{3a\sqrt{12ab}}{\sqrt{36a^2b^2}} \quad Simplify.\\ &= \frac{3a\sqrt{4 \cdot 3ab}}{6ab}\\ &= \frac{6a\sqrt{3ab}}{b} \quad Cancel.\\ &= \frac{\sqrt{3ab}}{b}\end{aligned}$$

Notice that b does not cancel in this example. Do not cancel factors inside a radical with those that are outside.

Answer:

$\frac{\sqrt{3ab}}{b}$

Exercise 5.4.2

Rationalize the denominator: $\sqrt{\frac{9x}{2y}}$.

Answer

$\frac{3\sqrt{2xy}}{2y}$

www.youtube.com/v/h-lAW8kI2rA

Up to this point, we have seen that multiplying a numerator and a denominator by a square root with the exact same radicand results in a rational denominator. In general, this is true only when the denominator contains a square root. However, this is not the case for a cube root. For example,

$$\frac{1}{\sqrt[3]{x}} \cdot \frac{\sqrt[3]{x}}{\sqrt[3]{x}} = \frac{\sqrt[3]{x}}{\sqrt[3]{x^2}}$$

Note that multiplying by the same factor in the denominator does not rationalize it. In this case, if we multiply by 1 in the form of $\frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}}$, then we can write the radicand in the denominator as a power of 3. Simplifying the result then yields a rationalized denominator.

$$\frac{1}{\sqrt[3]{x}} = \frac{1}{\sqrt[3]{x}} \cdot \frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}} = \frac{\sqrt[3]{x^2}}{\sqrt[3]{x^3}} = \frac{\sqrt[3]{x^2}}{x}$$

Therefore, to rationalize the denominator of a radical expression with one radical term in the denominator, begin by factoring the radicand of the denominator. The factors of this radicand and the index determine what we should multiply by. Multiply the numerator and denominator by the nth root of factors that produce nth powers of all the factors in the radicand of the denominator.

Example 5.4.12:

Rationalize the denominator: $\frac{\sqrt[3]{2}}{\sqrt[3]{25}}$.

Solution

The radical in the denominator is equivalent to $\sqrt[3]{5^2}$. To rationalize the denominator, we need: $\sqrt[3]{5^3}$. To obtain this, we need one more factor of 5. Therefore, multiply by 1 in the form of $\frac{\sqrt[3]{5}}{\sqrt[3]{5}}$.

$$\begin{aligned}
\frac{\sqrt[3]{2}}{\sqrt[3]{25}} &= \frac{\sqrt[3]{2}}{\sqrt[3]{5^2}} \cdot \frac{\sqrt[3]{5}}{\sqrt[3]{5}} && \textit{Multiply by the cube root of factors that result in powers of 3.} \\
&= \frac{\sqrt[3]{10}}{\sqrt[3]{5^3}} && \textit{Simplify.} \\
&= \frac{\sqrt[3]{10}}{5}
\end{aligned}$$

Answer:

$\frac{\sqrt[3]{10}}{5}$

Example 5.4.13:

Rationalize the denominator: $\sqrt[3]{\frac{27a}{2b^2}}$.

Solution

In this example, we will multiply by 1 in the form $\frac{\sqrt[3]{2^2b}}{\sqrt[3]{2^2b}}$.

$$\begin{aligned}
\sqrt[3]{\frac{27a}{2b^2}} &= \frac{\sqrt[3]{3^3a}}{\sqrt[3]{2b^2}} && \textit{Apply the quotient rule for radicals.} \\
&= \frac{3\sqrt[3]{a}}{\sqrt[3]{2b^2}} \cdot \frac{\sqrt[3]{2^2b}}{\sqrt[3]{2^2b}} && \textit{Multiply by the cube root of factors that result in powers.} \\
&= \frac{3\sqrt[3]{2^2ab}}{\sqrt[3]{2^3b^3}} && \textit{Simplify.} \\
&= \frac{3\sqrt[3]{4ab}}{2b}
\end{aligned}$$

Answer:

$\frac{3\sqrt[3]{4ab}}{2b}$

Example 5.4.14:

Rationalize the denominator: $\frac{2x\sqrt[5]{5}}{\sqrt[5]{4x^3y}}$

Solution

In this example, we will multiply by 1 in the form $\frac{\sqrt[5]{2^3x^2y^4}}{\sqrt[5]{2^3x^2y^4}}$

$$\begin{aligned}
\frac{2x\sqrt[5]{5}}{\sqrt[5]{4x^3y}} &= \frac{2x\sqrt[5]{5}}{\sqrt[5]{2^2x^3y}} \cdot \frac{\sqrt[5]{2^3x^2y^4}}{\sqrt[5]{2^3x^2y^4}} && \textit{Multiply by the fifth root of factors that result in pairs.} \\
&= \frac{2x\sqrt[5]{5 \cdot 2^3x^2y^4}}{\sqrt[5]{2^5x^5y^5}} && \textit{Simplify.} \\
&= \frac{2x\sqrt[5]{40x^2y^4}}{2xy} \\
&= \frac{\sqrt[5]{40x^2y^4}}{y}
\end{aligned}$$

Answer:

$\frac{\sqrt[5]{40x^2y^4}}{y}$

When two terms involving square roots appear in the denominator, we can rationalize it using a very special technique. This technique involves multiplying the numerator and the denominator of the fraction by the conjugate of the denominator. Recall that multiplying a radical expression by its conjugate produces a rational number.

Example 5.4.15:

Rationalize the denominator: $\frac{1}{\sqrt{5}-\sqrt{3}}$.

Solution

In this example, the conjugate of the denominator is $\sqrt{5}+\sqrt{3}$. Therefore, multiply by 1 in the form $\frac{(\sqrt{5}+\sqrt{3})}{(\sqrt{5}+\sqrt{3})}$.

$$\begin{aligned} \frac{1}{\sqrt{5}-\sqrt{3}} &= \frac{1}{(\sqrt{5}-\sqrt{3})}\frac{(\sqrt{5}+\sqrt{3})}{(\sqrt{5}+\sqrt{3})} & \text{Multiply numerator and denominator by the conjugate of the denominator.} \\ &= \frac{\sqrt{5}+\sqrt{3}}{\sqrt{25}+\sqrt{15}-\sqrt{15}-\sqrt{9}} & \text{Simplify.} \\ &= \frac{\sqrt{5}+\sqrt{3}}{5-3} \\ &= \frac{\sqrt{5}+\sqrt{3}}{2} \end{aligned}$$

Answer:

$\frac{\sqrt{5}+\sqrt{3}}{2}$

Notice that the terms involving the square root in the denominator are eliminated by multiplying by the conjugate. We can use the property $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a-b$ to expedite the process of multiplying the expressions in the denominator.

Example 5.4.16:

Rationalize the denominator: $\frac{\sqrt{10}}{\sqrt{2}+\sqrt{6}}$.

Solution

Multiply by 1 in the form $\frac{\sqrt{2}-\sqrt{6}}{\sqrt{2}-\sqrt{6}}$.

$$\begin{aligned} \frac{\sqrt{10}}{\sqrt{2}+\sqrt{6}} &= \frac{(\sqrt{10})}{(\sqrt{2}+\sqrt{6})}\frac{(\sqrt{2}-\sqrt{6})}{(\sqrt{2}-\sqrt{6})} & \text{Multiple by the conjugate.} \\ &= \frac{\sqrt{20}-\sqrt{60}}{2-6} & \text{Simplify.} \\ &= \frac{\sqrt{4\cdot 5}-\sqrt{4\cdot 15}}{-4} \\ &= \frac{2\sqrt{5}-2\sqrt{15}}{-4} \\ &= \frac{2(\sqrt{5}-\sqrt{15})}{-4} \\ &= \frac{\sqrt{5}-\sqrt{15}}{-2} = -\frac{\sqrt{5}-\sqrt{15}}{2} = \frac{-\sqrt{5}+\sqrt{15}}{2} \end{aligned}$$

Answer:

$\frac{\sqrt{15}-\sqrt{5}}{2}$

Example 5.4.17:

Rationalize the denominator: $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$.

Solution

In this example, we will multiply by 1 in the form $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}-\sqrt{y}}$.

$$\begin{aligned}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} &= \frac{(\sqrt{x}-\sqrt{y})}{(\sqrt{x}+\sqrt{y})}\frac{(\sqrt{x}-\sqrt{y})}{(\sqrt{x}-\sqrt{y})} && \textit{Multiply by the conjugate of the denominator.}\\ &= \frac{\sqrt{x^2}-\sqrt{xy}-\sqrt{xy}+\sqrt{y^2}}{x-y} && \textit{Simplify.}\\ &= \frac{x-2\sqrt{xy}+y}{x-y}\end{aligned}$$

Answer:

$\frac{x-2\sqrt{xy}+y}{x-y}$

Exercise 5.4.3

Rationalize the denominator: $\frac{2\sqrt{3}}{5-\sqrt{3}}$

Answer

$\frac{5\sqrt{3}+3}{11}$

www.youtube.com/v/gYNvQ3NVxCc

Key Takeaways

- To multiply two single-term radical expressions, multiply the coefficients and multiply the radicands. If possible, simplify the result.
- Apply the distributive property when multiplying a radical expression with multiple terms. Then simplify and combine all like radicals.
- Multiplying a two-term radical expression involving square roots by its conjugate results in a rational expression.
- It is common practice to write radical expressions without radicals in the denominator. The process of finding such an equivalent expression is called rationalizing the denominator.
- If an expression has one term in the denominator involving a radical, then rationalize it by multiplying the numerator and denominator by the nth root of factors of the radicand so that their powers equal the index.
- If a radical expression has two terms in the denominator involving square roots, then rationalize it by multiplying the numerator and denominator by the conjugate of the denominator.

Exercise 5.4.4

Multiply. (Assume all variables represent non-negative real numbers.)

1. $\sqrt{3}\cdot\sqrt{7}$
2. $\sqrt{2}\cdot\sqrt{5}$
3. $\sqrt{6}\cdot\sqrt{12}$
4. $\sqrt{10}\cdot\sqrt{15}$
5. $\sqrt{2}\cdot\sqrt{6}$
6. $\sqrt{5}\cdot\sqrt{15}$
7. $\sqrt{7}\cdot\sqrt{7}$
8. $\sqrt{12}\cdot\sqrt{12}$
9. $2\sqrt{5}\cdot 7\sqrt{10}$
10. $3\sqrt{15}\cdot 2\sqrt{6}$
11. $(2\sqrt{5})^2$
12. $(6\sqrt{2})^2$
13. $\sqrt{2x}\cdot\sqrt{2x}$
14. $\sqrt{5y}\cdot\sqrt{5y}$
15. $\sqrt{3a}\cdot\sqrt{12}$
16. $\sqrt{3a}\cdot\sqrt{2a}$
17. $4\sqrt{2x}\cdot 3\sqrt{6x}$
18. $5\sqrt{10y}\cdot 2\sqrt{2y}$
19. $\sqrt[3]{3}\cdot\sqrt[3]{9}$

20. $\sqrt[3]{4}\cdot\sqrt[3]{16}$
21. $\sqrt[3]{15}\cdot\sqrt[3]{25}$
22. $\sqrt[3]{100}\cdot\sqrt[3]{50}$
23. $\sqrt[3]{4}\cdot\sqrt[3]{10}$
24. $\sqrt[3]{18}\cdot\sqrt[3]{6}$
25. $(5\sqrt[3]{9})(2\sqrt[3]{6})$
26. $(2\sqrt[3]{4})(3\sqrt[3]{4})$
27. $(2\sqrt[3]{2})^3$
28. $(3\sqrt[3]{4})^3$
29. $\sqrt[3]{3a^2}\cdot\sqrt[3]{9a}$
30. $\sqrt[3]{7b}\cdot\sqrt[3]{49b^2}$
31. $\sqrt[3]{6x^2}\cdot\sqrt[3]{4x^2}$
32. $\sqrt[3]{12y}\cdot\sqrt[3]{9y^2}$
33. $\sqrt[3]{20x^2y}\cdot\sqrt[3]{10x^2y^2}$
34. $\sqrt[3]{63xy}\cdot\sqrt[3]{12x^4y^2}$
35. $\sqrt{5}(3-\sqrt{5})$
36. $\sqrt{2}(\sqrt{3}-\sqrt{2})$
37. $3\sqrt{7}(2\sqrt{7}-\sqrt{3})$
38. $2\sqrt{5}(6-3\sqrt{10})$
39. $\sqrt{6}(\sqrt{3}-\sqrt{2})$
40. $\sqrt{15}(\sqrt{5}+\sqrt{3})$
41. $\sqrt{x}(\sqrt{x}+\sqrt{xy})$
42. $\sqrt{y}(\sqrt{xy}+\sqrt{y})$
43. $\sqrt{2ab}(\sqrt{14a}-2\sqrt{10b})$
44. $\sqrt{6ab}(5\sqrt{2a}-\sqrt{3b})$
45. $\sqrt[3]{6}(\sqrt[3]{9}-\sqrt[3]{20})$
46. $\sqrt[3]{12}(\sqrt[3]{36}+\sqrt[3]{14})$
47. $(\sqrt{2}-\sqrt{5})(\sqrt{3}+\sqrt{7})$
48. $(\sqrt{3}+\sqrt{2})(\sqrt{5}-\sqrt{7})$
49. $(2\sqrt{3}-4)(3\sqrt{6}+1)$
50. $(5-2\sqrt{6})(7-2\sqrt{3})$
51. $(\sqrt{5}-\sqrt{3})^2$
52. $(\sqrt{7}-\sqrt{2})^2$
53. $(2\sqrt{3}+\sqrt{2})(2\sqrt{3}-\sqrt{2})$
54. $(\sqrt{2}+3\sqrt{7})(\sqrt{2}-3\sqrt{7})$
55. $(\sqrt{a}-\sqrt{2b})^2$
56. $(\sqrt{ab}+1)^2$
57. What is the perimeter and area of a rectangle with length measuring $5\sqrt{3}$ centimeters and width measuring $3\sqrt{2}$ centimeters?
58. What is the perimeter and area of a rectangle with length measuring $2\sqrt{6}$ centimeters and width measuring $\sqrt{3}$ centimeters?
59. If the base of a triangle measures $6\sqrt{2}$ meters and the height measures $3\sqrt{2}$ meters, then calculate the area.
60. If the base of a triangle measures $6\sqrt{3}$ meters and the height measures $3\sqrt{6}$ meters, then calculate the area.

Answer

1. $\sqrt{21}$

3. $6\sqrt{2}$

5. $2\sqrt{3}$

7. 7

9. $70\sqrt{2}$

11. 20

13. $2x$

15. $6\sqrt{a}$

17. $24x\sqrt{3}$

19. 3

21. $5\sqrt[3]{3}$

23. $2\sqrt[3]{5}$

25. $30\sqrt[3]{2}$

27. 16

29. $3a$

31. $2x\sqrt[3]{3x}$

33. $2xy\sqrt[3]{25x}$

35. $3\sqrt{5}-5$

37. $42-3\sqrt{21}$

39. $3\sqrt{2}-2\sqrt{3}$

41. $x+x\sqrt{y}$

43. $2a\sqrt{7b}-4b\sqrt{5a}$

45. $3\sqrt[3]{2}-2\sqrt[3]{15}$

47. $\sqrt{6}+\sqrt{14}-\sqrt{15}-\sqrt{35}$

49. $18\sqrt{2}+2\sqrt{3}-12\sqrt{6}-4$

51. $8-2\sqrt{15}$

53. 10

55. $a-2\sqrt{2ab}+2b$

57. Perimeter: $(10\sqrt{3}+6\sqrt{2})$ centimeters; area $15\sqrt{6}$ square centimeters

59. 18 square meters

Exercise 5.4.5

Divide. (Assume all variables represent positive real numbers.)

1. $\frac{\sqrt{75}}{\sqrt{3}}$
2. $\frac{\sqrt{360}}{\sqrt{10}}$
3. $\frac{\sqrt{72}}{\sqrt{75}}$
4. $\frac{\sqrt{90}}{\sqrt{98}}$
5. $\frac{\sqrt{90x^5}}{\sqrt{2x}}$
6. $\frac{\sqrt{96y^3}}{\sqrt{3y}}$
7. $\frac{\sqrt{162x^7y^5}}{\sqrt{2xy}}$
8. $\frac{\sqrt{363x^4y^9}}{\sqrt{3xy}}$
9. $\frac{\sqrt[3]{16a^5b^2}}{\sqrt[3]{2a^2b^2}}$
10. $\frac{\sqrt[3]{192a^2b^7}}{\sqrt[3]{2a^2b^2}}$

Answer

1. 5

3. $\frac{2\sqrt{6}}{5}$
5. $3x^2\sqrt{5}$
7. $9x^3y^2$
9. $2a$

Exercise 5.4.6

Rationalize the denominator. (Assume all variables represent positive real numbers.)

1. $\frac{1}{\sqrt{5}}$
2. $\frac{1}{\sqrt{6}}$
3. $\frac{\sqrt{2}}{\sqrt{3}}$
4. $\frac{\sqrt{3}}{\sqrt{7}}$
5. $\frac{5}{2\sqrt{10}}$
6. $\frac{3}{5\sqrt{6}}$
7. $\frac{\sqrt{3}-\sqrt{5}}{\sqrt{3}}$
8. $\frac{\sqrt{6}-\sqrt{2}}{\sqrt{2}}$
9. $\frac{1}{\sqrt{7x}}$
10. $\frac{1}{\sqrt{3y}}$
11. $\frac{a}{5\sqrt{ab}}$
12. $\frac{3b^2}{2\sqrt{3ab}}$
13. $\frac{2}{\sqrt[3]{36}}$
14. $\frac{14}{\sqrt[3]{7}}$
15. $\frac{1}{\sqrt[3]{4x}}$
16. $\frac{1}{\sqrt[3]{3y^2}}$
17. $\frac{9x\sqrt[3]{2}}{\sqrt[3]{9xy^2}}$
18. $\frac{5y^2\sqrt[3]{x}}{\sqrt[3]{5x^2y}}$
19. $\frac{3a}{2\sqrt[3]{3a^2b^2}}$
20. $\frac{25n}{3\sqrt[3]{25m^2n}}$
21. $\frac{3}{\sqrt[5]{27x^2y}}$
22. $\frac{2}{\sqrt[5]{16xy^2}}$
23. $\frac{ab}{\sqrt[5]{9a^3b}}$
24. $\frac{abc}{\sqrt[5]{ab^2c^3}}$
25. $\sqrt[5]{\frac{3x}{8y^2z}}$
26. $\sqrt[5]{\frac{4xy^2}{9x^3yz^4}}$
27. $\frac{3}{\sqrt{10}-3}$
28. $\frac{2}{\sqrt{6}-2}$
29. $\frac{1}{\sqrt{5}+\sqrt{3}}$
30. $\frac{1}{\sqrt{7}-\sqrt{2}}$
31. $\frac{\sqrt{3}}{\sqrt{3}+\sqrt{6}}$
32. $\frac{\sqrt{5}}{\sqrt{5}+\sqrt{15}}$
33. $\frac{10}{5-3\sqrt{5}}$
34. $\frac{-2\sqrt{2}}{4-3\sqrt{2}}$
35. $\frac{\sqrt{3}+\sqrt{5}}{\sqrt{3}-\sqrt{5}}$
36. $\frac{\sqrt{10}-\sqrt{2}}{\sqrt{10}+\sqrt{2}}$
37. $\frac{2\sqrt{3}-3\sqrt{2}}{4\sqrt{3}+\sqrt{2}}$
38. $\frac{6\sqrt{5}+2}{2\sqrt{5}-\sqrt{2}}$

39. $\frac{x-y}{\sqrt{x}+\sqrt{y}}$
40. $\frac{x-y}{\sqrt{x}-\sqrt{y}}$
41. $\frac{x+\sqrt{y}}{x-\sqrt{y}}$
42. $\frac{x-\sqrt{y}}{x+\sqrt{y}}$
43. $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$
44. $\frac{\sqrt{ab}+\sqrt{2}}{\sqrt{ab}-\sqrt{2}}$
45. $\frac{\sqrt{x}}{5-2\sqrt{x}}$
46. $\frac{1}{\sqrt{x}-y}$
47. $\frac{\sqrt{x}+\sqrt{2y}}{\sqrt{2x}-\sqrt{y}}$
48. $\frac{\sqrt{3x}-\sqrt{y}}{\sqrt{x}+\sqrt{3y}}$
49. $\frac{\sqrt{2x+1}}{\sqrt{2x+1}-1}$
50. $\frac{\sqrt{x+1}}{1-\sqrt{x+1}}$
51. $\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}$
52. $\frac{\sqrt{2x+3}-\sqrt{2x-3}}{\sqrt{2x+3}+\sqrt{2x-3}}$
53. The radius of the base of a right circular cone is given by $r=\sqrt{\frac{3V}{\pi h}}$ where V represents the volume of the cone and h represents its height. Find the radius of a right circular cone with volume 50 cubic centimeters and height 4 centimeters. Give the exact answer and the approximate answer rounded to the nearest hundredth.
54. The radius of a sphere is given by $r=\sqrt[3]{\frac{3V}{4\pi}}$ where V represents the volume of the sphere. Find the radius of a sphere with volume 135 square centimeters. Give the exact answer and the approximate answer rounded to the nearest hundredth.

Answer

1. $\frac{\sqrt{5}}{5}$
3. $\frac{\sqrt{6}}{3}$
5. $\frac{\sqrt{10}}{4}$
7. $\frac{3-\sqrt{15}}{3}$
9. $\frac{\sqrt{7x}}{7x}$
11. $\frac{\sqrt{ab}}{5b}$
13. $\frac{\sqrt[3]{6}}{3}$
15. $\frac{\sqrt[3]{2x^2}}{2x}$
17. $\frac{3\sqrt[3]{6x^2y}}{y}$
19. $\frac{\sqrt[3]{9ab}}{2b}$
21. $\frac{\sqrt[5]{9x^3y^4}}{xy}$
23. $\frac{\sqrt[5]{27a^2b^4}}{3}$
25. $\frac{\sqrt[5]{12xy^3z^4}}{2yz}$
27. $3\sqrt{10}+9$
29. $\frac{\sqrt{5}-\sqrt{3}}{2}$
31. \(- 1 + \sqrt { 2 }\
33. $\frac{-5-3\sqrt{5}}{2}$
35. $-4-\sqrt{15}$
37. $\frac{15-7\sqrt{6}}{23}$
39. $\sqrt{x}-\sqrt{y}$

41. $\frac{x^2+2x\sqrt{y}+y}{x^2-y}$
43. $\frac{a-2\sqrt{ab+b}}{a-b}$
45. $\frac{5\sqrt{x}+2x}{25-4x}$
47. $\frac{x\sqrt{2}+3\sqrt{xy}+y\sqrt{2}}{2x-y}$
49. $\frac{2x+1+\sqrt{2x+1}}{2x}$
51. $x+\sqrt{x^2-1}$
53. $\frac{5\sqrt{6\pi}}{2\pi}$ centimeters; 3.45 centimeters

Exercise $5.4.7$

1. Research and discuss some of the reasons why it is a common practice to rationalize the denominator.
2. Explain in your own words how to rationalize the denominator.

Answer

1. Answer may vary

Footnotes

[18] The factors $(a+b)$ and $(a-b)$ are conjugates.

[19] The process of determining an equivalent radical expression with a rational denominator.

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5.5: Rational Exponents

Learning Objectives

- Write expressions with rational exponents in radical form.
- Write radical expressions with rational exponents.
- Perform operations and simplify expressions with rational exponents.
- Perform operations on radicals with different indices.

So far, exponents have been limited to integers. In this section, we will define what rational (or fractional) exponents mean and how to work with them. All of the rules for exponents developed up to this point apply. In particular, recall the product rule for exponents. Given any rational numbers m and n, we have

$$x^{m} \cdot x^{n} = x^{m+n}$$

For example, if we have an exponent of $1/2$, then the product rule for exponents implies the following:

$$5^{1/2} \cdot 5^{1/2} = 5^{1/2+1/2} = 5^{1} = 5$$

Here $5^{1/2}$ is one of two equal factors of 5; hence it is a square root of 5, and we can write

$$5^{1/2} = \sqrt{5}$$

Furthermore, we can see that $2^{1/3}$ is one of three equal factors of 2.

$$2^{1/3} \cdot 2^{1/3} \cdot 2^{1/3} = 2^{1/3+1/3+1/3} = 2^{3/3} = 2^{1} = 2$$

Therefore, $2^{1/3}$ is a cube root of 2, and we can write

$$2^{1/3} = \sqrt[3]{2}$$

This is true in general, given any nonzero real number a and integer $n \geq 2$,

$$a^{1/n} = \sqrt[n]{a}$$

In other words, the denominator of a fractional exponent determines the index of an nth root.

Example $5.5.1$:

Rewrite as a radical.

a. $6^{1/2}$
b. $6^{1/3}$

Solution

a. $6^{1/2} = \sqrt[2]{6} = \sqrt{6}$
b. $6^{1/3} = \sqrt[3]{6}$

Example $5.5.2$:

Rewrite as a radical and then simplify.

1. $16^{1/2}$
2. $16^{1/4}$

Solution

1. $16^{1/2} = \sqrt{16} = \sqrt{4^{2}} = 4$
2. $16^{1/4} = \sqrt[4]{16} = \sqrt[4]{2^{4}} = 2$

Example 5.5.3:

Rewrite as a radical and then simplify.

1. $(64x^3)^{1/3}$
2. $(-32x^5y^{10})^{1/5}$

Solution

1.

$$\begin{aligned}(64x^3)^{1/3} &= \sqrt[3]{64x^3}\\ &= \sqrt[3]{4^3x^3}\\ &= 4x\end{aligned}$$

2.

$$\begin{aligned}(-32x^5y^{10})^{1/5} &= \sqrt[5]{-32x^5y^{10}}\\ &= \sqrt[5]{(-2)^5x^5\left(y^2\right)^5}\\ &= -2xy^2\end{aligned}$$

Next, consider fractional exponents where the numerator is an integer other than 1. For example, consider the following:

$$5^{2/3}\cdot 5^{2/3}\cdot 5^{2/3} = 5^{2/3+2/3+2/3} = 5^{6/3} = 5^2$$

This shows that $5^{2/3}$ is one of three equal factors of 5^2. In other words, $5^{2/3}$ is a cube root of 5^2 and we can write:

$$5^{2/3} = \sqrt[3]{5^2}$$

In general, given any nonzero real number a where m and n are positive integers $(n \geq 2)$,

$$a^{m/n} = \sqrt[n]{a^m}$$

An expression with a **rational exponent**[20] is equivalent to a radical where the denominator is the index and the numerator is the exponent. Any radical expression can be written with a rational exponent, which we call **exponential form**[21].

$$\begin{array}{ccc}\textit{Radical form} & & \textit{Exponential form}\\ \sqrt[5]{x^2} & = & x^{2/5}\end{array}$$

Example 5.5.4:

Rewrite as a radical.

1. $6^{2/5}$
2. $3^{3/4}$

Solution

1. $6^{2/5} = \sqrt[5]{6^2} = \sqrt[5]{36}$
2. $3^{3/4} = \sqrt[4]{3^3} = \sqrt[4]{27}$

Example 5.5.5:

Rewrite as a radical and then simplify.

1. $27^{2/3}$
2. $(12)^{5/3}$

Solution

We can often avoid very large integers by working with their prime factorization.

1.

$$\begin{aligned}27^{2/3} &= \sqrt[3]{27^2} \\ &= \sqrt[3]{\left(3^3\right)^2} && \textit{Replace 27 with } 3^3 \\ &= \sqrt[3]{3^6} && \textit{Simplify.} \\ &= 3^2 \\ &= 9\end{aligned}$$

2.

$$\begin{aligned}(12)^{5/3} &= \sqrt[3]{(12)^5} && \textit{Replace 12 with } 2^2 \cdot 3. \\ &= \sqrt[3]{\left(2^2 \cdot 3\right)^5} && \textit{Apply the rules for exponents.} \\ &= \sqrt[3]{2^{10} \cdot 3^5} && \textit{Simplify.} \\ &= \sqrt[3]{2^9 \cdot 2 \cdot 3^3 \cdot 3^2} \\ &= 2^3 \cdot 3 \cdot \sqrt[3]{2 \cdot 3^2} \\ &= 24\sqrt[3]{18}\end{aligned}$$

Given a radical expression, we might want to find the equivalent in exponential form. Assume all variables are positive.

Example 5.5.6:

Rewrite using rational exponents: $\sqrt[5]{x^3}$.

Solution

Here the index is 5 and the power is 3. We can write

$$\sqrt[5]{x^3} = x^{3/5}$$

Answer:

$x^{3/5}$

Example 5.5.7:

Rewrite using rational exponents: $\sqrt[6]{y^3}$.

Solution

Here the index is 6 and the power is 3. We can write

$$\begin{aligned}\sqrt[6]{y^3} &= y^{3/6} \\ &= y^{1/2}\end{aligned}$$

Answer:

$y^{1/2}$

It is important to note that the following are equivalent.

$$a^{n/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

In other words, it does not matter if we apply the power first or the root first. For example, we can apply the power before the nth root:

$$27^{2/3} = \sqrt[3]{27^2} = \sqrt[3]{\left(3^3\right)^2} = \sqrt[3]{3^6} = 3^2 = 9$$

Or we can apply the nth root before the power:

$$27^{2/3} = (\sqrt[3]{27})^2 = \left(\sqrt[3]{3^3}\right)^2 = (3)^2 = 9$$

The results are the same.

Example 5.5.8:

Rewrite as a radical and then simplify: $(-8)^{2/3}$.

Solution

Here the index is 3 and the power is 2. We can write

$$(-8)^{2/3} = (\sqrt[3]{-8})^2 = (-2)^2 = 4$$

Answer:

4

Exercise 5.5.1

Rewrite as a radical and then simplify: $100^{3/2}$.

Answer

$1,000$

www.youtube.com/v/39ImsSbFD5o

Some calculators have a caret button Missing superscript or subscript argument which is used for entering exponents. If so, we can calculate approximations for radicals using it and rational exponents. For example, to calculate $\sqrt{2} = 2^{1/2} = 2\wedge(1/2) \approx 1.414$ we make use of the parenthesis buttons and type

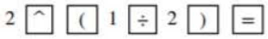

Figure $5.5.1$

To calculate $\sqrt[3]{2^2} = 2^{2/3} = 2\wedge(2/3) \approx 1.587$, we would type

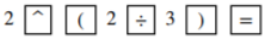

Figure $5.5.2$

Operations Using the Rules of Exponents

In this section, we review all of the rules of exponents, which extend to include rational exponents. If given any rational numbers m and n, then we have

Table $5.5.1$

Product rule for exponents:	$x^m \cdot x^n = x^{m+n}$
Quotient rule for exponents:	$\frac{x^m}{x^n} = x^{m-n}, x \neq 0$
Power rule for exponents:	$(x^m)^n = x^{m \cdot n}$
Power rule for a product:	$(xy)^n = x^n y^n$
Power rule for a quotient:	$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}, y \neq 0$
Negative exponents:	$x^{-n} = \frac{1}{x^n}$
Zero exponent:	$x^0 = 1, x \neq 0$

These rules allow us to perform operations with rational exponents.

Example 5.5.9:

Simplify: $7^{1/3} \cdot 7^{4/9}$.

Solution

$$\begin{aligned} 7^{1/3} \cdot 7^{49} &= 7^{1/3+49} && \textit{Apply the product rule } x^m \cdot x^n = x^{m+n}. \\ &= 7^{3/9+4/9} \\ &= 7^{7/9} \end{aligned}$$

Answer:

$7^{7/9}$

Example 5.5.10:

Simplify: $\frac{x^{3/2}}{x^{2/3}}$.

Solution

$$\begin{aligned} \frac{x^{3/2}}{x^{2/3}} &= x^{3/2-2/3} && \textit{Apply the quotient rule } \frac{x^m}{x^n} = x^{m-n}. \\ &= x^{9/6-4/6} \\ &= x^{5/6} \end{aligned}$$

Example 5.5.11:

Simplify: $\left(y^{3/4}\right)^{2/3}$.

Solution

$$\begin{aligned} \left(y^{3/4}\right)^{2/3} &= y^{(3/4)(2/3)} && \textit{Apply the power rule } (x^m)^n = x^{m \cdot n}. \\ &= y^{6/12} && \textit{Multiply the exponents and reduce.} \\ &= y^{1/2} \end{aligned}$$

Answer:

$y^{1/2}$

Example 5.5.12:

Simplify: $\left(81a^8b^{12}\right)^{3/4}$.

Solution

$$\begin{aligned} \left(81a^8b^{12}\right)^{3/4} &= \left(3^4a^8b^{12}\right)^{3/4} && \textit{Rewrite } 81 \textit{ as } 3^4. \\ &= \left(3^4\right)^{3/4}\left(a^8\right)^{3/4}\left(b^{12}\right)^{3/4} && \textit{Apply the power rule for a product.} \\ &= 3^{4(3/4)}a^{8(3/4)}b^{12(3/4)} && \textit{Apply the power rule to each factor.} \\ &= 3^3a^6b^9 && \textit{Simplify.} \\ &= 27a^6b^9 \end{aligned}$$

Answer:

$27a^6b^9$

Example 5.5.13:

Simplify: $\left(9x^4\right)^{-3/2}$.

Solution

$$\begin{aligned}(9x^4)^{-3/2} &= \frac{1}{(9x^4)^{3/2}} && \textit{Apply the definition of negative exponents } x^{-n}=\frac{1}{x^n}. \\ &= \frac{1}{(3^2x^4)^{3/2}} && \textit{Write 9 as } 3^2 \textit{ and apply the rules of exponents.} \\ &= \frac{1}{3^{2(3/2)}x^{4(3/2)}} \\ &= \frac{1}{3^3\cdot x^6} \\ &= \frac{1}{27x^6}\end{aligned}$$

Answer:

$\frac{1}{27x^6}$

Exercise 5.5.2

Simplify: $\frac{(125a^{1/4}b^6)^{2/3}}{a^{1/6}}$.

Answer

$25b^4$

www.youtube.com/v/lEQKFwYoMuc

Radical Expressions with Different Indices

To apply the product or quotient rule for radicals, the indices of the radicals involved must be the same. If the indices are different, then first rewrite the radicals in exponential form and then apply the rules for exponents.

Example 5.5.14:

Multiply: $\sqrt{2}\cdot\sqrt[3]{2}$.

Solution

In this example, the index of each radical factor is different. Hence the product rule for radicals does not apply. Begin by converting the radicals into an equivalent form using rational exponents. Then apply the product rule for exponents.

$$\begin{aligned}\sqrt{2}\cdot\sqrt[3]{2} &= 2^{1/2}\cdot 2^{1/3} && \textit{Equivalents using rational exponents.} \\ &= 2^{1/2+1/3} && \textit{Apply the product rule for exponents.} \\ &= 2^{5/6} \\ &= \sqrt[6]{2^5}\end{aligned}$$

Answer:

$\sqrt[6]{2^5}$

Example 5.5.15:

Divide: $\frac{\sqrt[3]{4}}{\sqrt[5]{2}}$.

Solution

In this example, the index of the radical in the numerator is different from the index of the radical in the denominator. Hence the quotient rule for radicals does not apply. Begin by converting the radicals into an equivalent form using rational exponents and then apply the quotient rule for exponents.

$$\begin{aligned}\frac{\sqrt[3]{4}}{\sqrt[5]{2}} &= \frac{\sqrt[3]{2^2}}{\sqrt[5]{2}} \\ &= \frac{2^{2/3}}{2^{1/5}} && \color{#1c9fe0}{\textit{Equivalents using rational exponents.}} \\ &= 2^{2/3-1/5} && \color{#1c9fe0}{\textit{Apply the quotient rule for exponents.}} \\ &= 2^{7/15} \\ &= \sqrt[15]{2^7}\end{aligned}$$

Answer:

$\sqrt[15]{2^7}$

Example 5.5.16:

Simplify: $\sqrt{\sqrt[3]{4}}$.

Solution

Here the radicand of the square root is a cube root. After rewriting this expression using rational exponents, we will see that the power rule for exponents applies.

$$\begin{aligned}\sqrt{\sqrt[3]{4}} &= \sqrt{\sqrt[3]{2^2}} \\ &= \left(2^{2/3}\right)^{1/2} && \color{#1c9fe0}{\textit{Equivalents using rational exponents.}} \\ &= 2^{(2/3)(1/2)} && \color{#1c9fe0}{\textit{Apply the power rule for exponents.}} \\ &= 2^{1/3} \\ &= \sqrt[3]{2}\end{aligned}$$

Answer:

$\sqrt[3]{2}$

Key Takeaways

- Any radical expression can be written in exponential form: $\sqrt[n]{a^m} = a^{m/n}$.
- Fractional exponents indicate radicals. Use the numerator as the power and the denominator as the index of the radical.
- All the rules of exponents apply to expressions with rational exponents.
- If operations are to be applied to radicals with different indices, first rewrite the radicals in exponential form and then apply the rules for exponents.

Exercise 5.5.3

Express using rational exponents.

1. $\sqrt{10}$
2. $\sqrt{6}$
3. $\sqrt[3]{3}$
4. $\sqrt[4]{5}$
5. $\sqrt[3]{5^2}$
6. $\sqrt[4]{2^3}$
7. $\sqrt[3]{49}$
8. $\sqrt[3]{9}$
9. $\sqrt[5]{x}$
10. $\sqrt[6]{x}$
11. $\sqrt[6]{x^7}$
12. $\sqrt[5]{x^4}$
13. $\frac{1}{\sqrt{x}}$
14. $\frac{1}{\sqrt[3]{x^2}}$

Answer

1. $10^{1/2}$
3. $3^{1/3}$
5. $5^{2/3}$
7. $7^{2/3}$
9. $x^{1/5}$
11. $x^{7/6}$
13. $x^{-1/2}$

Exercise $5.5.4$

Express in radical form.

1. $10^{1/2}$
2. $11^{1/3}$
3. $7^{2/3}$
4. $2^{3/5}$
5. $x^{3/4}$
6. $x^{5/6}$
7. $x^{-1/2}$
8. $x^{-3/4}$
9. $\left(\frac{1}{x}\right)^{-1/3}$
10. $\left(\frac{1}{x}\right)^{-3/5}$
11. $(2x+1)^{2/3}$
12. $(5x-1)^{1/2}$

Answer

1. $\sqrt{10}$
3. $\sqrt[3]{49}$
5. $\sqrt[4]{x^3}$
7. $\frac{1}{\sqrt{x}}$
9. $\sqrt[3]{x}$
11. $\sqrt[3]{(2x+1)^2}$

Exercise $5.5.5$

Write as a radical and then simplify.

1. $64^{1/2}$
2. $49^{1/2}$
3. $\left(\frac{1}{4}\right)^{1/2}$
4. $\left(\frac{4}{9}\right)^{1/2}$
5. $4^{-1/2}$
6. $9^{-1/2}$
7. $\left(\frac{1}{4}\right)^{-1/2}$
8. $\left(\frac{1}{16}\right)^{-1/2}$
9. $8^{1/3}$
10. $125^{1/3}$
11. $\left(\frac{1}{27}\right)^{1/3}$
12. $\left(\frac{8}{125}\right)^{1/3}$

13. $(-27)^{1/3}$
14. $(-64)^{1/3}$
15. $16^{1/4}$
16. $625^{1/4}$
17. $81^{-1/4}$
18. $16^{-1/4}$
19. $100,000^{1/5}$
20. $(-32)^{1/5}$
21. $\left(\frac{1}{32}\right)^{1/5}$
22. $\left(\frac{1}{243}\right)^{1/5}$
23. $9^{3/2}$
24. $4^{3/2}$
25. $8^{5/3}$
26. $27^{2/3}$
27. $16^{3/2}$
28. $32^{2/5}$
29. $\left(\frac{1}{16}\right)^{3/4}$
30. $\left(\frac{1}{81}\right)^{3/4}$
31. $(-27)^{2/3}$
32. $(-27)^{4/3}$
33. $(-32)^{3/5}$
34. $(-32)^{4/5}$

Answer

1. 8

3. $\frac{1}{2}$

5. $\frac{1}{2}$

7. 2

9. 2

11. $frac13$

13. -3

15. 2

17. $\frac{1}{3}$

19. 10

21. $\frac{1}{2}$

23. 27

25. 32

27. 64

29. $\frac{1}{8}$

31. 9

33. -8

Exercise 5.5.6

Use a calculator to approximate an answer rounded to the nearest hundredth.

1. $2^{1/2}$
2. $2^{1/3}$
3. $2^{3/4}$
4. $3^{2/3}$
5. $5^{1/5}$
6. $7^{1/7}$
7. $(-9)^{3/2}$
8. $-9^{3/2}$
9. Explain why (-4)^$(3/2)$ gives an error on a calculator and -4^$(3/2)$ gives an answer of -8.
10. Marcy received a text message from Mark asking her age. In response, Marcy texted back “125^$(2/3)$ years old.” Help Mark determine Marcy’s age.

Answer

1. 1.41

3. 1.68

5. 1.38

7. Not a real number

9. Answer may vary

Exercise 5.5.7

Perform the operations and simplify. Leave answers in exponential form.

1. $5^{3/2} \cdot 5^{1/2}$
2. $3^{2/3} \cdot 3^{7/3}$
3. $5^{1/2} \cdot 5^{1/3}$
4. $2^{1/6} \cdot 2^{3/4}$
5. $y^{1/4} \cdot y^{2/5}$
6. $x^{1/2} \cdot x^{1/4}$
7. $\frac{5^{11/3}}{5^{2/3}}$
8. $\frac{2^{9/2}}{2^{1/2}}$
9. $\frac{2a^{2/3}}{a^{1/6}}$
10. $\frac{3b^{1/2}}{b^{1/3}}$
11. $\left(8^{1/2}\right)^{2/3}$
12. $\left(3^{6}\right)^{2/3}$
13. $\left(x^{2/3}\right)^{1/2}$
14. $\left(y^{3/4}\right)^{4/5}$
15. $\left(y^{8}\right)^{-1/2}$
16. $\left(y^{6}\right)^{-2/3}$
17. $\left(4x^{2}y^{4}\right)^{1/2}$
18. $\left(9x^{6}y^{2}\right)^{1/2}$
19. $\left(2x^{1/3}y^{2/3}\right)^{3}$
20. $\left(8x^{3/2}y^{1/2}\right)^{2}$
21. $\left(36x^{4}y^{2}\right)^{-1/2}$
22. $\left(8x^{3}y^{6}z^{-3}\right)^{-1/3}$
23. $\left(\frac{a^{3/4}}{a^{1/2}}\right)^{4/3}$
24. $\left(\frac{b^{4/5}}{b^{1/10}}\right)^{10/3}$
25. $\left(\frac{4x^{2/3}}{y^{4}}\right)^{1/2}$

26. $\left(\frac{27x^{3/4}}{y^9}\right)^{1/3}$
27. $\frac{y^{1/2}y^{2/3}}{y^{1/6}}$
28. $\frac{x^{2/5}x^{1/2}}{x^{1/10}}$
29. $\frac{xy}{x^{1/2}y^{1/3}}$
30. $\frac{x^{5/4}y}{xy^{2/5}}$
31. $\frac{49a^{5/7}b^{3/2}}{7a^{3/7}b^{1/4}}$
32. $\frac{16a^{5/6}b^{5/4}}{8a^{1/2}b^{2/3}}$
33. $\frac{\left(9x^{2/3}y^6\right)^{3/2}}{x^{1/2}y}$
34. $\frac{\left(125x^3y^{3/5}\right)^{2/3}}{xy^{1/3}}$
35. $\frac{\left(27a^{1/4}b^{3/2}\right)^{2/3}}{a^{1/6}b^{1/2}}$
36. $\frac{\left(25a^{2/3}b^{4/3}\right)^{3/2}}{a^{1/6}b^{1/3}}$
37. $\left(16x^2y^{-1/3}z^{2/3}\right)^{-3/2}$
38. $\left(81x^8y^{-4/3}z^{-4}\right)^{-3/4}$
39. $\left(100a^{-2/3}b^4c^{-3/2}\right)^{-1/2}$
40. $\left(125a^9b^{-3/4}c^{-1}\right)^{-1/3}$

Answer

1. 25
3. $5^{5/6}$
5. $y^{13/20}$
7. 125
9. $2a^{1/2}$
11. 2
13. $x^{1/3}$
15. $\frac{1}{y^4}$
17. $2xy^2$
19. $8xy^2$
21. $\frac{1}{6x^2y}$
23. $a^{1/3}$
25. $\frac{2x^{1/3}}{y^2}$
27. y
29. $x^{1/2}y^{2/3}$
31. $7a^{2/7}b^{5/4}$
33. $27x^{1/2}y^8$
35. $9b^{1/2}$
37. $\frac{y^{1/2}}{64x^3z}$
39. $\frac{a^{1/3}b^{3/4}}{10b^2}$

Exercise 5.5.8

Perform the operations.

1. $\sqrt[3]{9}\cdot\sqrt[5]{3}$
2. $\sqrt{5}\cdot\sqrt[5]{25}$

3. $\sqrt{x} \cdot \sqrt[3]{x}$
4. $\sqrt{y} \cdot \sqrt[4]{y}$
5. $\sqrt[3]{x^2} \cdot \sqrt[4]{x}$
6. $\sqrt[5]{x^3} \cdot \sqrt[3]{x}$
7. $\frac{\sqrt[3]{100}}{\sqrt{10}}$
8. $\frac{\sqrt[5]{16}}{\sqrt[3]{4}}$
9. $\frac{\sqrt[3]{a^2}}{\sqrt{a}}$
10. $\frac{\sqrt[5]{b^4}}{\sqrt[3]{b}}$
11. $\frac{\sqrt[3]{x^2}}{\sqrt[5]{x^3}}$
12. $\frac{\sqrt[4]{x^3}}{\sqrt[3]{x^2}}$
13. $\sqrt{\sqrt[5]{16}}$
14. $\sqrt{\sqrt[3]{9}}$
15. $\sqrt[3]{\sqrt[5]{2}}$
16. $\sqrt[3]{\sqrt[5]{5}}$
17. $\sqrt[3]{\sqrt{7}}$
18. $\sqrt[3]{\sqrt{3}}$

Answer

1. $\sqrt[15]{3^{13}}$

3. $\sqrt[6]{x^5}$

5. $\sqrt[12]{x^{11}}$

7. $\sqrt[6]{10}$

9. $\sqrt[6]{a}$

11. $\sqrt[15]{x}$

13. $\sqrt[5]{4}$

15. $\sqrt[15]{2}$

17. $\sqrt[6]{7}$

Exercise $5.5.9$

1. Who is credited for devising the notation that allows for rational exponents? What are some of his other accomplishments?
2. When using text, it is best to communicate nth roots using rational exponents. Give an example.

Answer

1. Answer may vary

Footnotes

[20] The fractional exponent m/n that indicates a radical with index n and exponent m: $a^{m/n} = \sqrt[n]{a^m}$.

[21] An equivalent expression written using a rational exponent.

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5.6: Solving Radical Equations

Learning Objectives

- Solve equations involving square roots.
- Solve equations involving cube roots.

Radical Equations

A **radical equation**[22] is any equation that contains one or more radicals with a variable in the radicand. Following are some examples of radical equations, all of which will be solved in this section:

Table $5.6.1$

$\sqrt{2x-1}=3$	$\sqrt[3]{4x^2+7}-2=0$	$\sqrt{x+2}-\sqrt{x}=1$

We begin with the **squaring property of equality**[23]; given real numbers a and b, we have the following:

$$\text{If} \quad a=b, \quad \text{then } a^2=b^2$$

In other words, equality is retained if we square both sides of an equation.

$$\begin{aligned} -3=-3 \Rightarrow (-3)^2 &= (-3)^2 \\ 9 &= 9 \ \checkmark \end{aligned}$$

The converse, on the other hand, is not necessarily true,

$$\begin{aligned} 9 &= 9 \\ (-3)^2=(3)^2 &\Rightarrow -3 \neq 3 \ \text{✗} \end{aligned}$$

This is important because we will use this property to solve radical equations. Consider a very simple radical equation that can be solved by inspection,

$$\sqrt{x}=5$$

Here we can see that $x=25$ is a solution. To solve this equation algebraically, make use of the squaring property of equality and the fact that $(\sqrt{a})^2=\sqrt{a^2}=a$ when a is nonnegative. Eliminate the square root by squaring both sides of the equation as follows:

$$\begin{aligned} (\sqrt{x})^2 &= (5)^2 \\ x &= 25 \end{aligned}$$

As a check, we can see that $\sqrt{25}=5$ as expected. Because the converse of the squaring property of equality is not necessarily true, solutions to the squared equation may not be solutions to the original. Hence squaring both sides of an equation introduces the possibility of **extraneous solutions**[24], which are solutions that do not solve the original equation. For example,

$$\sqrt{x}=-5$$

This equation clearly does not have a real number solution. However, squaring both sides gives us a solution:

$$\begin{aligned} (\sqrt{x})^2 &= (-5)^2 \\ x &= 25 \end{aligned}$$

As a check, we can see that $\sqrt{25} \neq -5$. For this reason, we must check the answers that result from squaring both sides of an equation.

Example $5.6.1$:

Solve: $\sqrt{3x+1}=4$.

Solution

We can eliminate the square root by applying the squaring property of equality.

$$\begin{aligned} \sqrt{3x+1} &= 4 \\ (\sqrt{3x+1})^2 &= (4)^2 && \textit{Square both sides.} \\ 3x+1 &= 16 && \textit{Solve.} \\ 3x &= 15 \\ x &= 5 \end{aligned}$$

Next, we must check.

$$\begin{aligned} \sqrt{3(5)+1} &= 4 \\ \sqrt{15+1} &= 4 \\ \sqrt{16} &= 4 \\ 4 &= 4 \;\checkmark \end{aligned}$$

Answer:

The solution is 5.

There is a geometric interpretation to the previous example. Graph the function defined by $f(x) = \sqrt{3x+1}$ and determine where it intersects the graph defined by $g(x) = 4$.

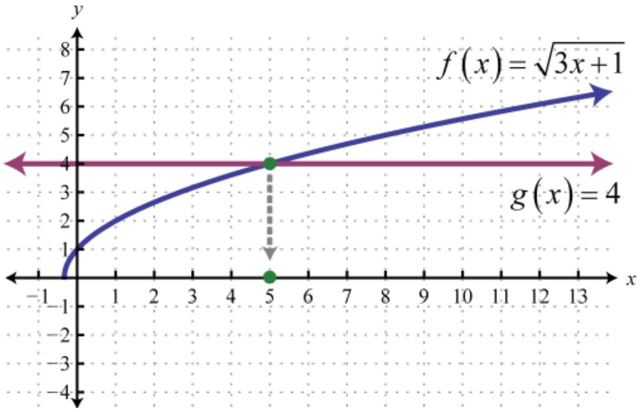

Figure $5.6.1$

As illustrated, $f(x) = g(x)$ where $x = 5$.

Example $5.6.2$:

Solve: $\sqrt{x-3} = x-5$.

Solution

Begin by squaring both sides of the equation.

$$\begin{aligned} \sqrt{x-3} &= x-5 \\ (\sqrt{x-3})^2 &= (x-5)^2 && \textit{Square both sides.} \\ x-3 &= x^2-10x+25 \end{aligned}$$

The resulting quadratic equation can be solved by factoring.

$$\begin{aligned} x-3 &= x^2-10x+25 \\ 0 &= x^2-11x+28 \\ 0 &= (x-4)(x-7) \end{aligned}$$

$$\begin{aligned} x-4 &= 0 \quad \text{or} \quad & x-7 &= 0 \\ x &= 4 & x &= 7 \end{aligned}$$

Checking the solutions after squaring both sides of an equation is not optional. Use the original equation when performing the check.

Table $5.6.2$

$Check x = 4$	$Check x = 7$
$\sqrt{x-3} = x-5$ $\sqrt{4-3} = 4-5$ $\sqrt{1} = -1$ $1 = -1$ ✗	$\sqrt{x-3} = x-5$ $\sqrt{7-3} = 7-5$ $\sqrt{4} = 2$ $2 = 2$ ✓

After checking, you can see that $x = 4$ is an extraneous solution; it does not solve the original radical equation. Disregard that answer. This leaves $x = 7$ as the only solution.

Answer:

The solution is 7.

Geometrically we can see that $f(x) = \sqrt{x+3}$ is equal to $g(x) = x - 5$ where $x = 7$.

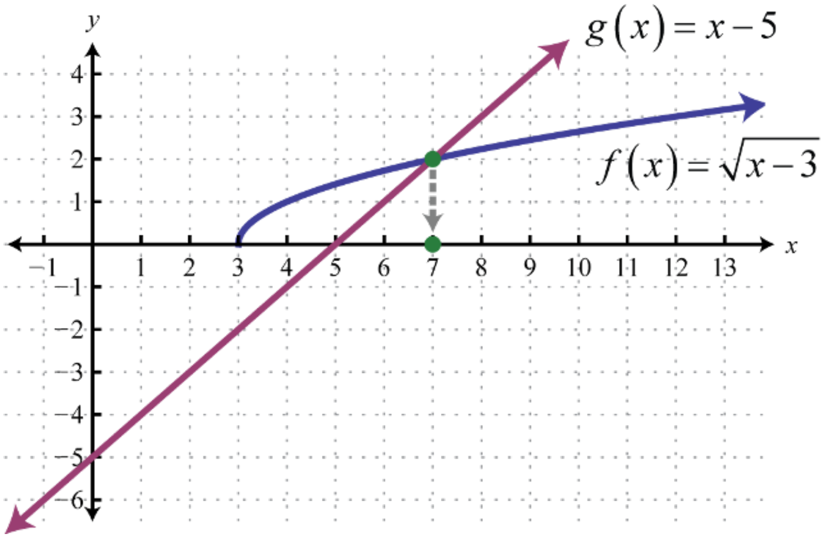

Figure $5.6.2$

In the previous two examples, notice that the radical is isolated on one side of the equation. Typically, this is not the case. The steps for solving radical equations involving square roots are outlined in the following example.

Example $5.6.3$:

Solve: $\sqrt{2x-1} + 2 = x$.

Solution

Step 1: Isolate the square root. Begin by subtracting 2 from both sides of the equation.

$$\begin{aligned} \sqrt{2x-1} + 2 &= x \\ \sqrt{2x-1} &= x - 2 \end{aligned}$$

Step 2: Square both sides. Squaring both sides eliminates the square root.

$$\begin{aligned} (\sqrt{2x-1})^2 &= (x-2)^2 \\ 2x - 1 &= x^2 - 4x + 4 \end{aligned}$$

Step 3: Solve the resulting equation. Here we are left with a quadratic equation that can be solved by factoring.

$$\begin{aligned} 2x - 1 &= x^2 - 4x + 4 \\ 0 &= x^2 - 6x + 5 \\ 0 &= (x-1)(x-5) \end{aligned}$$

$$\begin{aligned} x - 1 &= 0 \quad \text{or} \quad & x - 5 &= 0 \\ x &= 1 & x &= 5 \end{aligned}$$

Step 4: Check the solutions in the original equation. Squaring both sides introduces the possibility of extraneous solutions; hence the check is required.

Table $5.6.3$

$Check x = 1$	$Check x = 5$

$\sqrt{2x-1}+2=x$ $\sqrt{2(1)-1}+2=1$ $\sqrt{1}+2=1$ $1+2=1$ $3=1$ ✗	$\sqrt{2x-1}+2=x$ $\sqrt{2(5)-1}+2=5$ $\sqrt{9}+2=5$ $3+2=5$ $5=5$ ✓

After checking, we can see that $x=1$ is an extraneous solution; it does not solve the original radical equation. This leaves $x=5$ as the only solution.

Answer:

The solution is 5.

Sometimes there is more than one solution to a radical equation.

Example $5.6.4$:

Solve: $2\sqrt{2x+5}-x=4$.

Solution

Begin by isolating the term with the radical.

$$\begin{aligned} 2\sqrt{2x+5}-x&=4 \qquad \textit{Add } x \textit{ to both sides.} \\ 2\sqrt{2x+5}&=x+4 \end{aligned}$$

Despite the fact that the term on the left side has a coefficient, we still consider it to be isolated. Recall that terms are separated by addition or subtraction operators.

$$\begin{aligned} 2\sqrt{2x+5}&=x+4 \\ (2\sqrt{2x+5})^2&=(x+4)^2 \qquad \textit{Square both sides.} \\ 4(2x+5)&=x^2+8x+16 \end{aligned}$$

Solve the resulting quadratic equation.

$$\begin{aligned} 4(2x+5)&=x^2+8x+16 \\ 8x+20&=x^2+8x+16 \\ 0&=x^2-4 \\ 0&=(x+2)(x-2) \end{aligned}$$

$$\begin{aligned} x+2&=0 \quad \text{or } x-2=0 \\ x&=-2 \qquad\quad x=2 \end{aligned}$$

Since we squared both sides, we must check our solutions.

Table $5.6.4$

Check $x=-2$	*Check* $x=2$
$2\sqrt{2x+5}-x=4$ $2\sqrt{2(-2)+5}-(-2)=4$ $2\sqrt{-4+5}+2=4$ $2\sqrt{1}+2=4$ $2+2=4$ $4=4$ ✓	$2\sqrt{2x+5}-x=4$ $2\sqrt{2(2)+5}-(2)=4$ $2\sqrt{4+5}-2=4$ $2\sqrt{9}-2=4$ $6-2=4$ $4=4$ ✓

After checking, we can see that both are solutions to the original equation.

Answer:

The solutions are ± 2.

Sometimes both of the possible solutions are extraneous.

Example 5.6.5:

Solve: $\sqrt{4-11x}-x+2=0$.

Solution

Begin by isolating the radical.

$$\begin{aligned} \sqrt{4-11x}-x+2&=0 && \textit{Isolate the radical.} \\ \sqrt{4-11x}&=x-2 \\ (\sqrt{4-11x})^2&=(x-2)^2 && \textit{Square both sides.} \\ 4-11x&=x^2-4x+4 && \textit{Solve.} \\ 0&=x^2+7x \\ 0&=x(x+7) \\ x=0 \text{ or } x+7&=0 \\ x&=-7 \end{aligned}$$

Since we squared both sides, we must check our solutions.

Table 5.6.5

Check $x=0$	*Check* $x=-7$
$\sqrt{4-11x}-x+2=0$ $\sqrt{4-11(0)}-0+2=0$ $\sqrt{4}+2=0$ $2+2=0$ $4=0$ ✗	$\sqrt{4-11x}-x+2=0$ $\sqrt{4-11(-7)}-(-7)+2=0$ $\sqrt{4+77}+7+2=0$ $\sqrt{81}+9=0$ $9+9=0$ $18=0$ ✗

Since both possible solutions are extraneous, the equation has no solution.

Answer:

No solution, $\emptyset$

The squaring property of equality extends to any positive integer power n. Given real numbers a and b, we have the following:

$$\text{If} \quad a=b, \quad \text{then } a^n=b^n$$

This is often referred to as the **power property of equality**[25]. Use this property, along with the fact that $(\sqrt[n]{a})^n=\sqrt[n]{a^n}=a$, when a is nonnegative, to solve radical equations with indices greater than 2.

Example 5.6.6:

Solve $\sqrt[3]{4x^2+7}-2=0$.

Solution

Isolate the radical, and then cube both sides of the equation.

$$\begin{aligned} \sqrt[3]{4x^2+7}-2&=0 && \textit{Isolate the radical.} \\ \sqrt[3]{4x^2+7}&=2 \\ \left(\sqrt[3]{4x^2+7}\right)^3&=(2)^3 && \textit{Cube both sides.} \\ 4x^2+7&=8 && \textit{Solve.} \\ 4x^2-1&=0 \\ (2x+1)(2x-1)&=0 \end{aligned}$$

$$\begin{aligned} 2x+1&=0 \quad \text{or} & 2x-1&=0 \\ 2x&=-1 & 2x&=1 \\ x&=-\tfrac{1}{2} & x&=\tfrac{1}{2} \end{aligned}$$

Check.

Table 5.6.6

$\text{Check}x=-\frac{1}{2}$	$\text{Check}x=\frac{1}{2}$
$\sqrt[3]{4x^2+7}-2=0$ $\sqrt[3]{4\left(-\frac{1}{2}\right)^2+7}-2=0$ $\sqrt[3]{4\cdot\frac{1}{4}+7}-2=0$ $\sqrt[3]{8}-2=0$ $2-2=0$ $0=0$ ✓	$\sqrt[3]{4x^2+7}-2=0$ $\sqrt[3]{4\left(\frac{1}{2}\right)^2+7}-2=0$ $\sqrt[3]{4\cdot\frac{1}{4}+7}-2=0$ $\sqrt[3]{1+7}-2=0$ $\sqrt[3]{8}-2=0$ $2-2=0$ $0=0$ ✓

Answer:

The solutions are $\pm\frac{1}{2}$.

Exercise 5.6.1

$x-3\sqrt{3x+1}=3$

Answer

The solution is 33.

www.youtube.com/v/kR6X0qJBtbs

It may be the case that the equation has more than one term that consists of radical expressions.

Example 5.6.7:

Solve: $\sqrt{5x-3}=\sqrt{4x-1}$.

Solution

Both radicals are considered isolated on separate sides of the equation.

$$\begin{aligned}\sqrt{5x-3}&=\sqrt{4x-1}\\(\sqrt{5x-3})^2&=(\sqrt{4x-1})^2 && \textit{Square both sides.}\\5x-3&=4x-1 && \textit{Solve.}\\x&=2\end{aligned}$$

Check $x=2$.

$$\begin{aligned}\sqrt{5x-3}&=\sqrt{4x-1}\\\sqrt{5(2)-3}&=\sqrt{4(2)-1}\\\sqrt{10-3}&=\sqrt{8-1}\\\sqrt{7}&=\sqrt{7}\ \checkmark\end{aligned}$$

Answer:

The solution is 2.

Example 5.6.8:

Solve: $\sqrt[3]{x^2+x-14}=\sqrt[3]{x+50}$.

Solution

Eliminate the radicals by cubing both sides.

$$\begin{aligned}\sqrt[3]{x^2+x-14}&=\sqrt[3]{x+50}\\ \left(\sqrt[3]{x^2+x-14}\right)^3&=(\sqrt[3]{x+50})^3 && \textit{Cube both sides.}\\ x^2+x-14&=x+50 && \textit{Solve.}\\ x^2-64&=0\\ (x+8)(x-8)&=0\end{aligned}$$

$$\begin{aligned}x+8=0 \quad &\text{or} \quad x-8=0\\ x=-8 \quad & \qquad\quad x=8\end{aligned}$$

Check.

Table 5.6.7

$\textit{Check} x=-8$	$\textit{Check} x=8$
$\sqrt[3]{x^2+x-14}=\sqrt[3]{x+50}$ $\sqrt[3]{(-8)^2+(-8)-14}=\sqrt[3]{(-8)+50}$ $\sqrt[3]{64-8-14}=\sqrt[3]{42}$ $\sqrt[3]{42}=\sqrt[3]{42Table\backslash(5.6.7\backslash)}$ ✓	\($\sqrt[3]{x^2+x-14}=\sqrt[3]{x+50}$ $\sqrt[3]{(8)^2+(8)-14}=\sqrt[3]{(8)+50}$ $\sqrt[3]{64+8-14}=\sqrt[3]{58}$ $\sqrt[3]{58}=\sqrt[3]{58}$ ✓ (5.6.1) \

Answer:

The solutions are ± 8.

It may not be possible to isolate a radical on both sides of the equation. When this is the case, isolate the radicals, one at a time, and apply the squaring property of equality multiple times until only a polynomial remains.

Example 5.6.9:

Solve: $\sqrt{x+2}-\sqrt{x}=1$

Solution

Begin by isolating one of the radicals. In this case, add $\sqrt{x}$ to both sides of the equation.

$$\begin{aligned}\sqrt{x+2}-\sqrt{x}&=1\\ \sqrt{x+2}&=\sqrt{x}+1\end{aligned}$$

Next, square both sides. Take care to apply the distributive property to the right side.

$$\begin{aligned}(\sqrt{x+2})^2&=(\sqrt{x}+1)^2\\ x+2&=(\sqrt{x}+1)(\sqrt{x}+1)\\ x+2&=\sqrt{x^2}+\sqrt{x}+\sqrt{x}+1\\ x+2&=x+2\sqrt{x}+1\end{aligned}$$

At this point we have one term that contains a radical. Isolate it and square both sides again.

$$\begin{aligned}x+2&=x+2\sqrt{x}+1\\ 1&=2\sqrt{x}\\ (1)^2&=(2\sqrt{x})^2\\ 1&=4x\\ \frac{1}{4}&=x\end{aligned}$$

Check to see if $x=\frac{1}{4}$ satisfies the original equation $\sqrt{x+2}-\sqrt{x}=1$

$$\begin{aligned}\sqrt{\tfrac{1}{4}+2}-\sqrt{\tfrac{1}{4}}&=1\\ \sqrt{\tfrac{9}{4}}-\tfrac{1}{2}&=1\\ \tfrac{3}{2}-\tfrac{1}{2}&=1\\ \tfrac{2}{2}&=1\\ 1&=1\checkmark\end{aligned}$$

Answer

The solution is $\frac{1}{4}$.

Because $(A+B)^2 \neq A^2+B^2$, we cannot simply square each term. For example, it is incorrect to square each term as follows.

$$(\sqrt{x+2})^2-(\sqrt{x})^2=(1)^2$$
$$Incorrect!$$

This is a common mistake and leads to an incorrect result. When squaring both sides of an equation with multiple terms, we must take care to apply the distributive property.

Example 5.6.10:

Solve: $\sqrt{2x+10}-\sqrt{x+6}=1$

Solution

Begin by isolating one of the radicals. In this case, add $\sqrt{x+6}$ to both sides of the equation.

$$\begin{aligned}\sqrt{2x+10}-\sqrt{x+6}&=1\\ \sqrt{2x+10}&=\sqrt{x+6}+1\end{aligned}$$

Next, square both sides. Take care to apply the distributive property to the right side.

$$\begin{aligned}(\sqrt{2x+10})^2&=(\sqrt{x+6}+1)^2\\ 2x+10&=x+6+2\sqrt{x+6}+1\\ 2x+10&=x+7+2\sqrt{x+6}\end{aligned}$$

At this point we have one term that contains a radical. Isolate it and square both sides again.

$$\begin{aligned}2x+10&=x+7+2\sqrt{x+6}\\ x+3&=2\sqrt{x+6}\\ (x+3)^2&=(2\sqrt{x+6})^2\\ x^2+6x+9&=4(x+6)\\ x^2+6x+9&=4x+24\\ x^2+2x-15&=0\\ (x-3)(x+5)&=0\end{aligned}$$

$$\begin{aligned}x-3&=0 \quad \text{or} \quad x+5=0\\ x&=3 \qquad\qquad x=-5\end{aligned}$$

Check.

Table 5.6.8

$Check x=3$	$Check x=-5$
$\sqrt{2x+10}-\sqrt{x+6}=1$ $\sqrt{2(3)+10}-\sqrt{3+6}=1$ $\sqrt{16}-\sqrt{9}=1$ $4-3=1$ $1=1$ ✓	$\sqrt{2x+10}-\sqrt{x+6}=1$ $\sqrt{2(-5)+10}-\sqrt{-5+6}=1$ $\sqrt{0}-\sqrt{1}=1$ $0-1=1$ $-1=1$ ✗

Answer

The solution is 3.

Exercise 5.6.2

Solve: $\sqrt{4x+21} - \sqrt{2x+22} = 1$

Answer

The solution is 7.

www.youtube.com/v/BtMeQWPsMaY

Key Takeaways

- Solve equations involving square roots by first isolating the radical and then squaring both sides. Squaring a square root eliminates the radical, leaving us with an equation that can be solved using the techniques learned earlier in our study of algebra.
- Squaring both sides of an equation introduces the possibility of extraneous solutions. For this reason, you must check your solutions in the original equation.
- Solve equations involving nth roots by first isolating the radical and then raise both sides to the nth power. This eliminates the radical and results in an equation that may be solved with techniques you have already mastered.
- When more than one radical term is present in an equation, isolate them one at a time, and apply the power property of equality multiple times until only a polynomial remains.

Exercise 5.6.3

Solve

1. $\sqrt{x} = 7$
2. $\sqrt{x} = 4$
3. $\sqrt{x} + 8 = 9$
4. $\sqrt{x} - 4 = 5$
5. $\sqrt{x} + 7 = 4$
6. $\sqrt{x} + 3 = 1$
7. $5\sqrt{x} - 1 = 0$
8. $3\sqrt{x} - 2 = 0$
9. $\sqrt{3x+1} = 2$
10. $\sqrt{5x-4} = 4$
11. $\sqrt{7x+4} + 6 = 11$
12. $\sqrt{3x-5} + 9 = 14$
13. $2\sqrt{x-1} - 3 = 0$
14. $3\sqrt{x+1} - 2 = 0$
15. $\sqrt{x+1} = \sqrt{x} + 1$
16. $\sqrt{2x-1} = \sqrt{2x} - 1$
17. $\sqrt{4x-1} = 2\sqrt{x} - 1$
18. $\sqrt{4x-11} = 2\sqrt{x} - 1$
19. $\sqrt{x+8} = \sqrt{x} - 4$
20. $\sqrt{25x-1} = 5\sqrt{x} + 1$
21. $\sqrt[3]{x} = 3$
22. $\sqrt[3]{x} = -4$
23. $\sqrt[3]{2x+9} = 3$
24. $\sqrt[3]{4x-11} = 1$
25. $\sqrt[3]{5x+7} + 3 = 1$
26. $\sqrt[3]{3x-6} + 5 = 2$
27. $4 - 2\sqrt[3]{x+2} = 0$
28. $6 - 3\sqrt[3]{2x-3} = 0$
29. $\sqrt[5]{3(x+10)} = 2$
30. $\sqrt[5]{4x+3} + 5 = 4$

31. $\sqrt{8x+11}=3\sqrt{x+1}$
32. $2\sqrt{3x-4}=\sqrt{2(3x+1)}$
33. $\sqrt{2(x+10)}=\sqrt{7x-15}$
34. $\sqrt{5(x-4)}=\sqrt{x+4}$
35. $\sqrt[3]{5x-2}=\sqrt[3]{4x}$
36. $\sqrt[3]{9(x-1)}=\sqrt[3]{3(x+7)}$
37. $\sqrt[3]{3x+1}=\sqrt[3]{2(x-1)}$
38. $\sqrt[3]{9x}=\sqrt[3]{3(x-6)}$
39. $\sqrt[5]{3x-5}=\sqrt[5]{2x+8}$
40. $\sqrt[5]{x+3}=\sqrt[5]{2x+5}$
41. $\sqrt{4x+21}=x$
42. $\sqrt{8x+9}=x$
43. $\sqrt{4(2x-3)}=x$
44. $\sqrt{3(4x-9)}=x$
45. $2\sqrt{x-1}=x$
46. $3\sqrt{2x-9}=x$
47. $\sqrt{9x+9}=x+1$
48. $\sqrt{3x+10}=x+4$
49. $\sqrt{x-1}=x-3$
50. $\sqrt{2x-5}=x-4$
51. $\sqrt{16-3x}=x-6$
52. $\sqrt{7-3x}=x-3$
53. $3\sqrt{2x+10}=x+9$
54. $2\sqrt{2x+5}=x+4$
55. $3\sqrt{x-1}-1=x$
56. $2\sqrt{2x+2}-1=x$
57. $\sqrt{10x+41}-5=x$
58. $\sqrt{6(x+3)}-3=x$
59. $\sqrt{8x^{2}-4x+1}=2x$
60. $\sqrt{18x^{2}-6x+1}=3x$
61. $5\sqrt{x+2}=x+8$
62. $4\sqrt{2(x+1)}=x+7$
63. $\sqrt{x^{2}-25}=x$
64. $\sqrt{x^{2}+9}=x$
65. $3+\sqrt{6x-11}=x$
66. $2+\sqrt{9x-8}=x$
67. $\sqrt{4x+25}-x=7$
68. $\sqrt{8x+73}-x=10$
69. $2\sqrt{4x+3}-3=2x$
70. $2\sqrt{6x+3}-3=3x$
71. $2x-4=\sqrt{14-10x}$
72. $3x-6=\sqrt{33-24x}$
73. $\sqrt[3]{x^{2}-24}=1$
74. $\sqrt[3]{x^{2}-54}=3$
75. $\sqrt[3]{x^{2}+6x}+1=4$
76. $\sqrt[3]{x^{2}+2x}+5=7$
77. $\sqrt[3]{25x^{2}-10x-7}=-2$
78. $\sqrt[3]{9x^{2}-12x-23}=-3$
79. $\sqrt[3]{4x^{2}-1}-2=0$
80. $4\sqrt[3]{x^{2}}-1=0$
81. $\sqrt[5]{x(2x+1)}-1=0$
82. $\sqrt[5]{3x^{2}-20x}-2=0$

83. $\sqrt{2x^2-15x+25}=\sqrt{(x+5)(x-5)}$
84. $\sqrt{x^2-4x+4}=\sqrt{x(5-x)}$
85. $\sqrt[3]{2\left(x^2+3x-20\right)}=\sqrt[3]{(x+3)^2}$
86. $\sqrt[3]{3x^2+3x+40}=\sqrt[3]{(x-5)^2}$
87. $\sqrt{2x-5}+\sqrt{2x}=5$
88. $\sqrt{4x+13}-2\sqrt{x}=3$
89. $\sqrt{8x+17}-2\sqrt{2-x}=3$
90. $\sqrt{3x-6}-\sqrt{2x-3}=1$
91. $\sqrt{2(x-2)}-\sqrt{x-1}=1$
92. $\sqrt{2x+5}-\sqrt{x+3}=2$
93. $\sqrt{2(x+1)}-\sqrt{3x+4}-1=0$
94. $\sqrt{6-5x}+\sqrt{3-3x}-1=0$
95. $\sqrt{x-2}-1=\sqrt{2(x-3)}$
96. $\sqrt{14-11x}+\sqrt{7-9x}=1$
97. $\sqrt{x+1}=\sqrt{3}-\sqrt{2-x}$
98. $\sqrt{2x+9}-\sqrt{x+1}=2$
99. $x^{1/2}-10=0$
100. $x^{1/2}-6=0$
101. $x^{1/3}+2=0$
102. $x^{1/3}+4=0$
103. $(x-1)^{1/2}-3=0$
104. $(x+2)^{1/2}-6=0$
105. $(2x-1)^{1/3}+3=0$
106. $(3x-1)^{1/3}-2=0$
107. $(4x+15)^{1/2}-2x=0$
108. $(3x+2)^{1/2}-3x=0$
109. $(2x+12)^{1/2}-x=6$
110. $(4x+36)^{1/2}-x=9$
111. $2(5x+26)^{1/2}=x+10$
112. $3(x-1)^{1/2}=x+1$
113. $x^{1/2}+(3x-2)^{1/2}=2$
114. $(6x+1)^{1/2}-(3x)^{1/2}=1$
115. $(3x+7)^{1/2}+(x+3)^{1/2}-2=0$
116. $(3x)^{1/2}+(x+1)^{1/2}-5=0$

Answer

1. 49

3. 1

5. $\varnothing$

7. $\frac{1}{25}$

9. 1

11. 3

13. $\frac{13}{4}$

15. 0

17. $\frac{1}{4}$

19. $\varnothing$

21. 27

23. 9

25. -3

27. 6

29. $\frac{2}{3}$

31. 2

33. 7

35. 2

37. -3

39. 13

41. 7

43. $2, 6$

45. 2

47. $-1, 8$

49. 5

51. $\varnothing$

53. $-3, 3$

55. $2, 5$

57. $-4, 4$

59. $\frac{1}{2}$

61. $2, 7$

63. $\varnothing$

65. 10

67. $-6, -4$

69. $-\frac{1}{2}, \frac{3}{2}$

71. $\varnothing$

73. $-5, 5$

75. $-9, 3$

77. $\frac{1}{5}$

79. $-\frac{3}{2}, \frac{3}{2}$

81. $-1, \frac{1}{2}$

83. $5, 10$

85. $-7, 7$

87. $\frac{9}{2}$

89. 1

91. 10

93. $\varnothing$

95. 3

97. $1, 2$

99. 100

101. -8
103. 10
105. -13
107. $\frac{5}{2}$
109. $-6, -4$
111. $-2, 2$
113. 1
115. -2

Exercise 5.6.4

Determine the roots of the given functions. Recall that a root is a value in the domain that results in zero. In other words, find x where $f(x)=0$.

1. $f(x)=\sqrt{x+5}-2$
2. $f(x)=\sqrt{2x-3}-1$
3. $f(x)=2\sqrt{x+2}-8$
4. $f(x)=3\sqrt{x-7}-6$
5. $f(x)=\sqrt[3]{x+1}+2$
6. $f(x)=2\sqrt[3]{x-1}+6$

Answer

1. -1
3. 14
5. -9

Exercise 5.6.5

Solve for the indicated variable.

1. Solve for P $: r=\sqrt{P}-1$
2. Solve for $x : y=\sqrt{x-h}+k$
3. Solve for $s : t=\sqrt{\frac{2s}{g}}$
4. Solve for L $: T=2\pi\sqrt{\frac{L}{32}}$
5. Solve for R $: I=\sqrt{\frac{P}{R}}$
6. Solve for $h : r=\sqrt{\frac{3V}{\pi h}}$
7. Solve for $V : r=\sqrt[3]{\frac{3V}{4\pi}}$
8. Solve for $c : a=\sqrt[3]{\frac{b^2\pi}{2c}}$
9. The square root of 1 less than twice a number is equal to 2 less than the number. Find the number.
10. The square root of 4 less than twice a number is equal to 6 less than the number. Find the number.
11. The square root of twice a number is equal to one-half of that number. Find the number.
12. The square root of twice a number is equal to one-third of that number. Find the number.
13. The distance d in miles a person can see an object on the horizon is given by the formula $d=\frac{\sqrt{6h}}{2}$ where h represents the height in feet of the person's eyes above sea level. How high must a person's eyes be to see an object 5 miles away?
14. The current I measured in amperes is given by the formula $I=\sqrt{\frac{P}{R}}$ where P is the power usage measured in watts and R is the resistance measured in ohms. If a light bulb requires $1/2$ amperes of current and uses 60 watts of power, then what is the resistance through the bulb?

Answer

1. $P = (r + 1)^2$

3. $s = \frac{gt^2}{2}$

5. $R = \frac{P}{I^2}$

7. $V = \frac{4\pi r^3}{3}$

9. 5

11. $0, 8$

13. $16\frac{2}{3}$ feet

Exercise 5.6.6

The period of a pendulum T in seconds is given by the formula

$$T = 2\pi\sqrt{\frac{L}{32}}$$

where L represents the length in feet. Calculate the length of a pendulum given the period. Give the exact value and the approximate value rounded to the nearest tenth of a foot.

1. 1 second
2. 2 seconds
3. $\frac{1}{2}$ second
4. $\frac{1}{3}$ second

Answer

1. $\frac{8}{\pi^2}$ feet; 0.8 feet

3. $\frac{2}{\pi^2}$ feet; 0.2 feet

Exercise 5.6.7

The time t in seconds, an object is in free fall is given by the formula

$$t = \frac{\sqrt{s}}{4}$$

where s represents the distance it has fallen, in feet. Calculate the distance an object will fall given the amount of time.

1. 1 second
2. 2 seconds
3. $\frac{1}{2}$ second
4. $\frac{1}{4}$ second

Answer

1. 16 feet

3. 4 feet

Exercise 5.6.9

1. Discuss reasons why we sometimes obtain extraneous solutions when solving radical equations. Are there ever any conditions where we do not need to check for extraneous solutions? Why or why not?
2. If an equation has multiple terms, explain why squaring all of them is incorrect. Provide an example.

Answer

1. Answer may vary

Footnotes

[22] Any equation that contains one or more radicals with a variable in the radicand.

[23] Given real numbers a and b, where $a = b$, then $a^2 = b^2$.

[24] A properly found solution that does not solve the original equation.\

[25] Given any positive integer n and real numbers a and b where $a = b$, then $a^n = b^n$.

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5.7: Complex Numbers and Their Operations

Learning Objectives

- Define the imaginary unit and complex numbers.
- Add and subtract complex numbers.
- Multiply and divide complex numbers.

Introduction to Complex Numbers

Up to this point the square root of a negative number has been left undefined. For example, we know that $\sqrt{-9}$ is not a real number.

$$\sqrt{-9} = ? \quad \text{or} \quad (?)^2 = -9$$

There is no real number that when squared results in a negative number. We begin to resolve this issue by defining the **imaginary unit**[26], i, as the square root of -1.

$$i = \sqrt{-1} \quad \text{and} \quad i^2 = -1$$

To express a square root of a negative number in terms of the imaginary unit i, we use the following property where a represents any non-negative real number:

$$\sqrt{-a} = \sqrt{-1 \cdot a} = \sqrt{-1} \cdot \sqrt{a} = i\sqrt{a}$$

With this we can write.

$$\sqrt{-9} = \sqrt{-1 \cdot 9} = \sqrt{-1} \cdot \sqrt{9} = i \cdot 3 = 3i$$

If $\sqrt{-9} = 3i$, then we would expect that $3i$ squared will equal -9:

$$(3i)^2 = 9i^2 = 9(-1) = -9 \ ✓$$

In this way any square root of a negative real number can be written in terms of the imaginary unit. Such a number is often called an **imaginary number**[27].

Example 5.7.1

Rewrite in terms of the imaginary unit i.

a. $\sqrt{-7}$
b. $\sqrt{-25}$
c. $\sqrt{-72}$

Solution

a. $\sqrt{-7} = \sqrt{-1 \cdot 7} = \sqrt{-1} \cdot \sqrt{7} = i\sqrt{7}$
b. $\sqrt{-25} = \sqrt{-1 \cdot 25} = \sqrt{-1} \cdot \sqrt{25} = i \cdot 5 = 5i$
c. $\sqrt{-72} = \sqrt{-1 \cdot 36 \cdot 2} = \sqrt{-1} \cdot \sqrt{36} \cdot \sqrt{2} = i \cdot 6 \cdot \sqrt{2} = 6i\sqrt{2}$

> When an imaginary number involves a radical, we place i in front of the radical. Consider the following:
>
> $$6i\sqrt{2} = 6\sqrt{2}i$$
>
> Since multiplication is commutative, these numbers are equivalent. However, in the form $6\sqrt{2}i$, the imaginary unit i is often misinterpreted to be part of the radicand. To avoid this confusion, it is a best practice to place i in front of the radical and use $6i\sqrt{2}$

A **complex number**[28] is any number of the form,

$$a + bi$$

where a and b are real numbers. Here, a is called the **real part**[29] and b is called the **imaginary part**[30]. For example, $3 - 4i$ is a complex number with a real part of 3 and an imaginary part of -4. It is important to note that any real number is also a complex number. For example, 5 is a real number; it can be written as $5+0i$ with a real part of 5 and an imaginary part of 0. Hence, the set of real numbers, denoted $\mathbb{R}$, is a subset of the set of complex numbers, denoted $\mathbb{C}$.

$$C=\{a+bi|a,b\in\mathbb{R}\}$$

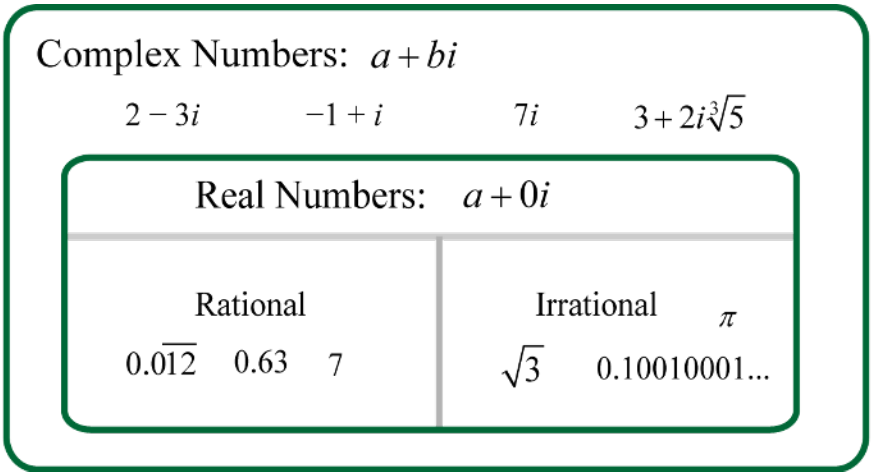

Figure $5.7.1$

Complex numbers are used in many fields including electronics, engineering, physics, and mathematics. In this textbook we will use them to better understand solutions to equations such as $x^2+4=0$. For this reason, we next explore algebraic operations with them.

Adding and Subtracting Complex Numbers

Adding or subtracting complex numbers is similar to adding and subtracting polynomials with like terms. We add or subtract the real parts and then the imaginary parts.

Example $5.7.2$

Add $(5-2i)+(7+3i)$.

Solution

Add the real parts and then add the imaginary parts.

$$\begin{aligned}(5-2i)+(7+3i)&=5-2i+7+3i\\&=5+7-2i+3i\\&=12+i\end{aligned}$$

Answer

$12+i$

To subtract complex numbers, we subtract the real parts and subtract the imaginary parts. This is consistent with the use of the distributive property.

Example $5.7.3$:

Subtract $(10-7i)-(9+5i)$.

Solution

Distribute the negative sign and then combine like terms.

$$\begin{aligned}(10-7i)-(9+5i)&=10-7i-9-5i\\&=10-9-7i-5i\\&=1-12i\end{aligned}$$

Answer:

$1-12i$

In general, given real numbers a, b, c, and d:

$$\begin{aligned}(a+bi)+(c+di)&=(a+c)+(b+d)i\\(a+bi)-(c+di)&=(a-c)+(b-d)i\end{aligned}$$

Example 5.7.4:

Simplify $(5+i)+(2-3i)-(4-7i)$.

Solution

$$\begin{aligned}(5+i)+(2-3i)-(4-7i)&=5+i+2-3i-4+7i\\&=3+5i\end{aligned}$$

Answer:

$3+5i$

In summary, adding and subtracting complex numbers results in a complex number.

Multiplying and Dividing Complex Numbers

Multiplying complex numbers is similar to multiplying polynomials. The distributive property applies. In addition, we make use of the fact that $i^2=-1$ to simplify the result into standard form $a+bi$.

Example 5.7.5:

Multiply $-6i(2-3i)$.

Solution

We begin by applying the distributive property.

$$\begin{aligned}-6i(2-3i)&=(-6i)\cdot 2-(-6i)\cdot 3i & &\textit{Distribute.}\\&=-12i+18i^2 & &\textit{Substitute } i^2=-1.\\&=-12i+18(-1) & &\textit{Simplify.}\\&=-12i-18\\&=-18-12i\end{aligned}$$

Answer:

$-18-12i$

Example 5.7.6:

Multiply $(3-4i)(4+5i)$.

Solution

$$\begin{aligned}(3-4i)(4+5i)&=3\cdot 4+3\cdot 5i-4i\cdot 4-4i\cdot 5i & &\textit{Distribute.}\\&=12+15i-16i-20i^2 & &\textit{Substitute } i^2=-1.\\&=12+15i-16i-20(-1)\\&=12-i+20\\&=32-i\end{aligned}$$

Answer:

$32-i$

In general, given real numbers a, b, c, and d:

$$\begin{aligned}(a+bi)(c+di)&=ac+adi+bci+bdi^2\\&=ac+adi+bci+bd(-1)\\&=ac+(ad+bc)i-bd\\&=(ac-bd)+(ad+bc)i\end{aligned}$$

Exercise 5.7.1

Simplify: $(3-2i)^2$.

Answer

$5-12i$

www.youtube.com/v/OVtqcB2z9v8

Given a complex number $a+bi$, its **complex conjugate**[31] is $a-bi$.We next explore the product of complex conjugates.

Example 5.7.7:

Multiply $(5+2i)(5-2i)$.

Solution

$$\begin{aligned}(5+2i)(5-2i) &= 5\cdot 5-5\cdot 2i+2i\cdot 5-2i\cdot 2i\\ &= 25-10i+10i-4i^2\\ &= 25-4(-1)\\ &= 25+4\\ &= 29\end{aligned}$$

Answer:

29

In general, the **product of complex conjugates**[32] follows:

$$\begin{aligned}(a+bi)(a-bi) &= a^2-a\cdot bi+bi\cdot a-b^2i^2\\ &= a^2-abi+abi-b^2(-1)\\ &= a^2+b^2\end{aligned}$$

Note that the result does not involve the imaginary unit; hence, it is real. This leads us to the very useful property

$$(a+bi)(a-bi)=a^2+b^2$$

To divide complex numbers, we apply the technique used to rationalize the denominator. Multiply the numerator and denominator by the conjugate of the denominator. The result can then be simplified into standard form $a+bi$.

Example 5.7.8:

Divide $\frac{1}{2-3i}$.

Solution

In this example, the conjugate of the denominator is $2+3i$. Therefore, we will multiply by 1 in the form $\frac{(2+3i)}{(2+3i)}$.

$$\begin{aligned}\frac{1}{2-3i} &= \frac{1}{(2-3i)}\cdot\frac{(2+3i)}{(2+3i)}\\ &= \frac{(2+3i)}{2^2+3^2}\\ &= \frac{2+3i}{4+9}\\ &= \frac{2+3i}{13}\end{aligned}$$

To write this complex number in standard form, we make use of the fact that 13 is a common denominator.

$$\begin{aligned}\frac{2+3i}{13} &= \frac{2}{13}+\frac{3i}{13}\\ &= \frac{2}{13}+\frac{3}{13}i\end{aligned}$$

Answer

$\frac{2}{13}+\frac{3}{13}i$

Example 5.7.9:

Divide: $\frac{1-5i}{4+i}$.

Solution

$$\begin{aligned}\frac{1-5i}{4+i} &= \frac{(1-5i)}{(4+i)} \cdot \frac{(4-i)}{(4-i)} \\ &= \frac{4-i-20i+5i^2}{4^2+1^2} \\ &= \frac{4-21i+5(-1)}{16+1} \\ &= \frac{4-21i-5}{16+1} \\ &= \frac{-1-21i}{17} \\ &= -\frac{1}{17}-\frac{21}{17}i\end{aligned}$$

Answer

$-\frac{1}{17}-\frac{21}{17}i$

In general, given real numbers a, b, c and d where c and d are not both 0:

$$\begin{aligned}\frac{(a+bi)}{(c+di)} &= \frac{(a+bi)}{(c+di)} \cdot \frac{(c-di)}{(c-di)} \\ &= \frac{ac-adi+bci-bdi^2}{c^2+d^2} \\ &= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} \\ &= \left(\frac{ac+bd}{c^2+d^2}\right)+\left(\frac{bc-ad}{c^2+d^2}\right)i\end{aligned}$$

Example 5.7.10:

Divide: $\frac{8-3i}{2i}$.

Solution

Here we can think of $2i=0+2i$ and thus we can see that its conjugate is $-2i=0-2i$.

$$\begin{aligned}\frac{8-3i}{2i} &= \frac{(8-3i)}{(2i)} \cdot \frac{(-2i)}{(-2i)} \\ &= \frac{-16i+6i^2}{-4i^2} \\ &= \frac{-16i+6(-1)}{-4(-1)} \\ &= \frac{-16i-6}{4} \\ &= \frac{-6-16i}{4} \\ &= \frac{-6}{4}-\frac{-16i}{4} \\ &= -\frac{3}{2}-4i\end{aligned}$$

Because the denominator is a monomial, we could multiply numerator and denominator by 1 in the form of $\frac{i}{i}$ and save some steps reducing in the end.

$$\begin{aligned}\frac{8-3i}{2i} &= \frac{(8-3i)}{(2i)}\cdot\frac{i}{i}\\ &= \frac{8i-3i^2}{2i^2}\\ &= \frac{8i-3(-1)}{2(-1)}\\ &= \frac{8i+3}{-2}\\ &= \frac{8i}{-2}+\frac{3}{-2}\\ &= -4i-\frac{3}{2}\end{aligned}$$

Answer

$-\frac{3}{2}-4i$

Exercise 5.7.2

Divide $\frac{3+2i}{1-i}$.

Answer

$\frac{1}{2}+\frac{5}{2}i$

www.youtube.com/v/4r2kolq-_T0

When multiplying and dividing complex numbers we must take care to understand that the product and quotient rules for radicals require that both a and b are positive. In other words, if $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are both real numbers then we have the following rules.

$$\begin{aligned} Product\ rule\ for\ radicals: &\quad \sqrt[n]{a\cdot b}=\sqrt[n]{a}\cdot\sqrt[n]{b}\\ Quotient\ rule\ for\ radicals: &\quad \sqrt[n]{\tfrac{a}{b}}=\tfrac{\sqrt[n]{a}}{\sqrt[n]{b}}\end{aligned}$$

For example, we can demonstrate that the product rule is true when a and b are both positive as follows:

$$\begin{aligned}\sqrt{4}\cdot\sqrt{9}&=\sqrt{36}\\ 2\cdot 3&=6\\ 6&=6\ \checkmark\end{aligned}$$

However, when a and b are both negative the property is not true.

$$\begin{aligned}\sqrt{-4}\cdot\sqrt{-9}&\overset{?}{=}\sqrt{36}\\ 2i\cdot 3i&=6\\ 6i^2&=6\\ -6&=6\ ✗\end{aligned}$$

Here $\sqrt{-4}$ and $\sqrt{-9}$ both are not real numbers and the product rule for radicals fails to produce a true statement. Therefore, to avoid some common errors associated with this technicality, ensure that any complex number is written in terms of the imaginary unit i before performing any operations.

Example 5.7.11:

Multiply $\sqrt{-6}\cdot\sqrt{-15}$.

Solution

Begin by writing the radicals in terms of the imaginary unit i.

$$\sqrt{-6}\cdot\sqrt{-15}=i\sqrt{6}\cdot i\sqrt{15}$$

Now the radicands are both positive and the product rule for radicals applies.

$$\begin{aligned}\sqrt{-6}\cdot\sqrt{-15} &= i\sqrt{6}\cdot i\sqrt{15}\\ &= i\sqrt{6\cdot 15}\\ &= (-1)\sqrt{90}\\ &= (-1)\sqrt{9\cdot 10}\\ &= (-1)\cdot 3\cdot\sqrt{10}\\ &= -3\sqrt{10}\end{aligned}$$

Answer

$-3\sqrt{10}$

Example 5.7.12:

Multiply: $\sqrt{-10}(\sqrt{-6}-\sqrt{10})$.

Solution

Begin by writing the radicals in terms of the imaginary unit i and then distribute.

$$\begin{aligned}\sqrt{-10}(\sqrt{-6}-\sqrt{10}) &= i\sqrt{10}(i\sqrt{6}-\sqrt{10})\\ &= i^2\sqrt{60}-i\sqrt{100}\\ &= (-1)\sqrt{4\cdot 15}-i\sqrt{100}\\ &= (-1)\cdot 2\cdot\sqrt{15}-i\cdot 10\\ &= -2\sqrt{15}-10i\end{aligned}$$

Answer:

$-2\sqrt{15}-10i$

In summary, multiplying and dividing complex numbers results in a complex number.

Exercise 5.7.3

Simplify: $(2i\sqrt{2})^2-(3-i\sqrt{5})^2$.

Answer

$-12+6i\sqrt{5}$

www.youtube.com/v/TONiI5oqTTg

Key Takeaways

- The imaginary unit i is defined to be the square root of negative one. In other words,$i=\sqrt{-1}$and $i^2=-1$.
- Complex numbers have the form $a+bi$ where a and b are real numbers.
- The set of real numbers is a subset of the complex numbers.
- The result of adding, subtracting, multiplying, and dividing complex numbers is a complex number.
- The product of complex conjugates, $a+bi$ and $a-bi$, is a real number. Use this fact to divide complex numbers. Multiply the numerator and denominator of a fraction by the complex conjugate of the denominator and then simplify.
- Ensure that any complex number is written in terms of the imaginary unit i before performing any operations.

Exercise 5.7.4

Rewrite in terms of imaginary unit i.

1. $\sqrt{-81}$
2. $\sqrt{-64}$
3. $-\sqrt{-4}$
4. $-\sqrt{-36}$
5. $\sqrt{-20}$
6. $\sqrt{-18}$

7. $\sqrt{-50}$
8. $\sqrt{-48}$
9. $-\sqrt{-45}$
10. $-\sqrt{-8}$
11. $\sqrt{-\frac{1}{16}}$
12. $\sqrt{-\frac{2}{9}}$
13. $\sqrt{-0.25}$
14. $\sqrt{-1.44}$

Answer

1. $9i$

3. $-2i$

5. $2i\sqrt{5}$

7. $5i\sqrt{2}$

9. $-3i\sqrt{5}$

11. $\frac{i}{4}$

13. $0.5i$

Exercise 5.7.5

Write the complex number in standard form $a+bi$.

1. $5-2\sqrt{-4}$
2. $3-5\sqrt{-9}$
3. $-2+3\sqrt{-8}$
4. $4-2\sqrt{-18}$
5. $\frac{3-\sqrt{-24}}{6}$
6. $\frac{2+\sqrt{-75}}{10}$
7. $\frac{\sqrt{-63}-\sqrt{5}}{-12}$
8. $\frac{-\sqrt{-72}+\sqrt{8}}{-24}$

Answer

1. $5-4i$

3. $-2+6i\sqrt{2}$

5. $\frac{1}{2}-\frac{\sqrt{6}}{3}i$

7. $\frac{\sqrt{5}}{12}-\frac{\sqrt{7}}{4}i$

Exercise 5.7.6

Given that $i^{12}=-1$ compute the following powers of i.

1. i^3
2. i^4
3. i^5
4. i^6
5. i^{15}
6. i^{24}

Answer

1. $-i$

3. i
5. $-i$

Exercise $5.7.7$

Perform the operations.

1. $(3+5i)+(7-4i)$
2. $(6-7i)+(-5-2i)$
3. $(-8-3i)+(5+2i)$
4. $(-10+15i)+(15-20i)$
5. $\left(\frac{1}{2}+\frac{3}{4}i\right)+\left(\frac{1}{6}-\frac{1}{8}i\right)$
6. $\left(\frac{2}{5}-\frac{1}{6}i\right)+\left(\frac{1}{10}-\frac{3}{2}i\right)$
7. $(5+2i)-(8-3i)$
8. $(7-i)-(-6-9i)$
9. $(-9-5i)-(8+12i)$
10. $(-11+2i)-(13-7i)$
11. $\left(\frac{1}{14}+\frac{3}{2}i\right)-\left(\frac{4}{7}-\frac{3}{4}i\right)$
12. $\left(\frac{3}{8}-\frac{1}{3}i\right)-\left(\frac{1}{2}-\frac{1}{2}i\right)$
13. $(2-i)+(3+4i)-(6-5i)$
14. $(7+2i)-(6-i)-(3-4i)$
15. $\left(\frac{1}{3}-i\right)-\left(1-\frac{1}{2}i\right)-\left(\frac{1}{6}+\frac{1}{6}i\right)$
16. $\left(1-\frac{3}{4}i\right)+\left(\frac{5}{2}+i\right)-\left(\frac{1}{4}-\frac{5}{8}i\right)$
17. $(5-3i)-(2+7i)-(1-10i)$
18. $(6-11i)+(2+3i)-(8-4i)$
19. $\sqrt{-16}-(3-\sqrt{-1})$
20. $\sqrt{-100}+(\sqrt{-9}+7)$
21. $(1+\sqrt{-1})-(1-\sqrt{-1})$
22. $(3-\sqrt{-81})-(5-3\sqrt{-9})$
23. $(5-2\sqrt{-25})-(-3+4\sqrt{-1})$
24. $(-12-\sqrt{-1})-(3-\sqrt{-49})$

Answer

1. $10+i$
3. $-3-i$
5. $\frac{2}{3}+\frac{5}{8}i$
7. $-3+5i$
9. $-17-17i$
11. $-\frac{1}{2}+\frac{9}{4}i$
13. $-1+8i$
15. $-\frac{5}{6}-\frac{2}{3}i$
17. 2
19. $-3+5i$
21. $2i$
23. $8-14i$

Exercise $5.7.8$

Perform the operations.

1. $i(1-i)$
2. $i(1+i)$
3. $2i(7-4i)$
4. $6i(1-2i)$
5. $-2i(3-4i)$
6. $-5i(2-i)$
7. $(2+i)(2-3i)$
8. $(3-5i)(1-2i)$
9. $(1-i)(8-9i)$
10. $(1+5i)(5+2i)$
11. $(4+3i)^2$
12. $(-1+2i)^2$
13. $(2-5i)^2$
14. $(5-i)^2$
15. $(1+i)(1-i)$
16. $(2-i)(2+i)$
17. $(4-2i)(4+2i)$
18. $(6+5i)(6-5i)$
19. $\left(\frac{1}{2}+\frac{2}{3}i\right)\left(\frac{1}{3}-\frac{1}{2}i\right)$
20. $\left(\frac{2}{3}-\frac{1}{3}i\right)\left(\frac{1}{2}-\frac{3}{2}i\right)$
21. $(2-i)^3$
22. $(1-3i)^3$
23. $\sqrt{-2}(\sqrt{-2}-\sqrt{6})$
24. $\sqrt{-1}(\sqrt{-1}+\sqrt{8})$
25. $\sqrt{-6}(\sqrt{10}-\sqrt{-6})$
26. $\sqrt{-15}(\sqrt{3}-\sqrt{-10})$
27. $(2-3\sqrt{-2})(2+3\sqrt{-2})$
28. $(1+\sqrt{-5})(1-\sqrt{-5})$
29. $(1-3\sqrt{-4})(2+\sqrt{-9})$
30. $(2-3\sqrt{-1})(1+2\sqrt{-16})$
31. $(2-3i\sqrt{2})(3+i\sqrt{2})$
32. $(-1+i\sqrt{3})(2-2i\sqrt{3})$
33. $\frac{-3}{i}$
34. $\frac{5}{i}$
35. $\frac{1}{5+4i}$
36. $\frac{1}{3-4i}$
37. $\frac{15}{1-2i}$
38. $\frac{29}{5+2i}$
39. $\frac{20i}{1-3i}$
40. $\frac{10i}{1+2i}$
41. $\frac{10-5i}{3-i}$
42. $\frac{5-2i}{1-2i}$
43. $\frac{5+10i}{3+4i}$
44. $\frac{2-4i}{5+3i}$
45. $\frac{26+13i}{2-3i}$
46. $\frac{4+2i}{1+i}$
47. $\frac{3-i}{2i}$
48. $\frac{-5+2i}{4i}$
49. $\frac{1}{a-bi}$

50. $\frac{1}{a+bi}$
51. $\frac{1-\sqrt{-1}}{1+\sqrt{-1}}$
52. $\frac{1+\sqrt{-9}}{1-\sqrt{-9}}$
53. $\frac{-\sqrt{-6}}{\sqrt{18}+\sqrt{-4}}$
54. $\frac{\sqrt{-12}}{\sqrt{2}-\sqrt{-27}}$

Answer

1. $1+i$

3. $8+14i$

5. $-8-6i$

7. $7-4i$

9. $-1-17i$

11. $7+24i$

13. $-21-20i$

15. 2

17. 20

19. $\frac{1}{2}-\frac{1}{36}i$

21. $2-11i$

23. $-2-2i\sqrt{3}$

25. $6+2i\sqrt{15}$

27. 22

29. $20-9i$

31. $12-7i\sqrt{2}$

33. $3i$

35. $\frac{5}{41}-\frac{4}{41}i$

37. $3+6i$

39. $-6+2i$

41. $\frac{7}{2}-\frac{1}{2}i$

43. $\frac{11}{5}-\frac{2}{5}i$

45. $1+8i$

47. $-\frac{1}{2}-\frac{3}{2}i$

49. $\frac{a}{a^2+b^2}+\frac{b}{a^2+b^2}i$

51. $-i$

53. $-\frac{\sqrt{6}}{11}-\frac{3\sqrt{3}}{11}i$

Exercise 5.7.9

Given that $i^{-n}=\frac{1}{i^n}$ compute the following powers of i.

1. i^{-1}
2. i^{-2}
3. i^{-3}
4. i^{-4}

Answer

1. $-i$

3. i

Exercise 5.7.10

Perform the operations and simplify.

1. $2i(2-i)-i(3-4i)$
2. $i(5-i)-3i(1-6i)$
3. $5-3(1-i)^2$
4. $2(1-2i)^2+3i$
5. $(1-i)^2-2(1-i)+2$
6. $(1+i)^2-2(1+i)+2$
7. $(2i\sqrt{2})^2+5$
8. $(3i\sqrt{5})^2-(i\sqrt{3})^2$
9. $(\sqrt{2}-i)^2-(\sqrt{2}+i)^2$
10. $(i\sqrt{3}+1)^2-(4i\sqrt{2})^2$
11. $\left(\frac{1}{1+i}\right)^2$
12. $\left(\frac{1}{1+i}\right)^3$
13. $(a-bi)^2-(a+bi)^2$
14. $\left(a^2+ai+1\right)\left(a^2-ai+1\right)$
15. Show that both $-2i$ and $2i$ satisfy $x^2+4=0$.
16. Show that both $-i$ and i satisfy $x^2+1=0$.
17. Show that both $3-2i$ and $3+2i$ satisfy $x^2-6x+13=0$.
18. Show that both $5-i$ and $5+i$ satisfy $x^2-10x+26=0$.
19. Show that $3, -2i$, and $2i$ are all solutions to $x^3-3x^2+4x-12=0$.
20. Show that $-2, 1, -i$, and $1+i$ are all solutions to $x^3-2x+4=0$.

Answer

1. $-2+i$

3. $5+6i$

5. 0

7. -3

9. $-4i\sqrt{2}$

11. $-\frac{i}{2}$

13. $-4abi$

15. Proof

17. Proof

19. Proof

Exercise 5.7.11

1. Research and discuss the history of the imaginary unit and complex numbers.
2. How would you define i^0 and why?
3. Research what it means to calculate the absolute value of a complex number $|a+bi|$. Illustrate your finding with an example.
4. Explore the powers of i. Look for a pattern and share your findings.

Answer

1. Answer may vary
3. Answer may vary

Footnotes

[26] Defined as $i=\sqrt{-1}$ where $i^2=-1$.

[27] A square root of any negative real number.

[28] A number of the form $a+bi$, where a and b are real numbers.

[29] The real number a of a complex number $a+bi$.

[30] The real number b of a complex number $a+bi$.

[31] Two complex numbers whose real parts are the same and imaginary parts are opposite. If given $a+bi$, then its complex conjugate is $a-bi$.

[32] The real number that results from multiplying complex conjugates: $(a+bi)(a-bi)=a^2+b^2$.

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5.E: Radical Functions and Equations (Exercises)

Exercise $5.E.1$

Simplify.

1. $-\sqrt{121}$
2. $\sqrt{(-7)^2}$
3. $\sqrt{(xy)^2}$
4. $\sqrt{(6x-7)^2}$
5. $\sqrt[3]{125}$
6. $\sqrt[3]{-27}$
7. $\sqrt[3]{(xy)^3}$
8. $\sqrt[3]{(6x+1)^3}$
9. Given $f(x)=\sqrt{x+10}$, find $f(-1)$ and $f(6)$.
10. Given $g(x)=\sqrt[3]{x-5}$, find $g(4)$ and $g(13)$.
11. Determine the domain of the function defined by $g(x)=\sqrt{5x+2}$.
12. Determine the domain of the function defined by $g(x)=\sqrt[3]{3x-1}$.

Answer

1. -11

3. $|xy|$

5. 5

7. xy

9. $f(-1)=3; f(6)=4$

11. $[-\frac{2}{5},\infty)$

Exercise $5.E.2$

Simplify.

1. $\sqrt[3]{250}$
2. $4\sqrt[3]{120}$
3. $-3\sqrt[3]{108}$
4. $10\sqrt[5]{\frac{1}{32}}$
5. $-6\sqrt[4]{\frac{81}{16}}$
6. $\sqrt[6]{128}$
7. $\sqrt[5]{-192}$
8. $-3\sqrt{420}$

Answer

1. $5\sqrt[3]{2}$

3. $-9\sqrt[3]{4}$

5. -9

7. $-2\sqrt[5]{6}$

Exercise $5.E.3$

Simplify.

1. $\sqrt{20x^4y^3}$
2. $-4\sqrt{54x^6y^3}$
3. $\sqrt{x^2-14x+49}$

4. $\sqrt{(x-8)^4}$

Answer

1. $2x^2|y|\sqrt{5y}$
3. $|x-7|$

Exercise $5.E.4$

Simplify. (Assume all variable expressions are nonzero.)

1. $\sqrt{100x^2y^4}$
2. $\sqrt{36a^6b^2}$
3. $\sqrt{\frac{8a^2}{b^4}}$
4. $\sqrt{\frac{72x^4y}{z^6}}$
5. $10x\sqrt{150x^7y^4}$
6. $-5n^2\sqrt{25m^{10}n^6}$
7. $\sqrt[3]{48x^6y^3z^2}$
8. $\sqrt[3]{270a^{10}b^8c^3}$
9. $\sqrt[3]{\frac{a^3b^5}{64c^6}}$
10. $\sqrt[5]{\frac{a^{26}}{32b^5c^{10}}}$
11. The period T in seconds of a pendulum is given by the formula $T=2\pi\sqrt{\frac{L}{32}}$ where L represents the length in feet of the pendulum. Calculate the period of a pendulum that is $2\frac{1}{2}$ feet long. Give the exact answer and the approximate answer to the nearest hundredth of a second.
12. The time in seconds an object is in free fall is given by the formula $t=\frac{\sqrt{s}}{4}$ where s represents the distance in feet the object has fallen. How long does it take an object to fall 28 feet? Give the exact answer and the approximate answer to the nearest tenth of a second.
13. Find the distance between $(-5,6)$ and $(-3,-4)$.
14. Find the distance between $\left(\frac{2}{3},-\frac{1}{2}\right)$ and $\left(1,-\frac{3}{4}\right)$.

Answer

1. $10xy^2$
3. $\frac{2a\sqrt{2}}{b^2}$
5. $50x^4y^2\sqrt{6x}$
7. $2x^2y\sqrt[3]{6z^2}$
9. $\frac{ab\sqrt[3]{b^2}}{4c^2}$
11. $\frac{\pi\sqrt{5}}{4}$ seconds; 1.76 seconds
13. $2\sqrt{26}$ units

Exercise $5.E.5$

Determine whether or not the three points form a right triangle. Use the Pythagorean theorem to justify your answer.

1. $(-4,5),(-3,-1),\text{ and }(3,0)$
2. $(-1,-1),(1,3),\text{ and }(-6,1)$

Answer

1. Right triangle

Exercise $5.E.6$

Simplify. Assume all radicands containing variables are nonnegative.

1. $7\sqrt{2}+5\sqrt{2}$
2. $8\sqrt{15}-2\sqrt{15}$
3. $14\sqrt{3}+5\sqrt{2}-5\sqrt{3}-6\sqrt{2}$
4. $22\sqrt{ab}-5a\sqrt{b}+7\sqrt{ab}-2a\sqrt{b}$
5. $7\sqrt{x}-(3\sqrt{x}+2\sqrt{y})$
6. $(8y\sqrt{x}-7x\sqrt{y})-(5x\sqrt{y}-12y\sqrt{x})$
7. $(3\sqrt{5}+2\sqrt{6})+(8\sqrt{5}-3\sqrt{6})$
8. $(4\sqrt[3]{3}-\sqrt[3]{12})-(5\sqrt[3]{3}-2\sqrt[3]{12})$
9. $(2-\sqrt{10x}+3\sqrt{y})-(1+2\sqrt{10x}-6\sqrt{y})$
10. $\left(3a\sqrt[3]{ab^2}+6\sqrt[3]{a^2b}\right)+\left(9a\sqrt[3]{ab^2}-12\sqrt[3]{a^2b}\right)$
11. $\sqrt{45}+\sqrt{12}-\sqrt{20}-\sqrt{75}$
12. $\sqrt{24}-\sqrt{32}+\sqrt{54}-2\sqrt{32}$
13. $2\sqrt{3x^2}+\sqrt{45x}-x\sqrt{27}+\sqrt{20x}$
14. $5\sqrt{6a^2b}+\sqrt{8a^2b^2}-2\sqrt{24a^2b}-a\sqrt{18b^2}$
15. $5y\sqrt{4x^2y}-\left(x\sqrt{16y^3}-2\sqrt{9x^2y^3}\right)$
16. $\left(2b\sqrt{9a^2c}-3a\sqrt{16b^2c}\right)-\left(\sqrt{64a^2b^2c}-9b\sqrt{a^2c}\right)$
17. $\sqrt[3]{216x}-\sqrt[3]{125xy}-\sqrt[3]{8x}$
18. $\sqrt[3]{128x^3}-2x\sqrt[3]{54}+3\sqrt[3]{2x^3}$
19. $\sqrt[3]{8x^3y}-2x\sqrt[3]{8y}+\sqrt[3]{27x^3y}+x\sqrt[3]{y}$
20. $\sqrt[3]{27a^3b}-3\sqrt[3]{8ab^3}+a\sqrt[3]{64b}-b\sqrt[3]{a}$
21. Calculate the perimeter of the triangle formed by the following set of vertices: $\{(-3,-2),(-1,1),(1,-2)\}$.
22. Calculate the perimeter of the triangle formed by the following set of vertices: $\{(0,-4),(2,0),(-3,0)\}$.

Answer

1. $12\sqrt{2}$

3. $9\sqrt{3}-\sqrt{2}$

5. $4\sqrt{x}-2\sqrt{y}$

7. $11\sqrt{5}-\sqrt{6}$

9. $1-3\sqrt{10x}+9\sqrt{y}$

11. $\sqrt{5}-3\sqrt{3}$

13. $-x\sqrt{3}+5\sqrt{5x}$

15. $12xy\sqrt{y}$

17. $4\sqrt[3]{x}-5\sqrt[3]{xy}$

19. $2x\sqrt[3]{y}$

21. $4+2\sqrt{13}$ units

Exercise $5.E.7$

Multiply.

1. $\sqrt{6}\cdot\sqrt{15}$
2. $(4\sqrt{2})^2$
3. $\sqrt{2}(\sqrt{2}-\sqrt{10})$
4. $(\sqrt{5}-\sqrt{6})^2$
5. $(5-\sqrt{3})(5+\sqrt{3})$
6. $(2\sqrt{6}+\sqrt{3})(\sqrt{2}-5\sqrt{3})$
7. $(\sqrt{a}-5\sqrt{b})^2$
8. $3\sqrt{xy}(\sqrt{x}-2\sqrt{y})$
9. $\sqrt[3]{3a^2}\cdot\sqrt[3]{18a}$
10. $\sqrt[3]{49a^2b}\cdot\sqrt[3]{7a^2b^2}$

Answer

1. $3\sqrt{10}$
3. $2-2\sqrt{5}$
5. 22
7. $a-10\sqrt{ab}+25b$
9. $3a\sqrt[3]{2}$

Exercise $5.E.8$

Divide. Assume all variables represent nonzero numbers and rationalize the denominator where appropriate.

1. $\frac{\sqrt{72}}{\sqrt{9}}$
2. $\frac{10\sqrt{48}}{\sqrt{64}}$
3. $\frac{5}{\sqrt{5}}$
4. $\frac{\sqrt{15}}{\sqrt{2}}$
5. $\frac{3}{2\sqrt{6}}$
6. $\frac{2+\sqrt{5}}{\sqrt{10}}$
7. $\frac{18}{\sqrt{3x}}$
8. $\frac{2\sqrt{3x}}{\sqrt{6xy}}$
9. $\frac{1}{\sqrt[3]{3x^2}}$
10. $\frac{5ab^2}{\sqrt[3]{5a^2b}}$
11. $\sqrt[3]{\frac{5xz^2}{49x^2y^2z}}$
12. $\frac{1}{\sqrt[5]{8x^4y^2z}}$
13. $\frac{9x^2y}{\sqrt[5]{81xy^2z^3}}$
14. $\sqrt[5]{\frac{27ab^3}{15a^4bc^2}}$
15. $\frac{1}{\sqrt{5}-\sqrt{3}}$
16. $\frac{\sqrt{3}}{\sqrt{2}+1}$
17. $\frac{-3\sqrt{6}}{2-\sqrt{10}}$
18. $\frac{\sqrt{xy}}{\sqrt{x}-\sqrt{y}}$
19. $\frac{\sqrt{2}-\sqrt{6}}{\sqrt{2}+\sqrt{6}}$
20. $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$
21. The base of a triangle measures $2\sqrt{6}$ units and the height measures $3\sqrt{15}$ units. Find the area of the triangle.
22. If each side of a square measures $5+2\sqrt{10}$ units, find the area of the square.

Answer

1. $2\sqrt{2}$
3. $\sqrt{5}$
5. $\frac{\sqrt{6}}{4}$
7. $\frac{6\sqrt{3x}}{x}$
9. $\frac{\sqrt[3]{9x}}{3x}$
11. $\frac{\sqrt[3]{35x^2yz}}{7xy}$
13. $\frac{3xy\sqrt[5]{3x^4y^3z^2}}{z}$
15. $\frac{\sqrt{5}+\sqrt{3}}{2}$
17. $\sqrt{6}+\sqrt{15}$
19. $-2+\sqrt{3}$

21. $9\sqrt{10}$ square units

Exercise $5.E.9$

Express in radical form.

1. $11^{1/2}$
2. $2^{2/3}$
3. $x^{3/5}$
4. $a^{-4/5}$

Answer

1. $\sqrt{11}$

3. $\sqrt[5]{x^3}$

Exercise $5.E.10$

Write as a radical and then simplify.

1. $16^{1/2}$
2. $72^{1/2}$
3. $8^{2/3}$
4. $32^{1/3}$
5. $\left(\frac{1}{9}\right)^{3/2}$
6. $\left(\frac{1}{216}\right)^{-1/3}$

Answer

1. 4

3. 4

5. $\frac{1}{27}$

Exercise $5.E.11$

Perform the operations and simplify. Leave answers in exponential form.

1. $6^{1/2} \cdot 6^{3/2}$
2. $3^{1/3} \cdot 3^{1/2}$
3. $\frac{6^{5/2}}{6^{3/2}}$
4. $\frac{4^{3/4}}{4^{1/4}}$
5. $\left(64x^6y^2\right)^{1/2}$
6. $\left(27x^{12}y^6\right)^{1/3}$
7. $\left(\frac{a^{4/3}}{a^{1/2}}\right)^{2/5}$
8. $\left(\frac{16x^{4/3}}{y^2}\right)^{1/2}$
9. $\frac{56x^{3/4}y^{3/2}}{14x^{1/2}y^{2/3}}$
10. $\frac{\left(4a^4b^{2/3}c^{4/3}\right)^{1/2}}{2a^2b^{1/6}c^{2/3}}$
11. $\left(9x^{-4/3}y^{1/3}\right)^{-3/2}$
12. $\left(16x^{-4/5}y^{1/2}z^{-2/3}\right)^{-3/4}$

Answer

1. 36

3. 6

5. $8x^3y$

7. $a^{1/3}$

9. $4x^{1/4}y^{5/6}$
11. $\frac{x^2}{27y^{1/2}}$

Exercise $5.E.12$

Perform the operations with mixed indices.

1. $\sqrt{y} \cdot \sqrt[5]{y^2}$
2. $\sqrt[3]{y} \cdot \sqrt[5]{y^3}$
3. $\frac{\sqrt[3]{y^2}}{\sqrt[3]{y}}$
4. $\sqrt{\sqrt[3]{y^2}}$

Answer

1. $\sqrt[10]{y^9}$
3. $\sqrt[15]{y^7}$

Exercise $5.E.13$

Solve.

1. $2\sqrt{x}+3=13$
2. $\sqrt{3x-2}=4$
3. $\sqrt{x-5}+4=8$
4. $5\sqrt{x+3}+7=2$
5. $\sqrt{4x-3}=\sqrt{2x+15}$
6. $\sqrt{8x-15}=x$
7. $x-1=\sqrt{13-x}$
8. $\sqrt{4x-3}=2x-3$
9. $\sqrt{x+5}=5-\sqrt{x}$
10. $\sqrt{x+3}=3\sqrt{x}-1$
11. $\sqrt{2(x+1)}-\sqrt{x+2}=1$
12. $\sqrt{6-x}+\sqrt{x-2}=2$
13. $\sqrt{3x-2}+\sqrt{x-1}=1$
14. $\sqrt{9-x}=\sqrt{x+16}-1$
15. $\sqrt[3]{4x-3}=2$
16. $\sqrt[3]{x-8}=-1$
17. $\sqrt[3]{x(3x+10)}=2$
18. $\sqrt[3]{2x^2-x}+4=5$
19. $\sqrt[3]{3(x+4)(x+1)}=\sqrt[3]{5x+37}$
20. $\sqrt[3]{3x^2-9x+24}=\sqrt[3]{(x+2)^2}$
21. $y^{1/2}-3=0$
22. $y^{1/3}+3=0$
23. $(x-5)^{1/2}-2=0$
24. $(2x-1)^{1/3}-5=0$
25. $(x-1)^{1/2}=x^{1/2}-1$
26. $(x-2)^{1/2}-(x-6)^{1/2}=2$
27. $(x+4)^{1/2}-(3x)^{1/2}=-2$
28. $(5x+6)^{1/2}=3-(x+3)^{1/2}$
29. Solve for $g: t=\sqrt{\frac{2s}{g}}$.
30. Solve for $x: y=\sqrt[3]{x+4}-2$,
31. The period in seconds of a pendulum is given by the formula $T=2\pi\sqrt{\frac{L}{32}}$ where L represents the length in feet of the pendulum. Find the length of a pendulum that has a period of $1\frac{1}{2}$ seconds. Find the exact answer and the approximate answer rounded off to the nearest tenth of a foot.

32. The outer radius of a spherical shell is given by the formula $r=\sqrt[3]{\frac{3V}{4\pi}}+2$ where V represents the inner volume in cubic centimeters. If the outer radius measures 8 centimeters, find the inner volume of the sphere.
33. The speed of a vehicle before the brakes are applied can be estimated by the length of the skid marks left on the road. On dry pavement, the speed v in miles per hour can be estimated by the formula $v=2\sqrt{6d}$, where d represents the length of the skid marks in feet. Estimate the length of a skid mark if the vehicle is traveling 30 miles per hour before the brakes are applied.
34. Find the real root of the function defined by $f(x)=\sqrt[3]{x-3}+2$.

Answer

1. 25

3. 21

5. 9

7. 4

9. 4

11. 7

13. 1

15. $\frac{11}{4}$

17. $-4, \frac{2}{3}$

19. $-5, \frac{5}{3}$

21. 9

23. 9

25. 1

27. 12

29. $g=\frac{2s}{t^2}$

31. $\frac{18}{\pi^2}$ feet; 1.8 feet

33. 37.5 feet

Exercise $5.E.14$

Write the complex number in standard form $a+bi$.

1. $5-\sqrt{-16}$
2. $-\sqrt{-25}-6$
3. $\frac{3+\sqrt{-8}}{10}$
4. $\frac{\sqrt{-12}-4}{6}$

Answer

1. $5-4i$

3. $\frac{3}{10}+\frac{\sqrt{2}}{5}i$

Exercise $5.E.15$

Perform the operations.

1. $(6-12i)+(4+7i)$
2. $(-3+2i)-(6-4i)$
3. $\left(\frac{1}{2}-i\right)-\left(\frac{3}{4}-\frac{3}{2}i\right)$
4. $\left(\frac{5}{8}-\frac{1}{5}i\right)+\left(\frac{3}{2}-\frac{2}{3}i\right)$

5. $(5-2i)-(6-7i)+(4-4i)$
6. $(10-3i)+(20+5i)-(30-15i)$
7. $4i(2-3i)$
8. $(2+3i)(5-2i)$
9. $(4+i)^2$
10. $(8-3i)^2$
11. $(3+2i)(3-2i)$
12. $(-1+5i)(-1-5i)$
13. $\frac{2+9i}{2i}$
14. $\frac{i}{1-2i}$
15. $\frac{4+5i}{2-i}$
16. $\frac{3-2i}{3+2i}$
17. $10-5(2-3i)^2$
18. $(2-3i)^2-(2-3i)+4$
19. $\left(\frac{1}{1-i}\right)^2$
20. $\left(\frac{1+2i}{3i}\right)^2$
21. $\sqrt{-8}(\sqrt{3}-\sqrt{-4})$
22. $(1-\sqrt{-18})(3-\sqrt{-2})$
23. $(\sqrt{-5}-\sqrt{-10})^2$
24. $(1-\sqrt{-2})^2-(1+\sqrt{-2})^2$
25. Show that both $-5i$ and $5i$ satisfy $x^2+25=0$.
26. Show that both $1-2i$ and $1+2i$ satisfy $x^2-2x+5=0$.

Answer

1. $10-5i$

3. $-\frac{1}{4}+\frac{1}{2}i$

5. $3+i$

7. $12+8i$

9. $15+8i$

11. 13

13. $\frac{9}{2}-i$

15. $\frac{3}{5}+\frac{14}{5}i$

17. $35+60i$

19. $\frac{1}{2}i$

21. $4\sqrt{2}+2i\sqrt{6}$

23. $-15+10\sqrt{2}$

25. Answer may vary

Sample Exam

Exercise $5.E.16$

Simplify. (Assume all variables are positive.)

1. $5x\sqrt{121x^2y^4}$
2. $2xy^2\sqrt[3]{-64x^6y^9}$
3. Calculate the distance between $(-5,-3)$ and $(-2,6)$.
4. The time in seconds an object is in free fall is given by the formula $t=\frac{\sqrt{s}}{4}$ where s represents the distance in feet that the object has fallen. If a stone is dropped into a 36-foot pit, how long will it take to hit the bottom of the pit?

Answer

1. $55x^2y^2$

3. $3\sqrt{10}$ units

Exercise $5.E.17$

Perform the operations and simplify. (Assume all variables are positive and rationalize the denominator where appropriate.)

1. $\sqrt{150xy^2} - 2\sqrt{18x^3} + y\sqrt{24x} + x\sqrt{128x}$
2. $3\sqrt[3]{16x^3y^2} - \left(2x\sqrt[3]{250y^2} - \sqrt[3]{54x^3y^2}\right)$
3. $2\sqrt{2}(\sqrt{2} - 3\sqrt{6})$
4. $(\sqrt{10} - \sqrt{5})^2$
5. $\frac{\sqrt{6}}{\sqrt{2}+\sqrt{3}}$
6. $\frac{2x}{\sqrt{2xy}}$
7. $\frac{1}{\sqrt[5]{8xy^2z^4}}$
8. Simplify: $81^{3/4}$.
9. Express in radical form: $x^{-3/5}$.

Answer

1. $7y\sqrt{6x} + 2x\sqrt{2x}$

3. $4 - 12\sqrt{3}$

5. $-2\sqrt{3} + 3\sqrt{2}$

7. $\frac{\sqrt[5]{4x^4y^3z}}{2xyz}$

9. $\frac{1}{\sqrt[5]{x^3}}$

Exercise $5.E.18$

Solve.

1. $\sqrt{x} - 5 = 1$
2. $\sqrt[3]{5x-2} + 6 = 4$
3. $5\sqrt{2x+5} - 2x = 11$
4. $\sqrt{4-3x} + 2 = x$
5. $\sqrt{2x+5} - \sqrt{x+3} = 2$
6. The time in seconds an object is in free fall is given by the formula $t = \frac{\sqrt{s}}{4}$ where s represents the distance in feet that the object has fallen. If a stone is dropped into a pit and it takes 4 seconds to reach the bottom, how deep is the pit?
7. The width in inches of a container is given by the formula $w = \frac{\sqrt[3]{4V}}{2} + 1$ where V represents the inside volume in cubic inches of the container. What is the inside volume of the container if the width is 6 inches?

Answer

2. $-\frac{6}{5}$

4. $\varnothing$

6. 256 feet

Exercise $5.E.19$

Perform the operations and write the answer in standard form.

1. $\sqrt{-3}(\sqrt{6} - \sqrt{-3})$
2. $\frac{4+3i}{2-i}$
3. $6 - 3(2-3i)^2$

Answer

1. $3+3i\sqrt{2}$
3. $21+36i$

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CHAPTER OVERVIEW

6: Solving Equations and Inequalities

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6.1: Extracting Square Roots and Completing the Square

Learning Objectives

- Solve certain quadratic equations by extracting square roots.
- Solve any quadratic equation by completing the square.

Extracting Square Roots

Recall that a quadratic equation is in **standard form**[1] if it is equal to 0:

$$ax^2+bx+c=0$$

where a, b, and c are real numbers and $a \neq 0$. A solution to such an equation is a root of the quadratic function defined by $f(x)=ax^2+bx+c$. Quadratic equations can have two real solutions, one real solution, or no real solution—in which case there will be two complex solutions. If the quadratic expression factors, then we can solve the equation by factoring. For example, we can solve $4x^2-9=0$ by factoring as follows:

$$\begin{aligned} 4x^2-9&=0\\ (2x+3)(2x-3)&=0 \end{aligned}$$

$$\begin{array}{ccc} 2x+3=0 & \text{or} & 2x-3=0 \\ 2x=-3 & & 2x=3 \\ x=-\frac{3}{2} & & x=\frac{3}{2} \end{array}$$

The two solutions are $\pm\frac{3}{2}$. Here we use $\pm$ to write the two solutions in a more compact form. The goal in this section is to develop an alternative method that can be used to easily solve equations where $b=0$, giving the form

$$ax^2+c=0$$

The equation $4x^2-9=0$ is in this form and can be solved by first isolating x^2.

$$\begin{aligned} 4x^2-9&=0\\ 4x^2&=9\\ x^2&=\frac{9}{4} \end{aligned}$$

If we take the square root of both sides of this equation, we obtain the following:

$$\begin{aligned} \sqrt{x^2}&=\sqrt{\frac{9}{4}}\\ |x|&=\frac{3}{2} \end{aligned}$$

Here we see that $x=\pm\frac{3}{2}$ are solutions to the resulting equation. In general, this describes the **square root property**[2]; for any real number k,

$$\text{If} \quad x^2=k, \quad \text{then} \quad x=\pm\sqrt{k}$$

Applying the square root property as a means of solving a quadratic equation is called **extracting the root**[3]. This method allows us to solve equations that do not factor.

Example 6.1.1:

Solve: $9x^2-8=0$.

Solution

Notice that the quadratic expression on the left does not factor. However, it is in the form $ax^2+c=0$ and so we can solve it by extracting the roots. Begin by isolating x^2.

$$\begin{aligned} 9x^2-8&=0\\ 9x^2&=8\\ x^2&=\frac{8}{9} \end{aligned}$$

Next, apply the square root property. Remember to include the $\pm$ and simplify.

$$\begin{aligned} x &= \pm\sqrt{\frac{8}{9}} \\ &= \pm\frac{2\sqrt{2}}{3} \end{aligned}$$

For completeness, check that these two real solutions solve the original quadratic equation.

Table 6.1.1

Check $x = -\frac{2\sqrt{2}}{3}$	Check $x = \frac{2\sqrt{2}}{3}$
$9x^2 - 8 = 0$ $9\left(-\frac{2\sqrt{2}}{3}\right)^2 - 8 = 0$ $9\left(\frac{4.2}{9}\right) - 8 = 0$ $8 - 8 = 0$ $0 = 0 \checkmark$	$9x^2 - 8 = 0$ $9\left(\frac{2\sqrt{2}}{3}\right)^2 - 8 = 0$ $9\left(\frac{4.2}{9}\right) - 8 = 0$ $8 - 8 = 0$ $0 = 0 \checkmark$

Answer:

Two real solutions, $\pm\frac{2\sqrt{2}}{3}$

Sometimes quadratic equations have no real solution. In this case, the solutions will be complex numbers.

Example 6.1.2:

Solve: $x^2 + 25 = 0$.

Solution

Begin by isolating x^2 and then apply the square root property.

$$\begin{aligned} x^2 + 25 &= 0 \\ x^2 &= -25 \\ x &= \pm\sqrt{-25} \end{aligned}$$

After applying the square root property, we are left with the square root of a negative number. Therefore, there is no real solution to this equation; the solutions are complex. We can write these solutions in terms of the imaginary unit $i = \sqrt{-1}$.

$$\begin{aligned} x &= \pm\sqrt{-25} \\ &= \pm\sqrt{-1 \cdot 25} \\ &= \pm i \cdot 5 \\ &= \pm 5i \end{aligned}$$

Table 6.1.2

Check $x = -5i$	Check $x = 5i$
$x^2 + 25 = 0$ $(-5i)^2 + 25 = 0$ $25i^2 + 25 = 0$ $25(-1) + 25 = 0$ $-25 + 25 = 0$ $0 = 0 \checkmark$	$x^2 + 25 = 0$ $(-5i)^2 + 25 = 0$ $25i^2 + 25 = 0$ $25(-1) + 25 = 0$ $-25 + 25 = 0$ $0 = 0 \checkmark$

Answer:

Two complex solutions, $\pm 5i$.

Exercise 6.1.1

Solve: $2x^2 - 3 = 0$.

Answer

The solutions are $\pm\frac{\sqrt{6}}{2}$.

www.youtube.com/v/9ff7QGhFytQ

Consider solving the following equation:

$$(x+5)^2=9$$

To solve this equation by factoring, first square $x+5$ and then put the equation in standard form, equal to zero, by subtracting 9 from both sides.

$$\begin{aligned}(x+5)^2&=9\\x^2+10x+25&=9\\x^2+10x+16&=0\end{aligned}$$

Factor and then apply the zero-product property.

$$\begin{aligned}x^2+10x+16&=0&&\\(x+8)(x+2)&=0&&\\x+8&=0\quad\text{or}\quad&x+2&=0\\x&=-8&x&=-2\end{aligned}$$

The two solutions are -8 and -2. When an equation is in this form, we can obtain the solutions in fewer steps by extracting the roots.

Example $6.1.3$:

Solve by extracting roots: $(x+5)^2=9$.

Solution

The term with the square factor is isolated so we begin by applying the square root property.

$$\begin{aligned}(x+5)^2&=9&&\textit{Apply the square root property.}\\x+5&=\pm\sqrt{9}&&\textit{Simplify.}\\x+5&=\pm\sqrt{9}&&\\x&=-5\pm 3&&\end{aligned}$$

At this point, separate the “plus or minus” into two equations and solve each individually.

$$\begin{aligned}x&=-5+3\quad\text{or}\quad&x&=-5-3\\x&=-2&x&=-8\end{aligned}$$

Answer:

The solutions are -2 and -8.

In addition to fewer steps, this method allows us to solve equations that do not factor.

Example $6.1.4$:

Solve: $2(x-2)^2-5=0$.

Solution

Begin by isolating the term with the square factor.

$$\begin{aligned}2(x-2)^2-5&=0\\2(x-2)^2&=5\\(x-2)^2&=\frac{5}{2}\end{aligned}$$

Next, extract the roots, solve for x, and then simplify.

$$\begin{aligned} x-2 &= \pm\sqrt{\frac{5}{2}} && \textit{Rationalize the denominator.} \\ x &= 2 \pm \frac{\sqrt{5}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\ x &= 2 \pm \frac{\sqrt{10}}{2} \\ x &= \frac{4 \pm \sqrt{10}}{2} \end{aligned}$$

Answer:

The solutions are $\frac{4-\sqrt{10}}{2}$ and $\frac{4+\sqrt{10}}{2}$.

Exercise 6.1.2

Solve: $2(3x-1)^2+9=0$.

Answer

The solutions are $\frac{1}{3} \pm \frac{\sqrt{2}}{2} i$.

www.youtube.com/v/RQrrtff3rw0

Completing the Square

In this section, we will devise a method for rewriting any quadratic equation of the form

$$ax^2+bx+c=0$$

as an equation of the form

$$(x-p)^2=q$$

This process is called **completing the square**[4]. As we have seen, quadratic equations in this form can be easily solved by extracting roots. We begin by examining perfect square trinomials:

$$\begin{array}{rcccc} (x+3)^2 = x^2 + & 6x & + & & 9 \\ & \downarrow & & & \uparrow \\ & \left(\frac{6}{2}\right)^2 & = (3)^2 = & & 9 \end{array}$$

The last term, 9, is the square of one-half of the coefficient of x. In general, this is true for any perfect square trinomial of the form x^2+bx+c.

$$\begin{aligned} \left(x+\frac{b}{2}\right)^2 &= x^2 + 2 \cdot \frac{b}{2} x + \left(\frac{b}{2}\right)^2 \\ &= x^2 + bx + \left(\frac{b}{2}\right)^2 \end{aligned}$$

In other words, any trinomial of the form x^2+bx+c will be a perfect square trinomial if

$$c=\left(\frac{b}{2}\right)^2$$

Note

It is important to point out that the leading coefficient must be equal to 1 for this to be true.

Example 6.1.5:

Complete the square: $x^2-6x+ ? =(x+ ?)^2$.

Solution

In this example, the coefficient b of the middle term is -6. Find the value that completes the square as follows:

$$\left(\frac{b}{2}\right)^2 = \left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$$

The value that completes the square is 9.

$$\begin{aligned} x^2-6x+9 &= (x-3)(x-3) \\ &= (x-3)^2 \end{aligned}$$

Answer:

$x^2-6x+9=(x-3)^2$

Example 6.1.6:

Complete the square: $x^2+x+?=(x+?)^2$.

Solution

Here $b=1$. Find the value that will complete the square as follows:

$$\left(\frac{b}{2}\right)^2=\left(\frac{1}{2}\right)^2=\frac{1}{4}$$

The value $\frac{1}{4}$ completes the square:

$$\begin{aligned} x^2+x+\frac{1}{4} &= \left(x+\frac{1}{2}\right)\left(x+\frac{1}{2}\right) \\ &= \left(x+\frac{1}{2}\right)^2 \end{aligned}$$

Answer:

$x^2+x+\frac{1}{4}=\left(x+\frac{1}{2}\right)^2$

We can use this technique to solve quadratic equations. The idea is to take any quadratic equation in standard form and complete the square so that we can solve it by extracting roots. The following are general steps for solving a quadratic equation with leading coefficient 1 in standard form by completing the square.

Example 6.1.7:

Solve by completing the square: $x^2-8x-2=0$.

Solution

It is important to notice that the leading coefficient is 1.

Step 1: Add or subtract the constant term to obtain an equation of the form $x^2+bx=c$. Here we add 2 to both sides of the equation.

$$\begin{aligned} x^2-8x-2 &= 0 \\ x^2-8x &= 2 \end{aligned}$$

Step 2: Use $\left(\frac{b}{2}\right)^2$ to determint the value that completes the square. In this case, $b=-8$.

$$\left(\frac{b}{2}\right)^2=\left(\frac{-8}{2}\right)^2=(-4)^2=16$$

Step 3: Add $\left(\frac{b}{2}\right)^2$ to both sides of the equation and complete the square.

$$\begin{aligned} x^2-8x &= 2 \\ x^2-8x+16 &= 2+16 \\ (x-4)(x-4) &= 18 \\ (x-4)^2 &= 18 \end{aligned}$$

Step 4: Solve by extracting roots.

$$\begin{aligned} (x-4)^2 &= 18 \\ x-4 &= \pm\sqrt{18} \\ x &= 4\pm\sqrt{9\cdot 2} \\ x &= 4\pm 3\sqrt{2} \end{aligned}$$

Answer:

The solutions are $4-3\sqrt{2}$ and $4+3\sqrt{2}$. The check is left to the reader.

Example 6.1.8:

Solve by completing the square: $x^2+2x-48=0$.

Solution

Begin by adding 48 to both sides

$$\begin{aligned} x^2+2x-48&=0 \\ x^2+2x&=48 \end{aligned}$$

Next, find the value that completes the square using $b=2$.

$$\left(\frac{b}{2}\right)^2=\left(\frac{2}{2}\right)^2=(1)^2=1$$

To complete the square, add 1 to both sides, complete the square, and then solve by extracting the roots.

$$\begin{aligned} x^2+2x&=48 && \textit{Complete the square.} \\ x^2+2x+1&=48+1 \\ (x+1)(x+1)&=49 \\ (x+1)^2&=49 && \textit{Extract the roots.} \\ x+1&=\pm\sqrt{49} \\ x+1&=\pm 7 \\ x&=-1\pm 7 \end{aligned}$$

At this point, separate the "plus or minus" into two equations and solve each individually.

$$\begin{aligned} x=-1-7 &\text{ or } x=-1+7 \\ x=-8 &\qquad x=6 \end{aligned}$$

Answer:

The solutions are -8 and 6.

Note

In the previous example the solutions are integers. If this is the case, then the original equation will factor.

$$\begin{aligned} x^2+2x-48&=0 \\ (x-6)(x+8)&=0 \end{aligned}$$

If an equation factors, we can solve it by factoring. However, not all quadratic equations will factor. Furthermore, equations often have complex solutions.

Example 6.1.9:

Solve by completing the square: $x^2-10x+26=0$.

Solution

Begin by subtracting 26 from both sides of the equation.

$$\begin{aligned} x^2-10x+26&=0 \\ x^2-10x&=-26 \end{aligned}$$

Here $b=-10$, and we determine the value that completes the square as follows:

$$\left(\frac{b}{2}\right)^2=\left(\frac{-10}{2}\right)^2=(-5)^2=25$$

To complete the square, add 25 to both sides of the equation.

$$\begin{aligned} x^2-10x&=-26 \\ x^2-10x+25&=-26+25 \\ x^2-10x+25&=-1 \end{aligned}$$

Factor and then solve by extracting roots.

$$\begin{aligned}x^2-10x+25&=-1\\(x-5)(x-5)&=-1\\(x-5)^2&=-1\\x-5&=\pm\sqrt{-1}\\x-5&=\pm i\\x&=5\pm i\end{aligned}$$

Answer:

The solutions are $5\pm i$.

Exercise 6.1.3

Solve by completing the square: $x^2-2x-17=0$.

Answer

The solutions are $x=1\pm 3\sqrt{2}$.

www.youtube.com/v/i8WVWpe-Ct0

The coefficient of x is not always divisible by 2.

Example 6.1.10:

Solve by completing the square: $x^2+3x+4=0$.

Solution

Begin by subtracting 4 from both sides.

$$\begin{aligned}x^2+3x+4&=0\\x^2+3x&=-4\end{aligned}$$

Use $b=3$ to find the value that completes the square:

$$\left(\frac{b}{2}\right)^2=\left(\frac{3}{2}\right)^2=\frac{9}{4}$$

To complete the square, add $\frac{9}{4}$ to both sides of the equation.

$$\begin{aligned}x^2+3x&=-4\\x^2+3x+\frac{9}{4}&=-4+\frac{9}{4}\\\left(x+\frac{3}{2}\right)\left(x+\frac{3}{2}\right)&=\frac{-16}{4}+\frac{9}{4}\\\left(x+\frac{3}{2}\right)^2&=\frac{-7}{4}\end{aligned}$$

Solve by extracting roots.

$$\begin{aligned}\left(x+\frac{3}{2}\right)^2&=-\frac{7}{4}\\x+\frac{3}{2}&=\pm\sqrt{\frac{-1\cdot 7}{4}}\\x+\frac{3}{2}&=\pm\frac{i\sqrt{7}}{2}\\x&=-\frac{3}{2}\pm\frac{\sqrt{7}}{2}i\end{aligned}$$

Answer:

The solutions are $-\frac{3}{2}\pm\frac{\sqrt{7}}{2}i$.

So far, all of the examples have had a leading coefficient of 1. The formula $\left(\frac{b}{2}\right)^2$ determines the value that completes the square only if the leading coefficient is 1. If this is not the case, then simply divide both sides by the leading coefficient before beginning the steps outlined for completing the square.

Example 6.1.11:

Solve by completing the square: $2x^2 + 5x - 1 = 0$.

Solution

Notice that the leading coefficient is 2. Therefore, divide both sides by 2 before beginning the steps required to solve by completing the square.

$$\begin{gathered}\frac{2x^2+5x-1}{2}=\frac{0}{2}\\ \frac{2x^2}{2}+\frac{5x}{2}-\frac{1}{2}=0\\ x^2+\frac{5}{2}x-\frac{1}{2}=0\end{gathered}$$

Add $\frac{1}{2}$ to both sides of the equation.

$$\begin{aligned}x^2+\frac{5}{2}x-\frac{1}{2}&=0\\ x^2+\frac{5}{2}x&=\frac{1}{2}\end{aligned}$$

Here $b = \frac{5}{2}$, and we can find the value that completes the square as follows:

$$\left(\frac{b}{2}\right)^2=\left(\frac{5/2}{2}\right)^2=\left(\frac{5}{2}\cdot\frac{1}{2}\right)^2=\left(\frac{5}{4}\right)^2=\frac{25}{16}$$

To complete the square, add $\frac{25}{16}$ to both sides of the equation.

$$\begin{aligned}x^2+\frac{5}{2}x&=\frac{1}{2}\\ x^2+\frac{5}{2}x+\frac{25}{16}&=\frac{1}{2}+\frac{25}{16}\\ \left(x+\frac{5}{4}\right)\left(x+\frac{5}{4}\right)&=\frac{8}{16}+\frac{25}{16}\\ \left(x+\frac{5}{4}\right)^2&=\frac{33}{16}\end{aligned}$$

Next, solve by extracting roots.

$$\begin{aligned}\left(x+\frac{5}{4}\right)^2&=\frac{33}{16}\\ x+\frac{5}{4}&=\pm\sqrt{\frac{33}{16}}\\ x+\frac{5}{4}&=\pm\frac{\sqrt{33}}{4}\\ x&=-\frac{5}{4}\pm\frac{\sqrt{33}}{4}\\ x&=\frac{-5\pm\sqrt{33}}{4}\end{aligned}$$

Answer:

The solutions are $\frac{-5\pm\sqrt{33}}{4}$.

Exercise 6.1.4

Solve by completing the square: $3x^2 - 2x + 1 = 0$.

Answer

The solutions are $x = \frac{1}{3} \pm \frac{\sqrt{2}}{3}i$

www.youtube.com/v/A-i6LKqNmtY

Key Takeaways

- Solve equations of the form $ax^2 + c = 0$ by extracting the roots.
- Extracting roots involves isolating the square and then applying the square root property. Remember to include "±" when taking the square root of both sides.

- After applying the square root property, solve each of the resulting equations. Be sure to simplify all radical expressions and rationalize the denominator if necessary.
- Solve any quadratic equation by completing the square.
- You can apply the square root property to solve an equation if you can first convert the equation to the form $(x-p)^2=q$.
- To complete the square, first make sure the equation is in the form $x^2+bx=c$. The leading coefficient must be 1. Then add the value $\left(\frac{b}{2}\right)^2$ to both sides and factor.
- The process for completing the square always works, but it may lead to some tedious calculations with fractions. This is the case when the middle term, b, is not divisible by 2.

Exercise $6.1.5$

Solve by factoring and then solve by extracting roots. Check answers.

1. $x^2-16=0$
2. $x^2-36=0$
3. $9y^2-1=0$
4. $4y^2-25=0$
5. $(x-2)^2-1=0$
6. $(x+1)^2-4=0$
7. $4(y-2)^2-9=0$
8. $9(y+1)^2-4=0$
9. $(u-5)^2-25=0$
10. $(u+2)^2-4=0$

Answer

1. $-4,4$
3. $-\frac{1}{3},\frac{1}{3}$
5. $1,3$
7. $\frac{1}{2},\frac{7}{2}$
9. \(0,10)

Exercise $6.1.6$

Solve by extracting the roots.

1. $x^2=81$
2. $x^2=1$
3. $y^2=\frac{1}{9}$
4. $y^2=\frac{1}{16}$
5. $x^2=12$
6. $x^2=18$
7. $16x^2=9$
8. $4x^2=25$
9. $2t^2=1$
10. $3t^2=2$
11. $x^2-40=0$
12. $x^2-24=0$
13. $x^2+1=0$
14. $x^2+100=0$
15. $5x^2-1=0$
16. $6x^2-5=0$
17. $8x^2+1=0$
18. $12x^2+5=0$

19. $y^2+4=0$
20. $y^2+1=0$
21. $x^2-\frac{4}{9}=0$
22. $x^2-\frac{9}{25}=0$
23. $x^2-8=0$
24. $t^2-18=0$
25. $x^2+8=0$
26. $x^2+125=0$
27. $5y^2-2=0$
28. $3x^2-1=0$
29. $(x+7)^2-4=0$
30. $(x+9)^2-36=0$
31. $(x-5)^2-20=0$
32. $(x+1)^2-28=0$
33. $(3t+2)^2+6=0$
34. $(3t-5)^2+10=0$
35. $4(3x+1)^2-27=0$
36. $9(2x-3)^2-8=0$
37. $2(3x-1)^2+3=0$
38. $5(2x-1)^2+2=0$
39. $3\left(y-\frac{2}{3}\right)^2-\frac{3}{2}=0$
40. $2\left(3y-\frac{1}{3}\right)^2-\frac{5}{2}=0$
41. $-3(t-1)^2+12=0$
42. $-2(t+1)^2+8=0$
43. Solve for $x: px^2-q=0, p, q>0$
44. Solve for $x(x-p)^2-q=0, p, q>0$
45. The diagonal of a square measures 3 centimeters. Find the length of each side.
46. The length of a rectangle is twice its width. If the diagonal of the rectangle measures 10 meters, then find the dimensions of the rectangle.
47. If a circle has an area of 50π square centimeters, then find its radius.
48. If a square has an area of 27 square centimeters, then find the length of each side.
49. The height in feet of an object dropped from an 18-foot stepladder is given by $h(t)=-16t^2+18$, where t represents the time in seconds after the object is dropped. How long does it take the object to hit the ground? (Hint: The height is 0 when the object hits the ground. Round to the nearest hundredth of a second.)
50. The height in feet of an object dropped from a 50-foot platform is given by $h(t)=-16t^2+50$, where t represents the time in seconds after the object is dropped. How long does it take the object to hit the ground? (Round to the nearest hundredth of a second.)
51. How high does a 22-foot ladder reach if its base is 6 feet from the building on which it leans? Round to the nearest tenth of a foot.
52. The height of a triangle is $\frac{1}{2}$ the length of its base. If the area of the triangle is 72 square meters, find the exact length of the triangle's base.

Answer

1. ± 9

3. $\pm\frac{1}{3}$

5. $\pm 2\sqrt{3}$

7. $\pm\frac{3}{4}$

9. $\pm\frac{\sqrt{2}}{2}$

11. $\pm 2\sqrt{10}$

13. $\pm i$

15. $\pm\frac{\sqrt{5}}{5}$
17. $\pm\frac{\sqrt{2}}{4}i$
19. $\pm 2i$
21. $\pm\frac{2}{3}$
23. $\pm 2\sqrt{2}$
25. $\pm 2i\sqrt{2}$
27. $\pm\frac{\sqrt{10}}{5}$
29. $-9, -5$
31. $5 \pm 2\sqrt{5}$
33. $-\frac{2}{3} \pm \frac{\sqrt{6}}{3}i$
35. $\frac{-2\pm 3\sqrt{3}}{6}$
37. $\frac{1}{3} \pm \frac{\sqrt{6}}{6}i$
39. $\frac{4\pm 3\sqrt{2}}{6}$
41. $-1, -3$
43. $x = \pm\frac{\sqrt{pq}}{p}$
45. $\frac{3\sqrt{2}}{2}$ centimeters
47. $5\sqrt{2}$ centimeters
49. 1.06 seconds
51. 21.2 feet

Exercise $6.1.7$

Complete the square.

1. $x^2 - 2x + ? = (x - ?)^2$
2. $x^2 - 4x + ? = (x - ?)^2$
3. $x^2 + 10x + ? = (x + ?)^2$
4. $x^2 + 12x + ? = (x + ?)^2$
5. $x^2 + 7x + ? = (x + ?)^2$
6. $x^2 + 5x + ? = (x + ?)^2$
7. $x^2 - x + ? = (x - ?)^2$
8. $x^2 - \frac{1}{2}x + ? = (x - ?)^2$
9. $x^2 + \frac{2}{3}x + ? = (x + ?)^2$
10. $x^2 + \frac{4}{5}x + ? = (x + ?)^2$

Answer

1. $x^2 - 2x + 1 = (x - 1)^2$
3. $x^2 + 10x + 25 = (x + 5)^2$
5. $x^2 + 7x + \frac{49}{4} = \left(x + \frac{7}{2}\right)^2$
7. $x^2 - x + \frac{1}{4} = \left(x - \frac{1}{2}\right)^2$
9. $x^2 + \frac{2}{3}x + \frac{1}{9} = \left(x + \frac{1}{3}\right)^2$

Exercise 6.1.8

Solve by factoring and then solve by completing the square. Check answers.

1. $x^2 + 2x - 8 = 0$
2. $x^2 - 8x + 15 = 0$
3. $y^2 + 2y - 24 = 0$
4. $y^2 - 12y + 11 = 0$
5. $t^2 + 3t - 28 = 0$
6. $t^2 - 7t + 10 = 0$
7. $2x^2 + 3x - 2 = 0$
8. $3x^2 - x - 2 = 0$
9. $2y^2 - y - 1 = 0$
10. $2y^2 + 7y - 4 = 0$

Answer

1. $-4, 2$
3. $-6, 4$
5. $-7, 4$
7. $-2, \frac{1}{2}$
9. $-\frac{1}{2}, 1$

Exercise 6.1.9

Solve by completing the square.

1. $x^2 + 6x - 1 = 0$
2. $x^2 + 8x + 10 = 0$
3. $x^2 - 2x - 7 = 0$
4. $x^2 - 6x - 3 = 0$
5. $y^2 - 2y + 4 = 0$
6. $y^2 - 4y + 9 = 0$
7. $t^2 + 10t - 75 = 0$
8. $t^2 + 12t - 108 = 0$
9. $u^2 - \frac{2}{3}u - \frac{1}{3} = 0$
10. $u^2 - \frac{4}{5}u - \frac{1}{5} = 0$
11. $x^2 + x - 1 = 0$
12. $x^2 + x - 3 = 0$
13. $y^2 + 3y - 2 = 0$
14. $y^2 + 5y - 3 = 0$
15. $x^2 + 3x + 5 = 0$
16. $x^2 + x + 1 = 0$
17. $x^2 - 7x + \frac{11}{2} = 0$
18. $x^2 - 9x + \frac{3}{2} = 0$
19. $t^2 - \frac{1}{2}t - 1 = 0$
20. $t^2 - \frac{1}{3}t - 2 = 0$
21. $4x^2 - 8x - 1 = 0$
22. $2x^2 - 4x - 3 = 0$
23. $3x^2 + 6x + 1 = 0$
24. $5x^2 + 10x + 2 = 0$
25. $3x^2 + 2x - 3 = 0$
26. $5x^2 + 2x - 5 = 0$
27. $4x^2 - 12x - 15 = 0$

28. $2x^2+4x-43=0$
29. $2x^2-4x+10=0$
30. $6x^2-24x+42=0$
31. $2x^2-x-2=0$
32. $2x^2+3x-1=0$
33. $3u^2+2u-2=0$
34. $3u^2-u-1=0$
35. $x^2-4x-1=15$
36. $x^2-12x+8=-10$
37. $x(x+1)-11(x-2)=0$
38. $(x+1)(x+7)-4(3x+2)=0$
39. $y^2=(2y+3)(y-1)-2(y-1)$
40. $(2y+5)(y-5)-y(y-8)=-24$
41. $(t+2)^2=3(3t+1)$
42. $(3t+2)(t-4)-(t-8)=1-10t$

Answer

1. $-3\pm\sqrt{10}$
3. $1\pm 2\sqrt{2}$
5. $1\pm i\sqrt{3}$
7. $-15, 5$
9. $-\frac{1}{3}, 1$
11. $\frac{-1\pm\sqrt{5}}{2}$
13. $\frac{-3\pm\sqrt{17}}{2}$
15. $-\frac{3}{2}\pm\frac{\sqrt{11}}{2}i$
17. $\frac{7\pm 3\sqrt{3}}{2}$
19. $\frac{1\pm\sqrt{17}}{4}$
21. $\frac{2\pm\sqrt{5}}{2}$
23. $\frac{-3\pm\sqrt{6}}{3}$
25. $\frac{-1\pm\sqrt{10}}{3}$
27. $\frac{3\pm 2\sqrt{6}}{2}$
29. $1\pm 2i$
31. $\frac{1\pm\sqrt{17}}{4}$
33. $\frac{-1\pm\sqrt{7}}{3}$
35. $2\pm 2\sqrt{5}$
37. $5\pm\sqrt{3}$
39. $\frac{1\pm\sqrt{5}}{2}$
41. $\frac{5\pm\sqrt{21}}{2}$

Exercise 6.1.10

Solve by completing the square and round the solutions to the nearest hundredth.

1. $(2x-1)^2=2x$
2. $(3x-2)^2=5-15x$

3. $(2x+1)(3x+1)=9x+4$
4. $(3x+1)(4x-1)=17x-4$
5. $9x(x-1)-2(2x-1)=-4x$
6. $(6x+1)^2-6(6x+1)=0$

Answer

1. $0.19, 1.31$
3. $-0.45, 1.12$
5. $0.33, 0.67$

Exercise 6.1.11

1. Create an equation of your own that can be solved by extracting the roots. Share it, along with the solution, on the discussion board.
2. Explain why the technique of extracting roots greatly expands our ability to solve quadratic equations.
3. Explain why the technique for completing the square described in this section requires that the leading coefficient be equal to 1.
4. Derive a formula for the diagonal of a square in terms of its sides.

Answer

1. Answer may vary
3. Answer may vary

Footnotes

[1] Any quadratic equation in the form $ax^2+bx+c=0$, where a, b, and c are real numbers and $a \neq 0$.

[2] For any real number k, if $x^2=k$, then $x=\pm\sqrt{k}$.

[3] Applying the square root property as a means of solving a quadratic equation.

[4] The process of rewriting a quadratic equation to be in the form $(x-p)^2=q$.

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6.2: Quadratic Formula

Learning Objectives

- Solve quadratic equations using the quadratic formula.
- Use the determinant to determine the number and type of solutions to a quadratic formula.

The Quadratic Formula

In this section, we will develop a formula that gives the solutions to any quadratic equation in standard form. To do this, we begin with a general quadratic equation in standard form and solve for x by completing the square. Here a, b, and c are real numbers and $a \neq 0$:

$$\begin{aligned} ax^2+bx+c&=0 && \textit{Standard form of a quadratic equation.} \\ \frac{ax^2+bx+c}{a}&=\frac{0}{a} && \textit{Divde both sides by a.} \\ x^2+\frac{b}{a}x+\frac{c}{a}&=0 && \textit{Subtract } \frac{c}{a} \textit{ from both sides.} \\ x^2+\frac{b}{a}x&=-\frac{c}{a} \end{aligned}$$

Determine the constant that completes the square: take the coefficient of x, divide it by 2, and then square it.

$$\left(\frac{b/a}{2}\right)^2=\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}$$

Add this to both sides of the equation to complete the square and then factor.

$$\begin{aligned} x^2+\frac{b}{a}x+\frac{b^2}{4a^2}&=-\frac{c}{a}+\frac{b^2}{4a^2} \\ \left(x+\frac{b}{2a}\right)\left(x+\frac{b}{2a}\right)&=-\frac{c}{a}+\frac{b^2}{4a^2} \\ \left(x+\frac{b}{2a}\right)^2&=-\frac{4ac}{4a^2}+\frac{b^2}{4a^2} \\ \left(x+\frac{b}{2a}\right)^2&=\frac{b^2-4ac}{4a^2} \end{aligned}$$

Solve by extracting roots.

$$\begin{aligned} \left(x+\frac{b}{2a}\right)^2&=\frac{b^2-4ac}{4a^2} \\ x+\frac{b}{2a}&=\pm\sqrt{\frac{b^2-4ac}{4a^2}} \\ x+\frac{b}{2a}&=\pm\frac{\sqrt{b^2-4ac}}{2a} \\ x&=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a} \\ x&=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \end{aligned}$$

This derivation gives us a formula that solves any quadratic equation in standard form. Given $ax^2+bx+c=0$, where a, b, and c are real numbers and $a \neq 0$, the solutions can be calculated using the **quadratic formula**[5]:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{6.2.1}$$

Example $6.2.1$:

Solve using the quadratic formula: $2x^2-7x-15=0$

Solution

Begin by identifying the coefficients of each term: a, b, and c.

$$a=2 \quad b=-7 \quad c=-15$$

Substitute these values into the quadratic formula (Equation 6.2.1) and then simplify.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-(-7) \pm \sqrt{(-7)^2-4(2)(-15)}}{2(2)} \\ &= \frac{7 \pm \sqrt{49+120}}{4} \\ &= \frac{7 \pm \sqrt{169}}{4} \\ &= \frac{7 \pm 13}{4} \end{aligned}$$

Separate the "plus or minus" into two equations and simplify further.

$$\begin{aligned} x = \tfrac{7-13}{4} &\text{ or } x = \tfrac{7+13}{4} \\ x = \tfrac{-6}{4} &\quad x = \tfrac{20}{4} \\ x = -\tfrac{3}{2} &\quad x = 5 \end{aligned}$$

Answer:

The solutions are $-\frac{3}{2}$ and 5.

The previous example can be solved by factoring as follows:

$$\begin{aligned} 2x^2-7x-15 &= 0 \\ (2x+3)(x-5) &= 0 \end{aligned}$$

$$\begin{aligned} 2x+3 = 0 &\quad \text{ or } x-5=0 \\ 2x = -3 &\quad x = 5 \\ x = -\frac{3}{2} & \end{aligned}$$

Of course, if the quadratic expression factors, then it is a best practice to solve the equation by factoring. However, not all quadratic polynomials factor so easily. The quadratic formula (Equation 6.2.1) provides us with a means to solve all quadratic equations.

Example 6.2.2:

Solve using the quadratic formula: $3x^2+6x-2=0$.

Solution

Begin by identifying a, b, and c.

$$a=3 \quad b=6 \quad c=-2$$

Substitute these values into the quadratic formula (Equation 6.2.1).

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-(6) \pm \sqrt{(6)^2-4(3)(-2)}}{2(3)} \\ &= \frac{-6 \pm \sqrt{36+24}}{6} \\ &= \frac{-6 \pm \sqrt{60}}{6} \end{aligned}$$

At this point we see that $60 = 4 \times 15$ and thus the fraction can be simplified further.

$$
\begin{aligned}
&= \frac{-6 \pm \sqrt{60}}{6} \\
&= \frac{-6 \pm \sqrt{4 \times 15}}{6} \\
&= \frac{-6 \pm 2\sqrt{15}}{6} \\
&= \frac{\cancel{2}(-3 \pm \sqrt{15})}{\cancel{6}} \\
&= \frac{-3 \pm \sqrt{15}}{3}
\end{aligned}
$$

It is important to point out that there are two solutions here:

$$x = \tfrac{-3-\sqrt{15}}{3} \quad \text{or} \quad x = \tfrac{-3+\sqrt{15}}{3}$$

We may use $\pm$ to write the two solutions in a more compact form.

Answer:

The solutions are $\frac{-3\pm\sqrt{15}}{3}$.

Sometimes terms are missing. When this is the case, use 0 as the coefficient.

Example $6.2.3$:

Solve using the quadratic formula: $x^2 - 45 = 0$

Solution

This equation is equivalent to

$$1x^2 + 0x - 45 = 0$$

And we can use the following coefficients:

$$a = 1 \quad b = 0 \quad c = -45$$

Substitute these values into the quadratic formula (Equation $6.2.1$).

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(0) \pm \sqrt{(0)^2 - 4(1)(-45)}}{2(1)} \\
&= \frac{0 \pm \sqrt{0 + 180}}{2} \\
&= \frac{\pm\sqrt{36}}{2} \\
&= \frac{\pm\sqrt{36}\sqrt{5}}{2} \\
&= \frac{\pm 6\sqrt{5}}{2} \\
&= \pm 3\sqrt{5}
\end{aligned}
$$

Since the coefficient of x was 0, we could have solved this equation by extracting the roots. As an exercise, solve it using this method and verify that the results are the same.

Answer:

The solutions are $\pm 3\sqrt{5}$.

Often solutions to quadratic equations are not real.

Example 6.2.4:

Solve using the quadratic formula: $x^2 - 4x + 29 = 0$.

Solution

Begin by identifying a, b, and c. Here

$$a = 1 \quad b = -4 \quad c = 29$$

Substitute these values into the quadratic formula (Equation 6.2.1) and then simplify.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(29)}}{2(1)} \\ &= \frac{4 \pm \sqrt{16 - 116}}{2} \\ &= \frac{4 \pm \sqrt{-100}}{2} \quad \textit{Negative radicand} \\ &= \frac{4 \pm 10i}{2} \quad \textit{Two complex solutions} \\ &= \frac{4}{2} \pm \frac{10i}{2} \\ &= 2 \pm 5i \end{aligned}$$

Check these solutions by substituting them into the original equation.

Table 6.2.1

Check $x = 2 - 5i$	**Check** $x = 2 + 5i$
$x^2 - 4x + 29 = 0$ $(2 - 5i)^2 - 4(2 - 5i) + 29 = 0$ $4 - 20i + 25i^2 - 8 + 20i + 29 = 0$ $25i^2 + 25 = 0$ $25(-1) + 25 = 0$ $-25 + 25 = 0$ ✓	$x^2 - 4x + 29 = 0$ $(2 + 5i)^2 - 4(2 + 5i) + 29 = 0$ $4 + 20i + 25i^2 - 8 - 20i + 29 = 0$ $25i^2 + 25 = 0$ $25(-1) + 25 = 0$ $-25 + 25 = 0$ ✓

Answer:

The solutions are $2 \pm 5i$.

The equation may not be given in standard form. The general steps for using the quadratic formula are outlined in the following example.

Example 6.2.5:

Solve: $(5x + 1)(x - 1) = x(x + 1)$

Solution

Step 1: Write the quadratic equation in standard form, with zero on one side of the equal sign.

$$\begin{aligned} (5x + 1)(x - 1) &= x(x + 1) \\ 5x^2 - 5x + x - 1 &= x^2 + x \\ 5x^2 - 4x - 1 &= x^2 + x \\ 4x^2 - 5x - 1 &= 0 \end{aligned}$$

Step 2: Identify a, b, and c for use in the quadratic formula (Equation 6.2.1). Here

$$a = 4 \quad b = -5 \quad c = -1$$

Step 3: Substitute the appropriate values into the quadratic formula (Equation 6.2.1) and then simplify.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-(-5) \pm \sqrt{(-5)^2-4(4)(-1)}}{2(4)} \\ &= \frac{5 \pm \sqrt{25+16}}{8} \\ &= \frac{5 \pm \sqrt{41}}{8} \end{aligned}$$

Answer:

The solution is $\frac{5\pm\sqrt{41}}{8}$.

Exercise 6.2.1

Solve: $(x+3)(x-5)=-19$

Answer

$1 \pm i\sqrt{3}$

www.youtube.com/v/r78S_kXUSOY

The Discriminant

If given a quadratic equation in standard form, $ax^2+bx+c=0$, where a, b, and c are real numbers and $a \neq 0$, then the solutions can be calculated using the quadratic formula:

$$x=\frac{-b \pm \sqrt{b^2-4ac}}{2a} \tag{6.2.2}$$

As we have seen, the solutions can be rational, irrational, or complex. We can determine the number and type of solutions by studying the **discriminant**[6], the expression inside the radical, b^2-4ac. If the value of this expression is negative, then the equation has two complex solutions. If the discriminant is positive, then the equation has two real solutions. And if the discriminant is 0, then the equation has one real solution, a double root.

Example 6.2.6:

Determine the type and number of solutions: $2x^2+x+3=0$

Solution

We begin by identifying a, b, and c. Here

$$a=2 \quad b=1 \quad c=3$$

Substitute these values into the discriminant (Equation 6.2.2) and simplify.

$$\begin{aligned} b^2-4ac &= (1)^2-4(2)(3) \\ &= 1-24 \\ &= -23 \end{aligned}$$

Since the discriminant is negative, we conclude that there are no real solutions. They are complex.

Answer:

The complex solutions.

If we use the quadratic formula in the previous example, we find that a negative radicand introduces the imaginary unit and we are left with two complex solutions.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-(1) \pm \sqrt{-23}}{2(2)} \\ &= \frac{-1 \pm i\sqrt{23}}{4} \\ &= -\frac{1}{4} \pm \frac{\sqrt{23}}{4} i \quad \textit{Two complex solutions} \end{aligned}$$

> Irrational and complex solutions of quadratic equations always appear in conjugate pairs.

Example 6.2.7:

Determine the type and number of solutions: $6x^2 - 5x - 1 = 0$.

Solution

In this example,

$$a = 6 \quad b = -5 \quad c = -1$$

Substitute these values into the discriminant (Equation 6.2.2) and simplify.

$$\begin{aligned} b^2 - 4ac &= (-5)^2 - 4(6)(-1) \\ &= 25 + 24 \\ &= 49 \end{aligned}$$

Since the discriminant is positive, we conclude that the equation has two real solutions. Furthermore, since the discriminant is a perfect square, we obtain two rational solutions.

Answer:

Two rational solutions

Because the discriminant is a perfect square, we could solve the previous quadratic equation by factoring or by using the quadratic formula.

Table 6.2.2

Solve by factoring:	Solve using the quadratic formula:
$6x^2 - 5x - 1 = 0$ $(6x+1)(x-1) = 0$ $6x + 1 = 0 \quad \text{or} \quad x - 1 = 0$ $6x = -1 \quad x = 1$ $x = -\frac{1}{6}$	$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ $= \frac{-(-5) \pm \sqrt{49}}{2(6)}$ $= \frac{5 \pm 7}{12}$ $x = \frac{5-7}{12} \quad \text{or} \quad x = \frac{5+7}{12}$ $x = \frac{-2}{12} \quad x = \frac{12}{12}$ $x = -\frac{1}{6} \quad x = 1$

Given the special condition where the discriminant is 0, we obtain only one solution, a double root.

Example 6.2.8:

Determine the type and number of solutions: $25x^2 - 20x + 4 = 0$.

Solution

Here $a = 25, b = -20$, and $c = 4$, and we have (via (Equation 6.2.2))

$$\begin{aligned} b^2 - 4ac &= (-20)^2 - 4(25)(4) \\ &= 400 - 400 \\ &= 0 \end{aligned}$$

Since the discriminant is 0, we conclude that the equation has only one real solution, a double root.

Answer:

One rational solution

Since 0 is a perfect square, we can solve the equation above by factoring.

$$\begin{aligned} 25x^2-20x+4&=0\\ (5x-2)(5x-2)&=0 \end{aligned}$$

$$\begin{array}{rl} 5x-2=0 \text{ or } & 5x-2=0\\ 5x=2 & 5x=2\\ x=\frac{2}{5} & x=\frac{2}{5} \end{array}$$

Here $\frac{2}{5}$ is a solution that occurs twice; it is a double root.

Example $6.2.9$:

Determine the type and number of solutions: $x^2-2x-4=0$.

Solution

Here $a=1, b=-2$, and $c=-4$, and we have

$$\begin{aligned} b^2-4ac&=(-2)^2-4(1)(-4)\\ &=4+16\\ &=20 \end{aligned} \tag{6.2.3}$$

Since the discriminant is positive, we can conclude that the equation has two real solutions. Furthermore, since 20 is not a perfect square, both solutions are irrational.

Answer

Two irrational solutions.

If we use the quadratic formula in the previous example, we find that a positive radicand in the quadratic formula leads to two real solutions.

$$\begin{aligned} x&=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\ &=\frac{-(-2)\pm\sqrt{20}}{2(1)} && \textit{Positive discriminant}\\ &=\frac{2\pm\sqrt{4\times 5}}{2}\\ &=\frac{2\pm 2\sqrt{5}}{2}\\ &=\frac{\cancel{2}(1\pm\sqrt{5}}{\cancel{2}}\\ &=1\pm\sqrt{5} && \textit{Two irrational solutions} \end{aligned}$$

Two real solutions are $1-\sqrt{5}$ and $1+\sqrt{5}$. Note that these solutions are irrational; we can approximate the values con a calculator.

$$1-\sqrt{5}\approx -1.24 \text{ and } 1+\sqrt{5}\approx 3.24$$

In summary, if given any quadratic equation in standard form, $ax^2+bx+c=0$, where a, b, and c are real numbers and $a\neq 0$, then we have the following:

- **Positive discriminant:** $b^2-4ac>0$ Two real solutions
- **Zero discriminant:** $b^2-4ac=0$ One real solution
- **Negative discriminant**: $b^2-4ac<0$ Two complex solutions

Furthermore, if the discriminant is nonnegative and a perfect square, then the solutions to the equation are rational; otherwise they are irrational. As we will see, knowing the number and type of solutions ahead of time helps us determine which method is best for solving a quadratic equation.

Exercise 6.2.2

Determine the number and type of solutions: $2x^2 = x - 2$.

Answer

Two complex solutions.

www.youtube.com/v/Km05jHRG-vM

Key Takeaways

- We can use the quadratic formula to solve any quadratic equation in standard form.
- To solve any quadratic equation, we first rewrite it in standard form $ax^2 + bx + c = 0$, substitute the appropriate coefficients into the quadratic formula, $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, and then simplify.
- We can determine the number and type of solutions to any quadratic equation in standard form using the discriminant, $b^2 - 4ac$. If the value of this expression is negative, then the equation has two complex solutions. If the discriminant is positive, then the equation has two real solutions. And if the discriminant is 0, then the equation has one real solution, a double root.
- We can further classify real solutions into rational or irrational numbers. If the discriminant is a perfect square, the roots are rational and the equation will factor. If the discriminant is not a perfect square, the roots are irrational.

Exercise 6.2.3

Identify the coefficients, a, b, and c, used in the quadratic formula. Do not solve.

1. $x^2 - x + 3 = 0$
2. $5x^2 - 2x - 8 = 0$
3. $4x^2 - 9 = 0$
4. $x^2 + 3x = 0$
5. $-x^2 + 2x - 7 = 0$
6. $-2x^2 - 5x + 2 = 0$
7. $px^2 - qx - 1 = 0$
8. $p^2x^2 - x + 2q = 0$
9. $(x - 5)^2 = 49$
10. $(2x + 1)^2 = 2x - 1$

Answer

1. $a = 1; b = -1; c = 3$

3. $a = 4; b = 0; c = -9$

5. $a = -1; b = 2; c = -7$

7. $a = p, b = -q; c = -1$

9. $a = 1; b = -10; c = -24$

Exercise 6.2.4

Solve by factoring and then solve using the quadratic formula. Check answers.

1. $x^2 - 6x - 16 = 0$
2. $x^2 - 3x - 18 = 0$
3. $2x^2 + 7x - 4 = 0$
4. $3x^2 + 5x - 2 = 0$
5. $4y^2 - 9 = 0$
6. $9y^2 - 25 = 0$
7. $5t^2 - 6t = 0$
8. $t^2 + 6t = 0$

9. $-x^2+9x-20=0$
10. $-2x^2-3x+5=0$
11. $16y^2-24y+9=0$
12. $4y^2-20y+25=0$

Answer

1. $-2, 8$
3. $-4, \frac{1}{2}$
5. $\pm\frac{3}{2}$
7. $0, \frac{6}{5}$
9. $4, 5$
11. $\frac{3}{4}$

Exercise 6.2.5

Solve by extracting the roots and then solve using the quadratic formula. Check answers.

1. $x^2-18=0$
2. $x^2-12=0$
3. $x^2+12=0$
4. $x^2+20=0$
5. $3x^2+2=0$
6. $5x^2+3=0$
7. $(x+2)^2+9=0$
8. $(x-4)^2+1=0$
9. $(2x+1)^2-2=0$
10. $(3x+1)^2-5=0$

Answer

1. $\pm 3\sqrt{2}$
3. $\pm 2i\sqrt{3}$
5. $\pm\frac{i\sqrt{6}}{3}$
7. $-2\pm 3i$
9. $\frac{-1\pm\sqrt{2}}{2}$

Exercise 6.2.6

Solve using the quadratic formula.

1. $x^2-5x+1=0$
2. $x^2-7x+2=0$
3. $x^2+8x+5=0$
4. $x^2-4x+2=0$
5. $y^2-2y+10=0$
6. $y^2-4y+13=0$
7. $2x^2-10x-1=0$
8. $2x^2-4x-3=0$
9. $3x^2-x+2=0$
10. $4x^2-3x+1=0$
11. $5u^2-2u+1=0$

12. $8u^2-20u+13=0$
13. $-y^2+16y-62=0$
14. $-y^2+14y-46=0$
15. $-2t^2+4t+3=0$
16. $-4t^2+8t+1=0$
17. $\frac{1}{2}y^2+5y+\frac{3}{2}=0$
18. $3y^2+\frac{1}{2}y-\frac{1}{3}=0$
19. $2x^2-\frac{1}{2}x+\frac{1}{4}=0$
20. $3x^2-\frac{2}{3}x+\frac{1}{3}=0$
21. $1.2x^2-0.5x-3.2=0$
22. $0.4x^2+2.3x+1.1=0$
23. $2.5x^2-x+3.6=0$
24. $-0.8x^2+2.2x-6.1=0$
25. $-2y^2=3(y-1)$
26. $3y^2=5(2y-1)$
27. $(t+1)^2=2t+7$
28. $(2t-1)^2=73-4t$
29. $(x+5)(x-1)=2x+1$
30. $(x+7)(x-2)=3(x+1)$
31. $2x(x-1)=-1$
32. $x(2x+5)=3x-5$
33. $3t(t-2)+4=0$
34. $5t(t-1)=t-4$
35. $(2x+3)^2=16x+4$
36. $(2y+5)^2-12(y+1)=0$

Answer

1. $\frac{5\pm\sqrt{21}}{2}$

3. $-4\pm\sqrt{11}$

5. $1\pm 3i$

7. $\frac{5\pm 3\sqrt{3}}{2}$

9. $\frac{1}{6}\pm\frac{\sqrt{23}}{6}i$

11. $\frac{1}{5}\pm\frac{2}{5}i$

13. $8\pm\sqrt{2}$

15. $\frac{2\pm\sqrt{10}}{2}$

17. $-5\pm\sqrt{22}$

19. $\frac{1}{8}\pm\frac{\sqrt{7}}{8}i$

21. $x\approx-1.4$ or $x\approx 1.9$

23. $x\approx 0.2\pm 1.2i$

25. $\frac{-3\pm\sqrt{33}}{4}$

27. $\pm\sqrt{6}$

29. $-1\pm\sqrt{7}$

31. $\frac{1}{2}\pm\frac{1}{2}i$

33. $1\pm\frac{\sqrt{3}}{3}i$

35. $\frac{1}{2}\pm i$

Exercise 6.2.7

Assume p and q are nonzero integers and use the quadratic formula to solve for x.

1. $px^2 + x + 1 = 0$
2. $x^2 + px + 1 = 0$
3. $x^2 + x - p = 0$
4. $x^2 + px + q = 0$
5. $p^2x^2 + 2px + 1 = 0$
6. $x^2 - 2qx + q^2 = 0$

Answer

1. $x = \frac{-1 \pm \sqrt{1-4p}}{2p}$

3. $x = \frac{-1 \pm \sqrt{1+4p}}{2}$

5. $x = -\frac{1}{p}$

Exercise 6.2.8

Solve using algebra.

1. The height in feet reached by a baseball tossed upward at a speed of 48 feet per second from the ground is given by \($h(t) = -16t^2 + 48t$\), where t represents time in seconds after the ball is tossed. At what time does the baseball reach 24 feet? (Round to the nearest tenth of a second.)
2. The height in feet of a projectile launched upward at a speed of 32 feet per second from a height of 64 feet is given by \($h(t) = -16t^2 + 32t + 64$\). At what time after launch does the projectile hit the ground? (Round to the nearest tenth of a second.)
3. The profit in dollars of running an assembly line that produces custom uniforms each day is given by \($P(t) = -40t^2 + 960t - 4,000$\) where t represents the number of hours the line is in operation. Determine the number of hours the assembly line should run in order to make a profit of $\$1,760$ per day.
4. A manufacturing company has determined that the daily revenue R in thousands of dollars is given by \($R(n) = 12n - 0.6n^2$\) where n represents the number of pallets of product sold. Determine the number of pallets that must be sold in order to maintain revenues at 60 thousand dollars per day.
5. The area of a rectangle is 10 square inches. If the length is 3 inches more than twice the width, then find the dimensions of the rectangle. (Round to the nearest hundredth of an inch.)
6. The area of a triangle is 2 square meters. If the base is 2 meters less than the height, then find the base and the height. (Round to the nearest hundredth of a meter.)
7. To safely use a ladder, the base should be placed about $\frac{1}{4}$ of the ladder's length away from the wall. If a 32-foot ladder is used safely, then how high against a building does the top of the ladder reach? (Round to the nearest tenth of a foot.)
8. The length of a rectangle is twice its width. If the diagonal of the rectangle measures 10 centimeters, then find the dimensions of the rectangle. (Round to the nearest tenth of a centimeter.)
9. Assuming dry road conditions and average reaction times, the safe stopping distance in feet of a certain car is given by $d(x) = \frac{1}{20}x^2 + x$ where x represents the speed of the car in miles per hour. Determine the safe speed of the car if you expect to stop in 50 feet. (Round to the nearest mile per hour.)
10. The width of a rectangular solid is 2.2 centimeters less than its length and the depth measures 10 centimeters.

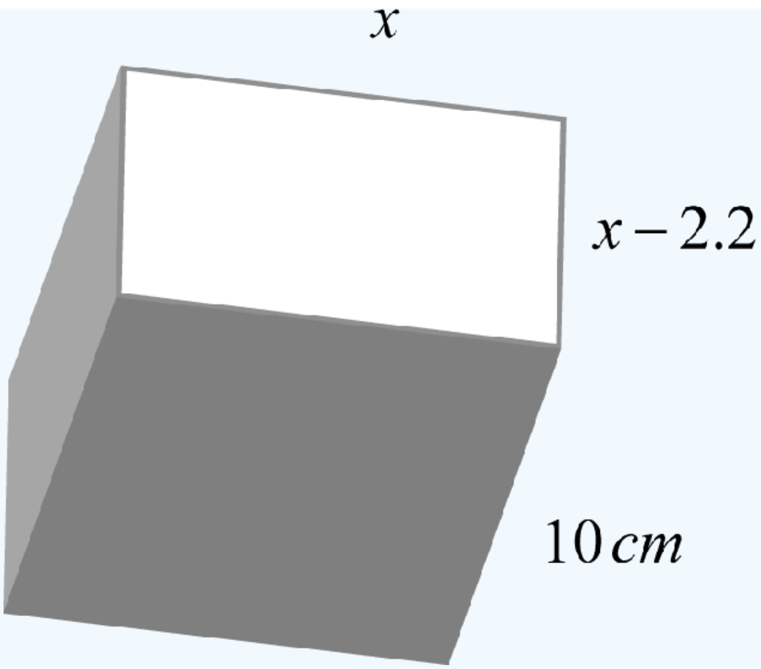

Figure $6.2.1$

Determine the length and width if the total volume of the solid is 268.8 cubic centimeters.

11. An executive traveled 25 miles in a car and then another 30 miles on a helicopter. If the helicopter was 10 miles per hour less than twice as fast as the car and the total trip took 1 hour, then what was the average speed of the car? (Round to the nearest mile per hour.)

12. Joe can paint a typical room in 1.5 hours less time than James. If Joe and James can paint 2 rooms working together in an 8-hour shift, then how long does it take James to paint a single room? (Round to the nearest tenth of an hour.)

Answer

1. 0.6 seconds and 2.4 seconds
3. 12 hours
5. Length: 6.22 inches; width; 1.61 inches
7. 31.0 feet
9. 23 miles per hour
11. 42 miles per hour

Exercise $6.2.9$

Calculate the discriminant and use it to determine the number and type of solutions. Do not solve.

1. $x^2 - x + 1 = 0$
2. $x^2 + 2x + 3 = 0$
3. $x^2 - 2x - 3 = 0$
4. $x^2 - 5x - 5 = 0$
5. $3x^2 - 1x - 2 = 0$
6. $3x^2 - 1x + 2 = 0$
7. $9y^2 + 2 = 0$
8. $9y^2 - 2 = 0$
9. $2x^2 + 3x = 0$
10. $4x^2 - 5x = 0$
11. $\frac{1}{2}x^2 - 2x + \frac{5}{2} = 0$

12. $\frac{1}{2}x^2 - x - \frac{1}{2} = 0$
13. $-x^2 - 3x + 4 = 0$
14. $-x^2 - 5x + 3 = 0$
15. $25t^2 + 30t + 9 = 0$
16. $9t^2 - 12t + 4 = 0$

Answer

1. -3; two complex solutions
3. 16; two rational solutions
5. 25; two rational solutions
7. -72; two complex solutions
9. 9; two rational solutions
11. -1; two complex solutions
13. 25; two rational solutions
15. 0; one rational solution

Exercise 6.2.10

Find a nonzero integer p so that the following equations have one real solution. (Hint: If the discriminant is zero, then there will be one real solution.)

1. $px^2 - 4x - 1 = 0$
2. $x^2 - 8x + p = 0$
3. $x^2 + px + 25 = 0$
4. $x^2 - 2x + p^2 = 0$

Answer

1. $p = -4$
3. $p = \pm 10$

Exercise 6.2.11

1. When talking about a quadratic equation in standard form $ax^2 + bx + c = 0$, why is it necessary to state that $a \neq 0$? What would happen if a is equal to zero?
2. Research and discuss the history of the quadratic formula and solutions to quadratic equations.
3. Solve $mx^2 + nx + p = 0$ for x by completing the square.

Answer

1. Answer may vary
3. Answer may vary

Footnotes

[5] The formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, which gives the solutions to any quadratic equation in the standard form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$.

[6] The expression inside the radical of the quadratic formula, $b^2 - 4ac$.

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6.3: Solving Equations Quadratic in Form

Learning Objectives

- Develop a general strategy for solving quadratic equations.
- Solve equations that are quadratic in form.

Use the coefficients of a quadratic equation to help decide which method is most appropriate for solving it. While the quadratic formula always works, it is sometimes not the most efficient method. If given any quadratic equation in standard form,

$$ax^2 + bx + c = 0$$

where $c = 0$, then it is best to factor out the GCF and solve by factoring.

Example 6.3.1:

Solve: $12x^2 - 3x = 0$

Solution

In this case, $c = 0$ and we can solve by factoring out the GCF $3x$.

$$\begin{aligned} 12x^2 - 3x &= 0 \\ 3x(4x-1) &= 0 \end{aligned}$$

Then apply the zero-product property and set each factor equal to zero.

$$\begin{array}{rlr} 3x = 0 & \text{or} & 4x - 1 = 0 \\ x = 0 & & 4x = 1 \\ & & x = \dfrac{1}{4} \end{array}$$

Answer

The solutions are 0 and $\frac{1}{4}$.

If $b = 0$, then we can solve by extracting the roots.

Example 6.3.2:

Solve $5x^2 + 8 = 0$

Solution

In this case, $b = 0$ and we can solve by extracting the roots. Begin by isolating the square.

$$\begin{aligned} 5x^2 + 8 &= 0 \\ 5x^2 &= -8 \\ x^2 &= -\frac{8}{5} \end{aligned}$$

Next, apply the square root property. Remember to include the $\pm$.

$$\begin{aligned} x &= \pm\sqrt{-\frac{8}{5}} & \color{cyan}{\textit{Rationalize the denominator.}} \\ &= \pm\frac{\sqrt{-4 \cdot 2}}{\sqrt{5}} \cdot \color{cyan}{\frac{\sqrt{5}}{\sqrt{5}}} & \color{cyan}{\textit{Simplify.}} \\ &= \pm\frac{2i\sqrt{2} \cdot \sqrt{5}}{5} \\ &= \pm\frac{2i\sqrt{10}}{5} \end{aligned}$$

Answer

The solutions are $\pm\frac{2i\sqrt{10}}{5}$.

Howto: General Guidelines for Solving Quadratic Equations

When given a quadratic equation in standard form where a, b, and c are all nonzero, determine the value for the discriminant using the formula $b^2 - 4ac$.

1. If the discriminant is a perfect square, then solve by factoring.
2. If the discriminant is not a perfect square, then solve using the quadratic formula.

Recall that if the discriminant is not a perfect square and positive, the quadratic equation will have two irrational solutions. And if the discriminant is negative, the quadratic equation will have two complex conjugate solutions.

Example 6.3.3:

Solve $(3x+5)(3x+7) = 6x+10$.

Solution

Begin by rewriting the quadratic equation in standard form.

$$\begin{aligned}(3x+5)(3x+7) &= 6x+10\\ 9x^2+21x+15x+35 &= 6x+10\\ 9x^2+36x+35 &= 6x+10\\ 9x^2+30x+25 &= 0\end{aligned}$$

Substitute, $a = 9, b = 30$, and $c = 25$ into the discriminant formula.

$$\begin{aligned}b^2-4ac &= (30)^2-4(9)(25)\\ &= 900-900\\ &= 0\end{aligned}$$

Since the discriminant is 0, solve by factoring and expect one real solution, a double root.

$$\begin{aligned}9x^2+30x+25 &= 0\\ (3x+5)(3x+5) &= 0\end{aligned}$$

$$\begin{aligned}3x+5 &= 0 \quad \text{or} & 3x+5 &= 0\\ 3x &= -5 & 3x &= -5\\ x &= -\frac{5}{3} & x &= -\frac{5}{3}\end{aligned}$$

Answer

The solution is $-\frac{5}{3}$.

It is good to know that the quadratic formula will work to find the solutions to all of the examples in this section. However, it is not always the best solution. If the equation can be solved by factoring or by extracting the roots, you should use that method.

Solving Equations Quadratic in Form

In this section we outline an algebraic technique that is used extensively in mathematics to transform equations into familiar forms. We begin by defining **quadratic form**[7],

$$au^2+bu+c=0$$

Here u represents an algebraic expression. Some examples follow:

$$\begin{aligned}\left(\frac{t+2}{t}\right)^2+8\left(\frac{t+2}{t}\right)+7=0 &\overset{u=\frac{t+2}{t}}{\Longrightarrow} u^2+8u+7=0\\ x^{2/3}-3x^{1/3}-10=0 &\overset{u=x^{1/3}}{\Longrightarrow} u^2-3u-10=0\\ 3y^{-2}+7y^{-1}-6=0 &\overset{u=y^{-1}}{\Longrightarrow} 3u^2+7u-6=0\end{aligned}$$

If we can express an equation in quadratic form, then we can use any of the techniques used to solve quadratic equations. For example, consider the following fourth-degree polynomial equation,

$$x^4-4x^2-32=0$$

If we let $u = x^2$ then $u^2 = (x^2)^2 = x^4$ and we can write

$$\begin{aligned} x^4-4x^2-32=0 \Rightarrow \left(x^2\right)^2 &- 4\left(x^2\right) - 32 = 0 \\ \downarrow \quad &\quad \downarrow \\ u^2 \quad &- \quad 4u \quad - 32 = 0 \end{aligned}$$

This substitution transforms the equation into a familiar quadratic equation in terms of u which, in this case, can be solved by factoring.

$$\begin{aligned} u^2-4u-32&=0 \\ (u-8)(u+4)&=0 \\ u=8 \quad \text{or} \quad u&=-4 \end{aligned}$$

Since $u=x^2$ we can back substitute and then solve for x.

$$\begin{array}{ll} u=8 \quad \text{or} & u=-4 \\ \downarrow & \quad\quad\downarrow \\ x^2=8 & x^2=-4 \\ x=\pm\sqrt{8} & x=\pm\sqrt{-4} \\ x=\pm 2\sqrt{2} & x=\pm 2i \end{array}$$

Therefore, the equation $x^4-4x^2-32=0$ has four solutions $\{\pm 2\sqrt{2}, \pm 2i\}$, two real and two complex. This technique, often called a **u-substitution**[8], can also be used to solve some non-polynomial equations.

Example 6.3.4:

Solve: $x-2\sqrt{x}-8=0$.

Solution

This is a radical equation that can be written in quadratic form. If we let $u=\sqrt{x}$ then $u^2=(\sqrt{x})^2=x$ and we can write

$$\begin{aligned} x - 2\sqrt{x} &- 8 = 0 \\ \downarrow \quad \downarrow & \\ u^2 \quad - 2u &- 8 = 0 \end{aligned}$$

Solve for u.

$$\begin{aligned} u^2-2u-8&=0 \\ (u-4)(u+2)&=0 \\ u=4 \text{ or } u&=-2 \end{aligned}$$

Back substitute $u=\sqrt{x}$ and solve for x.

$$\begin{array}{ccc} \sqrt{x}=4 & \text{or} & \sqrt{x}=-2 \\ (\sqrt{x})^2=(4)^2 & & (\sqrt{x})^2=(-2)^2 \\ x=16 & & x=4 \end{array}$$

Recall that squaring both sides of an equation introduces the possibility of extraneous solutions. Therefore we must check our potential solutions.

$$\begin{array}{r|l} \textit{Check } x=16 & \textit{Check } x=4 \\ x-2\sqrt{x}-8=0 & x-2\sqrt{x}-8=0 \\ 16-2\sqrt{16}-18=0 & 4-2\sqrt{4}-8=0 \\ 16-2\cdot 4-8=0 & 4-2\cdot 2-8=0 \\ 16-8-8=0 & 4-4-8=0 \\ 0=0\checkmark & -8=0 ✗ \end{array}$$

Because $x=4$ is extraneous, there is only one solution, $x=16$.

Answer

The solution is 16.

Example 6.3.5:

Solve: $x^{2/3}-3x^{1/3}-10=0$.

Solution

If we let $u=x^{1/3}$ then $u^2=\left(x^{1/3}\right)^2=x^{2/3}$ and we can write

$$\begin{array}{c} x^{2/3}-3x^{1/3}-10=0 \\ \downarrow \qquad \downarrow \\ u^2 \quad -\ 3u-10=0 \end{array}$$

Solve for u.

$$\begin{aligned} u^2-3u-10&=0 \\ (u-5)(u+2)&=0 \\ u=5 \quad \text{or} \quad u&=-2 \end{aligned}$$

Back substitute $u=x^{1/3}$ and solve for x.

$$\begin{array}{rr} x^{1/3}=5 \quad \text{or} & x^{1/3}=-2 \\ \left(x^{1/3}\right)^3=(5)^3 & \left(x^{1/3}\right)^3=(-2)^3 \\ x=125 \qquad & x=-8 \qquad \end{array}$$

Check

$$\begin{array}{r|r} \textit{Check } x=125 & \textit{Check } x=-8 \\ x^{2/3}-3x^{1/3}-10=0 & x^{2/3}-3x^{1/3}-10=0 \\ (125)^{2/3}-3(125)^{1/3}-10=0 & (-8)^{2/3}-3(-8)^{1/3}-10=0 \\ \left(5^3\right)^{2/3}-3\left(5^3\right)^{1/3}-10=0 & \left[(-2)^3\right]^{2/3}-3\left[(-2)^3\right]^{1/3}-10=0 \\ 5^2-3\cdot 5-10=0 & (-2)^2-3\cdot(-2)-10=0 \\ 25-15-10=0 & 4+6-10=0 \\ 0=0\checkmark & 0=0\checkmark \end{array}$$

Answer

The solutions are $-8, 125$.

Example 6.3.6:

Solve: $3y^{-2}+7y^{-1}-6=0$

Solution

If we let $u=y^{-1}$, then $u^2=\left(y^{-1}\right)^2=y^{-2}$ and we can write

$$\begin{array}{c} 3y^{-2}+7y^{-1}-6=0 \\ \downarrow \qquad \downarrow \\ 3u^2+7u-6=0 \end{array}$$

Solve for u.

$$\begin{aligned} 3u^2+7u-6&=0 \\ (3u-2)(u+3)&=0 \\ u=\tfrac{2}{3} \quad \text{or} \quad u&=-3 \end{aligned}$$

Back substitute $u=y^{-1}$ and solve for y.

$$\begin{array}{rr} y^{-1}=\dfrac{2}{3} \text{ or} & y^{-1}=-3 \\ \dfrac{1}{y}=\dfrac{2}{3} \quad & \dfrac{1}{y}=-3 \\ y=\dfrac{3}{2} \quad & y=-\dfrac{1}{3} \end{array}$$

The original equation is actually a rational equation where $y\neq 0$. In this case, the solutions are not restrictions; they solve the original equation.

Answer

The solutions are $-\frac{1}{3}, \frac{3}{2}$.

Example 6.3.7:

Solve: $\left(\frac{t+2}{t}\right)^2 + 8\left(\frac{t+2}{t}\right) + 7 = 0$

Solution

If we let $u = \frac{t+2}{t}$, then $u^2 = \left(\frac{t+2}{t}\right)^2$ and we can write

$$\begin{array}{c} \left(\frac{t+2}{t}\right)^2 + 8\left(\frac{t+2}{t}\right) + 7 = 0 \\ \downarrow \qquad\quad \downarrow \\ u^2 + \quad 8u \quad + 7 = 0 \end{array}$$

Solve for u.

$$\begin{aligned} u^2 + 8u + 7 &= 0 \\ (u+1)(u+7) &= 0 \\ u = -1 \quad \text{or} &\quad u = -7 \end{aligned}$$

Back substitute $u = \frac{t+2}{t}$, and solve for t.

$$\begin{aligned} \frac{t+2}{t} &= -1 \text{ or } \frac{t+2}{t} = -7 \\ t+2 &= -t \quad t+2 = -7t \\ 2t &= -2 \qquad 8t = -2 \\ t &= -1 \qquad\qquad\quad t = -\frac{1}{4} \end{aligned}$$

Answer

The solutions are $-1, -\frac{1}{4}$. The check is left to the reader.

Exercise 6.3.1

Solve: $12x^{-2} - 16x^{-1} + 5 = 0$

Answer

The solutions are $\frac{6}{5}, 2$.

www.youtube.com/v/pAuR53nLNDo

So far all of the examples were of equations that factor. As we know, not all quadratic equations factor. If this is the case, we use the quadratic formula.

Example 6.3.8:

Solve $x^4 - 10x^2 + 23 = 0$. Approximate to the nearest hundredth.

Solution

If we let $u = x^2$, then $u^2 = (x^2)^2 = x^4$ and we can write

$$\begin{array}{c} x^4 - 10x^2 + 23 = 0 \\ \downarrow \quad \downarrow \\ u^2 - 10u + 23 = 0 \end{array}$$

This equation does not factor; therefore, use the quadratic formula to find the solutions for u. Here $a = 1, b = -10$, and $c = 23$.

$$\begin{aligned} u &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-(-10) \pm \sqrt{(-10)^2-4(1)(23)}}{2(1)} \\ &= \frac{10 \pm \sqrt{8}}{2} \\ &= \frac{10 \pm 2\sqrt{2}}{2} \\ &= 5 \pm \sqrt{2} \end{aligned}$$

Therefore, $u = 5 \pm \sqrt{2}$. Now back substitute $u = x^2$ and solve for x.

$$\begin{array}{ccc} u = 5-\sqrt{2} & \text{or} & u = 5+\sqrt{2} \\ \downarrow & & \downarrow \\ x^2 = 5-\sqrt{2} & & x^2 = 5+\sqrt{2} \\ x = \pm\sqrt{5-\sqrt{2}} & & x = \pm\sqrt{5+\sqrt{2}} \end{array}$$

Round the four solutions as follows.

$$\begin{array}{ll} x = -\sqrt{5-\sqrt{2}} \approx -1.89 & x = -\sqrt{5+\sqrt{2}} \approx -2.53 \\ x = \sqrt{5-\sqrt{2}} \approx 1.89 & x = \sqrt{5+\sqrt{2}} \approx 2.53 \end{array}$$

Answer

The solutions are approximately $\pm 1.89, \pm 2.53$.

If multiple roots and complex roots are counted, then the **fundamental theorem of algebra**[9] implies that every polynomial with one variable will have as many roots as its degree. For example, we expect $f(x) = x^3 - 8$ to have three roots. In other words, the equation

$$x^3 - 8 = 0$$

should have three solutions. To find them one might first think of trying to extract the cube roots just as we did with square roots,

$$\begin{aligned} x^3 - 8 &= 0 \\ x^3 &= 8 \\ x &= \sqrt[3]{8} \\ x &= 2 \end{aligned}$$

As you can see, this leads to one solution, the real cube root. There should be two others; let’s try to find them.

Example 6.3.9:

Find the set of all roots: $f(x) = x^3 - 8$.

Solution

Notice that the expression $x^3 - 8$ is a difference of cubes and recall that $a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$. Here $a = x$ and $b = 2$ and we can write

$$\begin{aligned} x^3 - 8 &= 0 \\ (x-2)\left(x^2 + 2x + 4\right) &= 0 \end{aligned}$$

Next apply the zero-product property and set each factor equal to zero. After setting the factors equal to zero we can then solve the resulting equation using the appropriate methods.

$$
\begin{array}{lcl}
x-2=0 & \text{or} & x^2+2x+4=0 \\
 & & x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a} \\
 & & =\dfrac{-(2)\pm\sqrt{b^2-4ac}}{2a} \\
x=2 & & =\dfrac{-2\pm\sqrt{-12}}{2} \\
 & & =\dfrac{-2\pm 2i\sqrt{3}}{2} \\
 & & =-1\pm i\sqrt{3}
\end{array}
$$

Using this method, we were able to obtain the set of all three roots $\{2, -1\pm i\sqrt{3}\}$, one real and two complex.

Answer

$\{2, -1\pm i\sqrt{3}\}$

Sometimes the roots of a function will occur multiple times. For example, $g(x)=(x-2)^3$ has degree three where the roots can be found as follows:

$$
\begin{gathered}
(x-2)^3=0 \\
(x-2)(x-2)(x-2)=0 \\[1ex]
\begin{array}{ccccc}
x-2=0 & \text{or} & x-2=0 & \text{or} & x-2=0 \\
x=2 & & x=2 & & x=2
\end{array}
\end{gathered}
$$

Even though g is of degree 3 there is only one real root $\{2\}$; it occurs 3 times.

Key Takeaways

- The quadratic formula can solve any quadratic equation. However, it is sometimes not the most efficient method.
- If a quadratic equation can be solved by factoring or by extracting square roots you should use that method.
- We can sometimes transform equations into equations that are quadratic in form by making an appropriate u-substitution. After solving the equivalent equation, back substitute and solve for the original variable.
- Counting multiple and complex roots, the fundamental theorem of algebra guarantees as many roots as the degree of a polynomial equation with one variable.

Exercise $6.3.2$

Solve

1. $x^2-9x=0$
2. $x^2+10x=0$
3. $15x^2+6x=0$
4. $36x^2-18x=0$
5. $x^2-90=0$
6. $x^2+48=0$
7. $2x^2+1=0$
8. $7x^2-1=0$
9. $6x^2-11x+4=0$
10. $9x^2+12x-5=0$
11. $x^2+x+6=0$
12. $x^2+2x+8=0$
13. $4t^2+28t+49=0$
14. $25t^2-20t+4=0$
15. $u^2-4u-1=0$
16. $u^2-2u-11=0$
17. $2(x+2)^2=11+4x-2x^2$
18. $(2x+1)(x-3)+2x^2=3(x-1)$
19. $(3x+2)^2=6(2x+1)$

20. $(2x-3)^2+5x^2=4(2-3x)$
21. $4(3x-1)^2-5=0$
22. $9(2x+3)^2-2=0$

Answer

1. $0, 9$
3. $-\frac{2}{5}, 0$
5. $\pm 3\sqrt{10}$
7. $\pm \frac{\sqrt{2}}{2} i$
9. $\frac{1}{2}, \frac{4}{3}$
11. $-\frac{1}{2} \pm \frac{\sqrt{23}}{2} i$
13. $-\frac{7}{2}$
15. $2 \pm \sqrt{5}$
17. $-\frac{3}{2}, \frac{1}{2}$
19. $\pm \frac{\sqrt{2}}{3}$
21. $\frac{2 \pm \sqrt{5}}{6}$

Exercise $6.3.3$

Find all solutions.

1. $x^4+x^2-72=0$
2. $x^4-17x^2-18=0$
3. $x^4-13x^2+36=0$
4. $4x^4-17x^2+4=0$
5. $x+2\sqrt{x}-3=0$
6. $x-\sqrt{x}-2=0$
7. $x-5\sqrt{x}+6=0$
8. $x-6\sqrt{x}+5=0$
9. $x^{2/3}+5x^{1/3}+6=0$
10. $x^{2/3}-2x^{1/3}-35=0$
11. $4x^{2/3}-4x^{1/3}+1=0$
12. $3x^{2/3}-2x^{1/3}-1=0$
13. $5x^{-2}+9x^{-1}-2=0$
14. $3x^{-2}+8x^{-1}-3=0$
15. $8x^{-2}+14x^{-1}-15=0$
16. $9x^{-2}-24x^{-1}+16=0$
17. $\left(\frac{x-3}{x}\right)^2-2\left(\frac{x-3}{x}\right)-24=0$
18. $\left(\frac{2x+1}{x}\right)^2+9\left(\frac{2x+1}{x}\right)-36=0$
19. $2\left(\frac{x}{x+1}\right)^2-5\left(\frac{x}{x+1}\right)-3=0$
20. $3\left(\frac{x}{3x-1}\right)^2+13\left(\frac{x}{3x-1}\right)-10=0$
21. $4y^{-2}-9=0$
22. $16y^{-2}+4y^{-1}=0$
23. $30y^{2/3}-15y^{1/3}=0$
24. $y^{2/3}-9=0$
25. $81y^4-1=0$
26. $5\left(\frac{1}{x+2}\right)^2-3\left(\frac{1}{x+2}\right)-2=0$
27. $12\left(\frac{x}{2x-3}\right)^2-11\left(\frac{x}{2x-3}\right)+2=0$
28. $10x^{-2}-19x^{-1}-2=0$

29. $x^{1/2}-3x^{1/4}+2=0$
30. $x+5\sqrt{x}-50=0$
31. $8x^{2/3}+7x^{1/3}-1=0$
32. $x^{4/3}-13x^{2/3}+36=0$
33. $y^4-14y^2+46=0$
34. $x^{4/3}-2x^{2/3}+1=0$
35. $2y^{-2}-y^{-1}-1=0$
36. $2x^{-2/3}-3x^{-1/3}-2=0$
37. $4x^{-1}-17x^{-1/2}+4=0$
38. $3x^{-1}-8x^{-1/2}+4=0$
39. $2x^{1/3}-3x^{1/6}+1=0$
40. $x^{1/3}-x^{1/6}-2=0$

Answer

1. $\pm 2\sqrt{2}, \pm 3i$

3. $\pm 2, \pm 3$

5. 1

7. $4, 9$

9. $-27, -8$

11. $\frac{1}{8}$

13. $-\frac{1}{2}, 5$

15. $-\frac{2}{5}, \frac{4}{3}$

17. $\pm\frac{3}{5}$

19. $-\frac{3}{2}, -\frac{1}{3}$

21. $\pm\frac{2}{3}$

23. $0, \frac{1}{8}$

25. $\pm\frac{1}{3}, \pm\frac{i}{3}$

27. $-\frac{3}{2}, 6$

29. $1, 16$

31. $-1, \frac{1}{512}$

33. $\pm\sqrt{7-\sqrt{3}}, \pm\sqrt{7+\sqrt{3}}$

35. $-2, 1$

37. $\frac{1}{16}, 16$

39. $\frac{1}{64}, 1$

Exercise 6.3.4

Find all solutions. Round your answers to the nearest hundredth.

1. $x^4-6x^2+7=0$
2. $x^4-6x^2+6=0$
3. $x^4-8x^2+14=0$
4. $x^4-12x^2+31=0$
5. $4x^4-16x^2+13=0$
6. $9x^4-30x^2+1=0$

Answer

1. $\pm 1.26, \pm 2.10$
3. $\pm 1.61, \pm 2.33$
5. $\pm 1.06, \pm 1.69$

Exercise 6.3.5

Find the set of all roots.

1. $f(x)=x^3-1$
2. $g(x)=x^3+1$
3. $f(x)=x^3-27$
4. $g(x)=x^4-16$
5. $h(x)=x^4-1$
6. $h(x)=x^6-1$
7. $f(x)=(2x-1)^3$
8. $g(x)=x^2(x-4)^2$
9. $f(x)=x^3-q^3, q>0$
10. $f(x)=x^3+q^3, q>0$

Answer

1. $\left\{1,-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i\right\}$
3. $\left\{3,-\frac{3}{2}\pm\frac{3\sqrt{3}}{2}i\right\}$
5. $\{\pm 1,\pm i\}$
7. $\{\frac{1}{2}\}$
9. $\left\{q,-\frac{q}{2}\pm\frac{q\sqrt{3}}{2}i\right\}$

Exercise 6.3.6

Find all solutions.

1. $x^6+7x^3-8=0$
2. $x^6-7x^3-8=0$
3. $x^6+28x^3+27=0$
4. $x^6+16x^3+64=0$
5. $|x^2+2x-5|=1$
6. $|x^2-2x-3|=3$
7. $|2x^2-5|=4$
8. $|3x^2-9x|=6$

Answer

1. $-2, 1, 1\pm i\sqrt{3}, -\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$
3. $-3, -1, \frac{3}{2}\pm\frac{3\sqrt{3}}{2}i, \frac{1}{2}\pm\frac{\sqrt{3}}{2}i$
5. $-1\pm\sqrt{7}, -1\pm\sqrt{5}$
7. $\pm\frac{\sqrt{2}}{2}, \pm\frac{3\sqrt{2}}{2}$

Exercise 6.3.7

Find a quadratic function with integer coefficients and the given set of roots. (Hint: if r_1 and r_2 are roots, then $(x-r_1)(x-r_2) = 0$.)

1. $\{\pm 3i\}$
2. $\{\pm i\sqrt{5}\}$
3. $\{\pm\sqrt{3}\}$
4. $\{\pm 2\sqrt{6}\}$
5. $\{1\pm\sqrt{3}\}$
6. $\{2\pm 3\sqrt{2}\}$
7. $\{1\pm 6i\}$
8. $\{2\pm 3i\}$

Answer

1. $f(x)=x^2+9$
3. $f(x)=x^2-3$
5. $f(x)=x^2-2x-2$
7. $f(x)=x^2-2x+37$

Exercise 6.3.8

1. On a note card, write out your strategy for solving a quadratic equation. Share your strategy on the discussion board.
2. Make up your own equation that is quadratic in form. Share it and the solution on the discussion board

Answer

1. Answer may vary

Footnotes

[7] An equation of the form $au^2+bu+c=0$ where a, b and c are real numbers and u represents an algebraic expression.

[8] A technique in algebra using substitution to transform equations into familiar forms.

[9] If multiple roots and complex roots are counted, then every polynomial with one variable will have as many roots as its degree.

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6.4: Quadratic Functions and Their Graphs

Learning Objectives

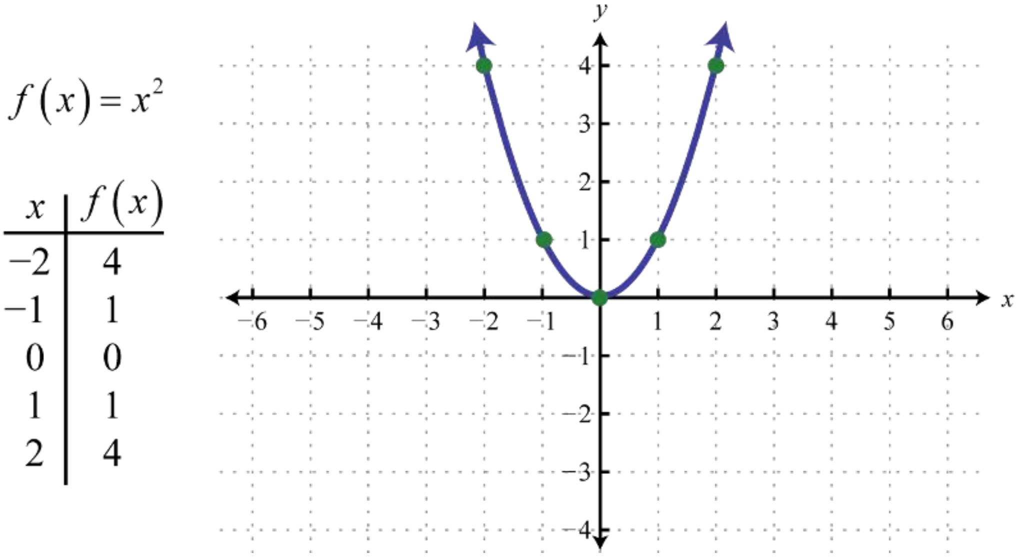
- Graph a parabola.
- Find the intercepts and vertex of a parabola.
- Find the maximum and minimum y-value.
- Find the vertex of a parabola by completing the square.

The Graph of a Quadratic Function

A **quadratic function** is a polynomial function of degree 2 which can be written in the general form,

$$f(x) = ax^2 + bx + c$$

Here a, b and c represent real numbers where $a \neq 0$.The squaring function $f(x) = x^2$ is a quadratic function whose graph follows.



$f(x) = x^2$

x	$f(x)$
-2	4
-1	1
0	0
1	1
2	4

Figure $6.4.1$

This general curved shape is called a **parabola**[10] and is shared by the graphs of all quadratic functions. Note that the graph is indeed a function as it passes the vertical line test. Furthermore, the domain of this function consists of the set of all real numbers $(-\infty, \infty)$ and the range consists of the set of nonnegative numbers $[0, \infty)$.

When graphing parabolas, we want to include certain special points in the graph. The y-intercept is the point where the graph intersects the y-axis. The x-intercepts are the points where the graph intersects the x-axis. The **vertex**[11] is the point that defines the minimum or maximum of the graph. Lastly, the **line of symmetry**[12] (also called the **axis of symmetry**[13]) is the vertical line through the vertex, about which the parabola is symmetric.

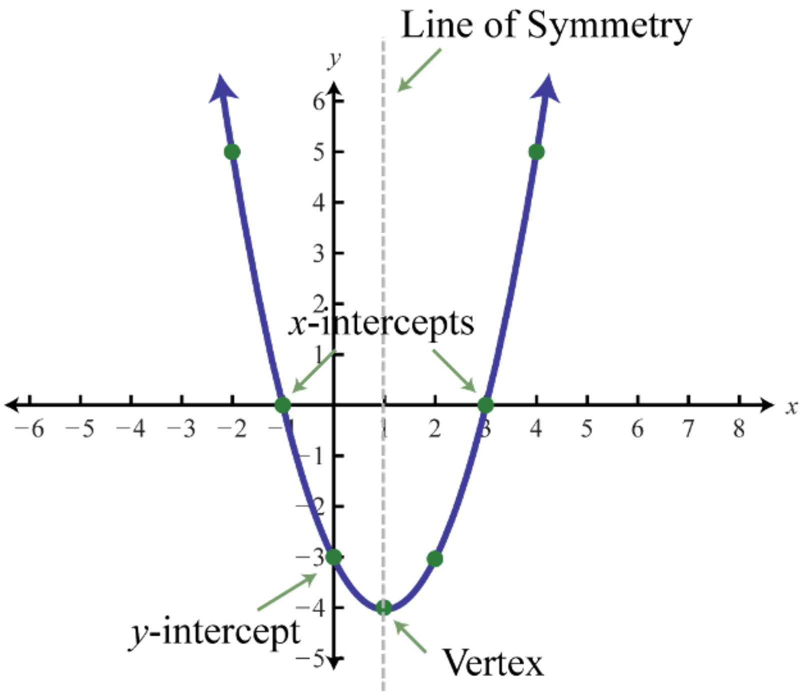


Figure $6.4.2$

For any parabola, we will find the vertex and y-intercept. In addition, if the x-intercepts exist, then we will want to determine those as well. Guessing at the x-values of these special points is not practical; therefore, we will develop techniques that will facilitate finding them. Many of these techniques will be used extensively as we progress in our study of algebra.

Given a quadratic function $f(x) = ax^2 + bx + c$, find the y-intercept by evaluating the function where $x = 0$. In general, $f(0) = a(0)^2 + b(0) + c = c$, and we have

$$\begin{gathered} y - intercept \\ (0, c) \end{gathered}$$

Next, recall that the x-intercepts, if they exist, can be found by setting $f(x) = 0$. Doing this, we have $a^2 + bx + c = 0$, which has general solutions given by the quadratic formula, \($x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$\).Therefore, the x-intercepts have this general form:

$$\begin{gathered} x - intercepts \\ \left(\frac{-b-\sqrt{b^2-4ac}}{2a}, 0\right) \text{ and } \left(\frac{-b+\sqrt{b^2-4ac}}{2a}, 0\right) \end{gathered}$$

Using the fact that a parabola is symmetric, we can determine the vertical line of symmetry using the x-intercepts. To do this, we find the x-value midway between the x-intercepts by taking an average as follows:

$$\begin{aligned} x &= \left(\frac{-b-\sqrt{b^2-4ac}}{2a} + \frac{-b+\sqrt{b^2-4ac}}{2a}\right) \div 2 \\ &= \left(\frac{-b-\cancel{\sqrt{b^2-4ac}} - b + \cancel{\sqrt{b^2-4ac}}}{2a}\right) \div \left(\frac{2}{1}\right) \\ &= \frac{-2b}{2a} \cdot \frac{1}{2} \\ &= -\frac{b}{2a} \end{aligned}$$

Therefore, the line of symmetry is the vertical line $x = -\frac{b}{2a}$. We can use the line of symmetry to find the the vertex.

$$\begin{array}{cc} \textit{Line of symmetry} & \textit{Vertex} \\ x = -\frac{b}{2a} & \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) \end{array}$$

Generally three points determine a parabola. However, in this section we will find five points so that we can get a better approximation of the general shape. The steps for graphing a parabola are outlined in the following example.

Example $6.4.1$:

Graph: $f(x)=-x^2-2x+3$.

Solution

Step 1: Determine the y-intercept. To do this, set x=0 and find $f(0)$.

$$\begin{aligned} f(x) &= -x^2-2x+3 \\ f(0) &= -(0)^2-2(0)+3 \\ &= 3 \end{aligned}$$

The y-intercept is $(0,3)$.

Step 2: Determine the x-intercepts if any. To do this, set $f(x)=0$ and solve for x.

$$\begin{aligned} f(x) &= -x^2-2x+3 && \textit{Set } f(x)=0. \\ 0 &= -x^2-2x+3 && \textit{Multiply both sides by } -1. \\ 0 &= x^2+2x-3 && \textit{Factor.} \\ 0 &= (x+3)(x-1) && \textit{Set each factor equal to zero.} \end{aligned}$$

$$\begin{aligned} x+3 &= 0 \quad \text{or} \quad x-1=0 \\ x &= -3 \qquad\qquad x=1 \end{aligned}$$

Here where $f(x)=0$, we obtain two solutions. Hence, there are two x-intercepts, $(-3,0)$ and $(1,0)$.

Step 3: Determine the vertex. One way to do this is to first use $x=-\frac{b}{2a}$ to find the x-value of the vertex and then substitute this value in the function to find the corresponding y-value. In this example, $a=-1$ and $b=-2$.

$$\begin{aligned} x &= \frac{-b}{2a} \\ &= \frac{-(-2)}{2(-1)} \\ &= \frac{2}{-2} \\ &= -1 \end{aligned}$$

Substitute -1 into the original function to find the corresponding y-value.

$$\begin{aligned} f(x) &= -x^2-2x+3 \\ f(-1) &= -(-1)^2-2(-1)+3 \\ &= -1+2+3 \\ &= 4 \end{aligned}$$

The vertex is $(-1,4)$.

Step 4: Determine extra points so that we have at least five points to plot. Ensure a good sampling on either side of the line of symmetry. In this example, one other point will suffice. Choose $x=-2$ and find the corresponding y-value.

Table $6.4.1$

x	y		Point
-2	3	$f(-2)=-(-2)^2-2(-2)+3$ $=-4+4+3=3$	$(-2,3)$

Our fifth point is $(-2,3)$.

Step 5: Plot the points and sketch the graph. To recap, the points that we have found are

y-intercept: $(0,3)$

x-intercepts: $(-3,0)$ and $(1,0)$

Vertex: $(-1,4)$

Extra point: $(-2,3)$

Answer:

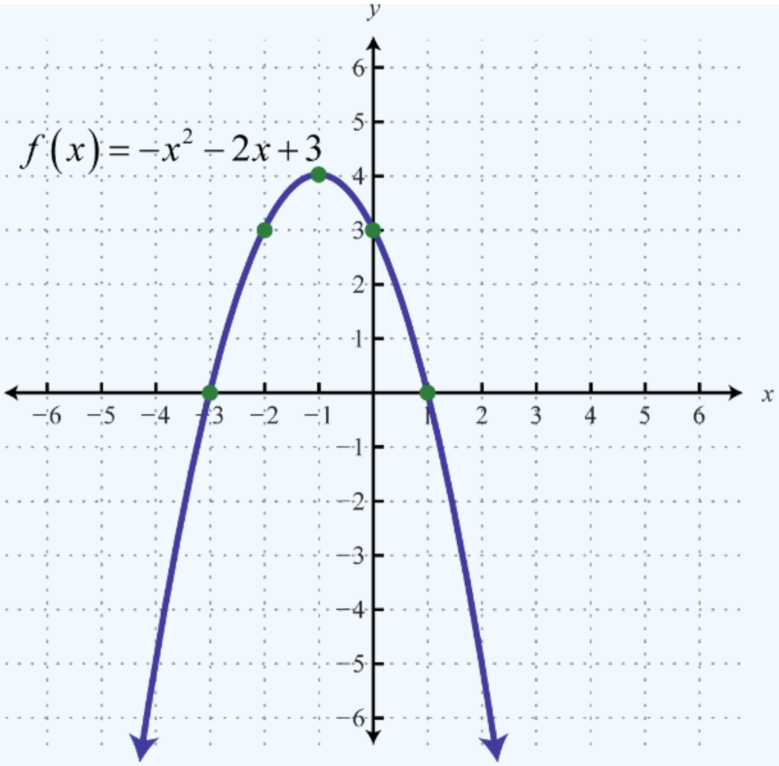


Figure $6.4.3$

The parabola opens downward. In general, use the leading coefficient to determine if the parabola opens upward or downward. If the leading coefficient is negative, as in the previous example, then the parabola opens downward. If the leading coefficient is positive, then the parabola opens upward.

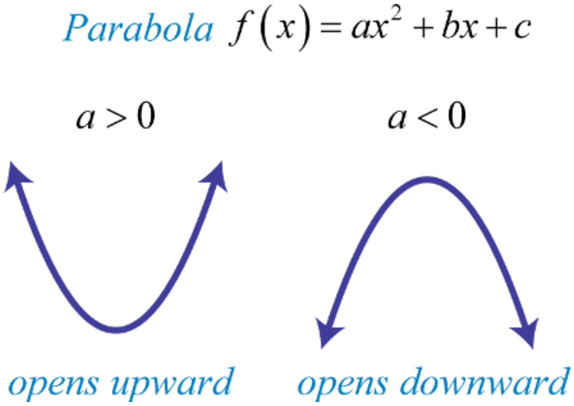


Figure $6.4.4$

All quadratic functions of the form $f(x) = ax^2 + bx + c$ have parabolic graphs with y-intercept $(0, c)$. However, not all parabolas have x-intercepts.

Example $6.4.2$:

Graph: $f(x) = 2x^2 + 4x + 5$.

Solution

Because the leading coefficient 2 is positive, we note that the parabola opens upward. Here $c = 5$ and the y-intercept is $(0, 5)$. To find the x-intercepts, set $f(x) = 0$.

$$\begin{aligned} f(x) &= 2x^2 + 4x + 5 \\ 0 &= 2x^2 + 4x + 5 \end{aligned}$$

In this case, $a=2, b=4$, and $c=5$. Use the discriminant to determine the number and type of solutions.

$$\begin{aligned} b^2-4ac &= (4)^2-4(2)(5) \\ &= 16-40 \\ &= -24 \end{aligned}$$

Since the discriminant is negative, we conclude that there are no real solutions. Because there are no real solutions, there are no x-intercepts. Next, we determine the x-value of the vertex.

$$\begin{aligned} x &= \frac{-b}{2a} \\ &= \frac{-(4)}{2(2)} \\ &= \frac{-4}{4} \\ &= -1 \end{aligned}$$

Given that the x-value of the vertex is -1, substitute -1 into the original equation to find the corresponding y-value.

$$\begin{aligned} f(x) &= 2x^2+4x+5 \\ f(-1) &= 2(-1)^2+4(-1)+5 \\ &= 2-4+5 \\ &= 3 \end{aligned}$$

The vertex is $(-1,3)$. So far, we have only two points. To determine three more, choose some x-values on either side of the line of symmetry, $x=-1$. Here we choose x-values $-3, -2$, and 1.

Table $6.4.2$

x	y		Points
-3	11	$f(-3)=2(-3)^2+4(-3)+5$ $=18-12+5=11$	$(-3,11)$
-2	5	$f(-2)=2(-2)^2+4(-2)+5=8$ $-8+5=5$	$(-2,5)$
1	11	$f(1)=2(1)^2+4(1)+5=2+4$ $+5=11$	$(1,11)$

To summarize, we have

y-intercept: $(0,5)$

x-intercepts: None

Vertex: $(-1,3)$

Extra points: $(-3,11),(-2,5),(1,11)$

Plot the points and sketch the graph.

Answer:

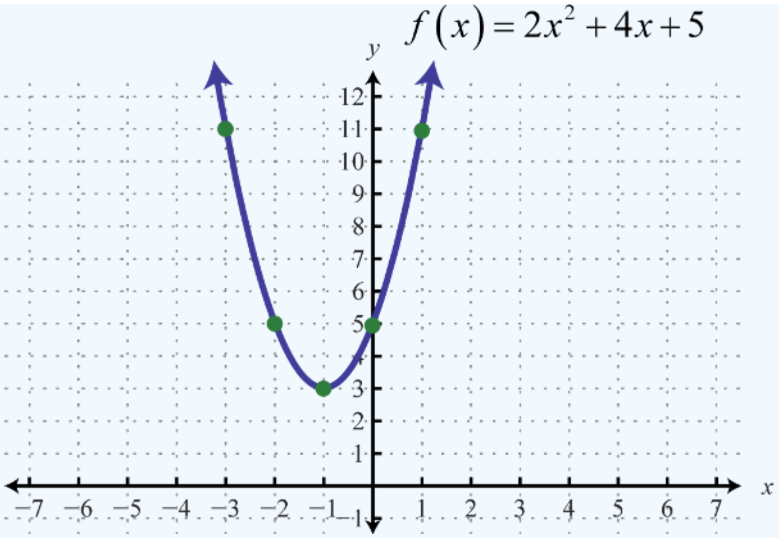


Figure 6.4.5

Example 6.4.3:

Graph: $f(x)=x^2-2x-1$.

Solution

Since $a=1$, the parabola opens upward. Furthermore, $c=-1$, so the y-intercept is $(0,-1)$. To find the x-intercepts, set $f(x)=0$.

$$\begin{aligned} f(x) &= x^2-2x-1 \\ 0 &= x^2-2x-1 \end{aligned}$$

In this case, solve using the quadratic formula with $a=1, b=-2$, and $c=-1$.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-(-2) \pm \sqrt{(-2)^2-4(1)(-1)}}{2(1)} \\ &= \frac{2 \pm \sqrt{8}}{2} \\ &= \frac{2 \pm 2\sqrt{2}}{2} \\ &= \frac{2(1 \pm \sqrt{2})}{2} \\ &= 1 \pm \sqrt{2} \end{aligned}$$

Here we obtain two real solutions for x, and thus there are two x-intercepts:

$$\begin{array}{lll} (1-\sqrt{2},0) \text{ and } (1+\sqrt{2},0) & \textit{Exact values} \\ (-0.41,0) \qquad (2.41,0) & \textit{Approximate values} \end{array}$$

Approximating the x-intercepts using a calculator will help us plot the points. However, we will present the exact x-intercepts on the graph. Next, find the vertex.

$$\begin{aligned} x &= \frac{-b}{2a} \\ &= \frac{-(-2)}{2(1)} \\ &= \frac{2}{2} \\ &= 1 \end{aligned}$$

Given that the x-value of the vertex is 1, substitute into the original equation to find the corresponding y-value.

$$\begin{aligned} y &= x^2 - 2x - 1 \\ &= (1)^2 - 2(1) - 1 \\ &= 1 - 2 - 1 \\ &= -2 \end{aligned}$$

The vertex is $(1, -2)$. We need one more point.

Table $6.4.3$

x	y		Point
2	-1	$f(2) = (2)^2 - 2(2) - 1 = 4 - 4 - 1 = -1$	$(2, -1)$

To summarize, we have

y-intercepts: $(0, 1)$

x-intercepts: $(1 - \sqrt{2}, 0)$ and $(1 + \sqrt{2}, 0)$

Vertex: $(1, -2)$

Extra point: $(2, -1)$

Plot the points and sketch the graph.

Answer:

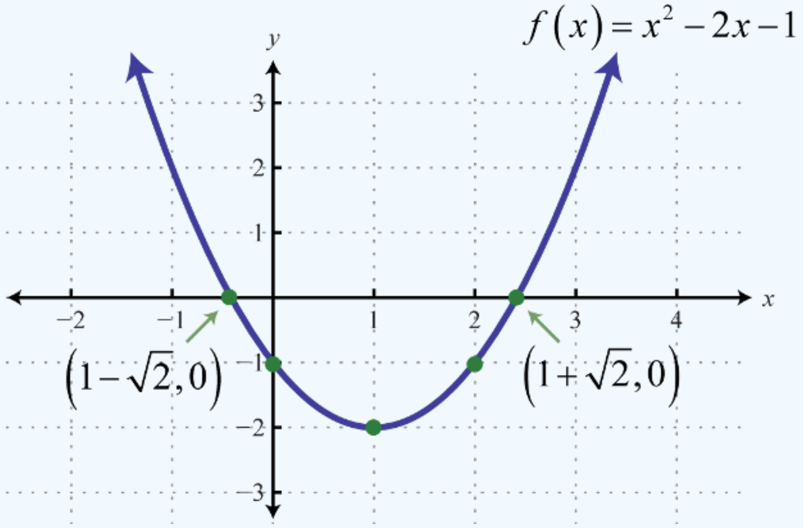


Figure $6.4.6$

Exercise $6.4.1$

Graph: $g(x) = -4x^2 + 12x - 9$.

Answer

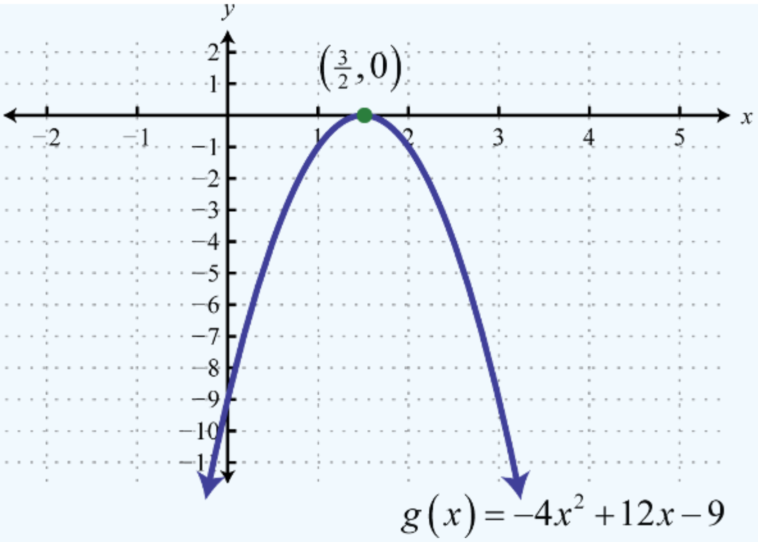


Figure $6.4.7$

www.youtube.com/v/tQiDXNhn7ik

Finding the Maximum or Minimum

It is often useful to find the maximum and/or minimum values of functions that model real-life applications. To find these important values given a quadratic function, we use the vertex. If the leading coefficient a is positive, then the parabola opens upward and there will be a minimum y-value. If the leading coefficient a is negative, then the parabola opens downward and there will be a maximum y-value.

Example $6.4.4$:

Determine the maximum or minimum: $y = -4x^2 + 24x - 35$.

Solution

Since $a = -4$, we know that the parabola opens downward and there will be a maximum y-value. To find it, first find the x-value of the vertex.

$$\begin{aligned} x &= -\frac{b}{2a} && \textit{x − value of the vertex.} \\ &= -\frac{24}{2(-4)} && \textit{Substitute a = −4 and b − 24.} \\ &= -\frac{24}{-8} && \textit{Simplify.} \\ &= 3 \end{aligned}$$

The x-value of the vertex is 3. Substitute this value into the original equation to find the corresponding y-value.

$$\begin{aligned} y &= -4x^2 + 24x - 35 && \textit{Substitute x = 3.} \\ &= -4(3)^2 + 24(3) - 35 && \textit{Simplify.} \\ &= -36 + 72 - 35 \\ &= 1 \end{aligned}$$

The vertex is $(3, 1)$. Therefore, the maximum y-value is 1, which occurs where $x = 3$, as illustrated below:

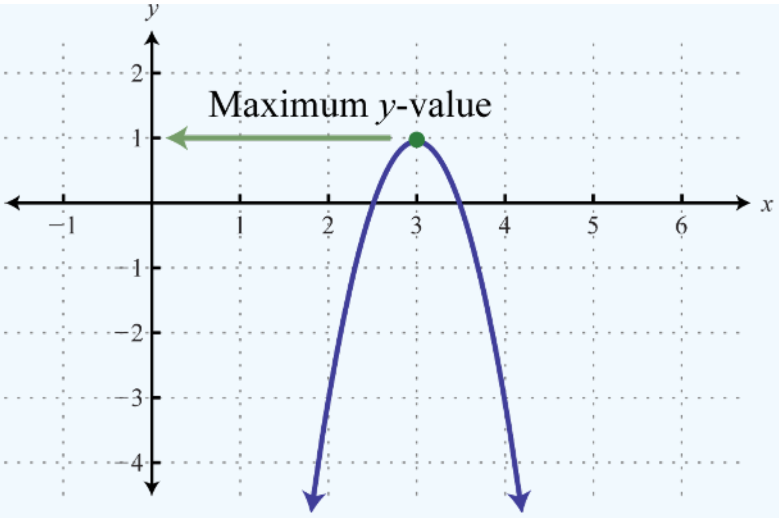


Figure $6.4.8$

Note: The graph is not required to answer this question.

Answer:

The maximum is 1.

Example $6.4.5$:

Determine the maximum or minimum: $y=4x^2-32x+62$.

Solution

Since $a=4$, the parabola opens upward and there is a minimum y-value. Begin by finding the x-value of the vertex.

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{-32}{2(4)} \quad \textit{Substitute } a=4 \textit{ and } b=-32. \\ &= -\frac{-32}{8} \quad \textit{Simplify.} \\ &= 4 \end{aligned}$$

Substitute $x=4$ into the original equation to find the corresponding y-value.

$$\begin{aligned} y &= 4x^2-32x+62 \\ &= 4(4)^2-32(4)+62 \\ &= 64-128+62 \\ &= -2 \end{aligned}$$

The vertex is $(4,-2)$. Therefore, the minimum y-value of -2 occurs where $x=4$, as illustrated below:

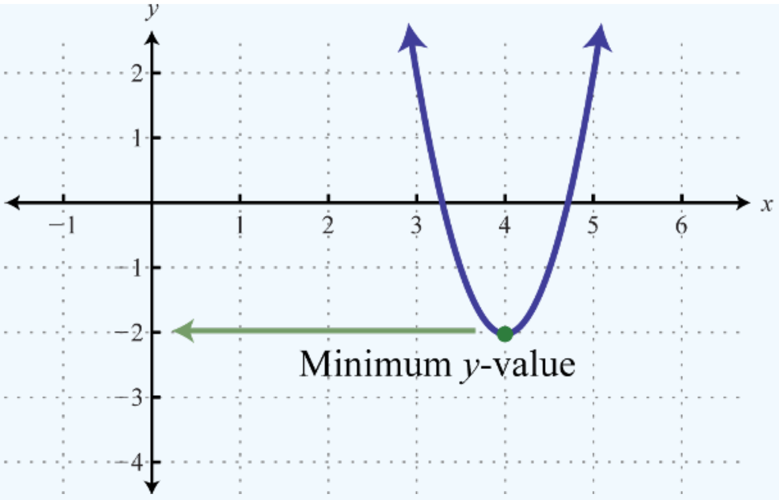


Figure $6.4.9$

Answer:

The minimum is -2.

Example $6.4.6$:

The height in feet of a projectile is given by the function $h(t)=-16t^2+72t$, where t represents the time in seconds after launch. What is the maximum height reached by the projectile?

Solution

Here $a=-16$, and the parabola opens downward. Therefore, the y-value of the vertex determines the maximum height. Begin by finding the time at which the vertex occurs.

$$t=-\frac{b}{2a}=-\frac{72}{2(-16)}=\frac{72}{32}=\frac{9}{4}$$

The maximum height will occur in $\frac{9}{4}$ seconds (or $2\frac{1}{4}$ seconds). Substitute this time into the function to determine the maximum height attained.

$$\begin{aligned} h\left(\frac{9}{4}\right) &= -16\left(\frac{9}{4}\right)^2+72\left(\frac{9}{4}\right) \\ &= -16\left(\frac{81}{16}\right)+72\left(\frac{9}{4}\right) \\ &= -81+162 \\ &= 81 \end{aligned}$$

Answer:

The maximum height of the projectile is 81 feet.

Finding the Vertex by Completing the Square

In this section, we demonstrate an alternate approach for finding the vertex. Any quadratic function $f(x)=ax^2+bx+c$ can be rewritten in **vertex form**[14],

$$f(x)=a(x-h)^2+k$$

In this form, the vertex is (h,k). To see that this is the case, consider graphing $f(x)=(x-2)^2+3$ using the transformations.

$$\begin{array}{ll} y=x^2 & \textit{Basic squaring function} \\ y=(x-2)^2 & \textit{Horizontal shift right 2 units} \\ y=(x-2)^2+3 & \textit{Vertical shift up 3 units} \end{array}$$

Use these translations to sketch the graph,

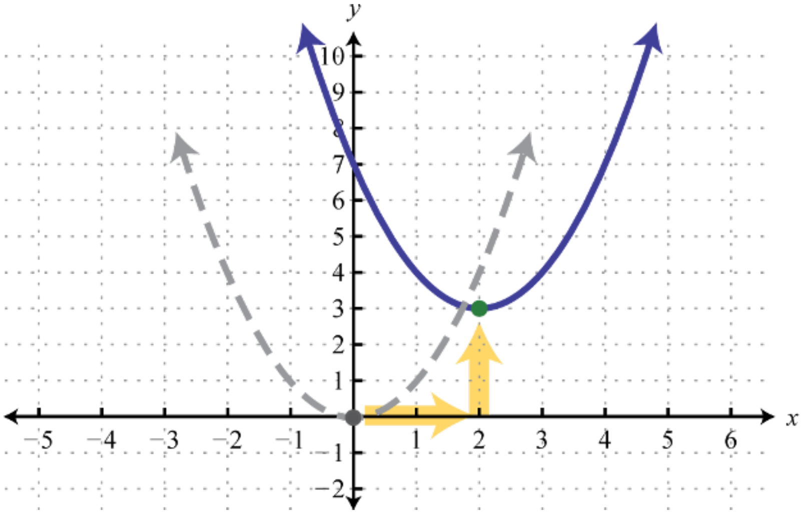


Figure 6.4.10

Here we can see that the vertex is $(2,3)$.

$$\begin{array}{c} f(x)=a(x-h)^2+k \\ \qquad\qquad\downarrow \quad \downarrow \\ f(x)=(x-2)^2\;+3 \end{array}$$

When the equation is in this form, we can read the vertex directly from it.

Example 6.4.7:

Determine the vertex: $f(x)=2(x+3)^2-2$.

Solution

Rewrite the equation as follows before determining h and k.

$$\begin{array}{c} f(x)=\;\; a\;\;(x\;-\;h\,)^2\;+\;\;k \\ \qquad\qquad\downarrow \qquad\quad \downarrow \\ f(x)=2[x-(-3)]^2+(-2) \end{array}$$

Here $h=-3$ and $k=-2$.

Answer:

The vertex is $(-3,-2)$.

Often the equation is not given in vertex form. To obtain this form, complete the square.

Example 6.4.8:

Rewrite in vertex form and determine the vertex: $f(x)=x^2+4x+9$.

Solution

Begin by making room for the constant term that completes the square.

$$\begin{aligned} f(x)&=x^2+4x+9 \\ &=x^2+4x+\_\_+9-\_\_ \end{aligned}$$

The idea is to add and subtract the value that completes the square, $\left(\frac{b}{2}\right)^2$, and then factor. In this case, add and subtract $\left(\frac{4}{2}\right)^2=(2)^2=4$.

$$\begin{aligned} f(x)&=x^2+4x+9 && \textit{Add and subtract 4.} \\ &=x^2+4x+4+9-4 && \textit{Factor.} \\ &=\left(x^2+4x+4\right)+5 \\ &=(x+3)(x+2)+5 \\ &=(x+2)^2+5 \end{aligned}$$

Adding and subtracting the same value within an expression does not change it. Doing so is equivalent to adding 0. Once the equation is in this form, we can easily determine the vertex.

$$\begin{array}{c} f(x)=a(x-h)^2 \ + \ k \\ \qquad\quad \downarrow \qquad \downarrow \\ f(x)=(x-(-2))^2+5 \end{array}$$

Here $h=-2$ and $k=5$.

Answer:

The vertex is $(-2,5)$.

If there is a leading coefficient other than 1, then we must first factor out the leading coefficient from the first two terms of the trinomial.

Example $6.4.9$:

Rewrite in vertex form and determine the vertex: $f(x)=2x^2-4x+8$.

Solution

Since $a=2$, factor this out of the first two terms in order to complete the square. Leave room inside the parentheses to add and subtract the value that completes the square.

$$\begin{aligned} f(x) &= 2x^2-4x+8 \\ &= 2\left(x^2-2x\right)+8 \end{aligned}$$

Now use -2 to determine the value that completes the square. In this case, $\left(\frac{-2}{2}\right)^2=(-1)^2=1$. Add and subtract 1 and factor as follows:

$$\begin{aligned} f(x) &= 2x^2-4x+8 \\ &= 2\left(x^2-2x+\underline{\quad}-\underline{\quad}\right)+8 && \textit{Add and subtract } 1. \\ &= 2(x^2-2x+1-1)+8 && \textit{Factor.} \\ &= 2[(x-1)(x-1)-1]+8 \\ &= 2\left[(x-1)^2-1\right]+8 && \textit{Distribute the } 2. \\ &= 2(x-1)^2-2+8 \\ &= 2(x-1)^2+6 \end{aligned}$$

In this form, we can easily determine the vertex.

$$\begin{array}{c} f(x)=a(x-h)^2+k \\ \qquad\quad \downarrow \quad \downarrow \\ f(x)=2(x-1)^2+6 \end{array}$$

Here $h=1$ and $k=6$.

Answer:

The vertex is $(1,6)$.

Exercise $6.4.1$

Rewrite in vertex form and determine the vertex: $f(x)=-2x^2-12x+3$.

Answer

$f(x)=-2(x+3)^2+21$; vertex: $(-3,21)$

www.youtube.com/v/cmwMn6oZ2dI

Key Takeaways

- The graph of any quadratic function $f(x)=ax^2+bx+c$, where a,b, and c are real numbers and $a\neq 0$, is called a parabola.

- When graphing a parabola always find the vertex and the y-intercept. If the x-intercepts exist, find those as well. Also, be sure to find ordered pair solutions on either side of the line of symmetry, $x=-\frac{b}{2a}$.
- Use the leading coefficient, a, to determine if a parabola opens upward or downward. If a is positive, then it opens upward. If a is negative, then it opens downward.
- The vertex of any parabola has an x-value equal to $-\frac{b}{2a}$. After finding the x-value of the vertex, substitute it into the original equation to find the corresponding y-value. This y-value is a maximum if the parabola opens downward, and it is a minimum if the parabola opens upward.
- The domain of a parabola opening upward or downward consists of all real numbers. The range is bounded by the y-value of the vertex.
- An alternate approach to finding the vertex is to rewrite the quadratic function in the form $f(x)=a(x-h)^2+k$. When in this form, the vertex is (h,k) and can be read directly from the equation. To obtain this form, take $f(x)=ax^2+bx+c$ and complete the square.

Exercise 6.4.3

Does the parabola open upward or downward? Explain.

1. $y=x^2-9x+20$
2. $y=x^2-12x+32$
3. $y=-2x^2+5x+12$
4. $y=-6x^2+13x-6$
5. $y=64-x^2$
6. $y=-3x+9x^2$

Answer

1. Upward

3. Downward

5. Downward

Exercise 6.4.4

Determine the x- and y-intercepts.

1. $y=x^2+4x-12$
2. $y=x^2-13x+12$
3. $y=2x^2+5x-3$
4. $y=3x^2-4x-4$
5. $y=-5x^2-3x+2$
6. $y=-6x^2+11x-4$
7. $y=4x^2-27$
8. $y=9x^2-50$
9. $y=x^2-x+1$
10. $y=x^2-6x+4$

Answer

1. x-intercepts: $(-6,0),(2,0)$; y-intercept: $(0,-12)$

3. x-intercepts: $(-3,0),\left(\frac{1}{2},0\right)$; y-intercept: $(0,-3)$

5. x-intercepts: $(-1,0),\left(\frac{2}{5},0\right)$; y-intercept: $(0,2)$

7. x-intercepts: $\left(-\frac{3\sqrt{3}}{2},0\right),\left(\frac{3\sqrt{3}}{2},0\right)$; y-intercept: $(0,-27)$

9. x-intercepts: none; y-intercept: $(0,1)$

Exercise 6.4.5

Find the vertex and the line of symmetry.

1. $y=-x^2+10x-34$
2. $y=-x^2-6x+1$
3. $y=-4x^2+12x-7$
4. $y=-9x^2+6x+2$
5. $y=4x^2-1$
6. $y=x^2-16$

Answer

1. Vertex: $(5,-9)$; line of symmetry: $x=5$

3. Vertex: $\left(\frac{3}{2},2\right)$; line of symmetry: $x=\frac{3}{2}$

5. Vertex: $(0,-1)$; line of symmetry: $x=0$

Exercise 6.4.6

Graph. Find the vertex and the y-intercept. In addition, find the x-intercepts if they exist.

1. $f(x)=x^2-2x-8$
2. $f(x)=x^2-4x-5$
3. $f(x)=-x^2+4x+12$
4. $f(x)=-x^2-2x+15$
5. $f(x)=x^2-10x$
6. $f(x)=x^2+8x$
7. $f(x)=x^2-9$
8. $f(x)=x^2-25$
9. $f(x)=1-x^2$
10. $f(x)=4-x^2$
11. $f(x)=x^2-2x+1$
12. $f(x)=x^2+4x+4$
13. $f(x)=-4x^2+12x-9$
14. $f(x)=-4x^2-4x+3$
15. $f(x)=x^2-2$
16. $f(x)=x^2-3$
17. $f(x)=-4x^2+4x-3$
18. $f(x)=4x^2+4x+3$
19. $f(x)=x^2-2x-2$
20. $f(x)=x^2-6x+6$
21. $f(x)=-2x^2+6x-3$
22. $f(x)=-4x^2+4x+1$
23. $f(x)=x^2+3x+4$
24. $f(x)=-x^2+3x-4$
25. $f(x)=-2x^2+3$
26. $f(x)=-2x^2-1$
27. $f(x)=2x^2+4x-3$
28. $f(x)=3x^2+2x-2$

Answer

1.

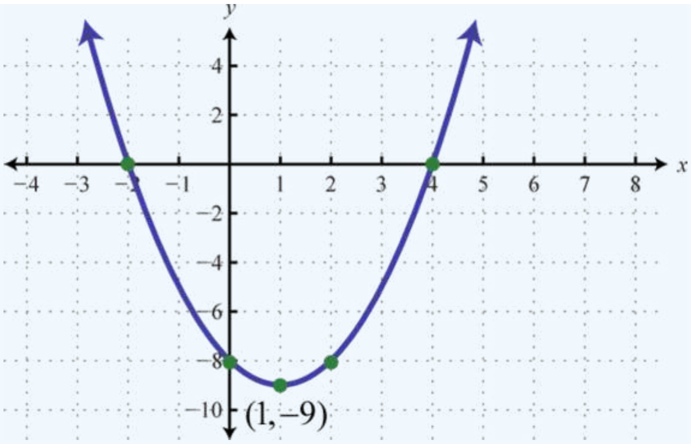


Figure 6.4.11

3.

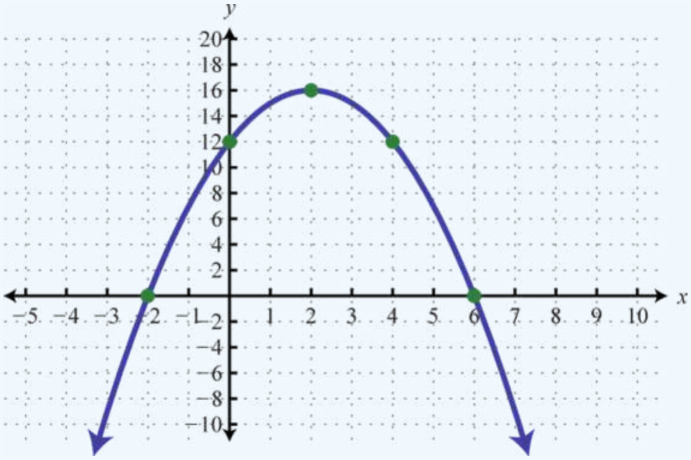


Figure 6.4.12

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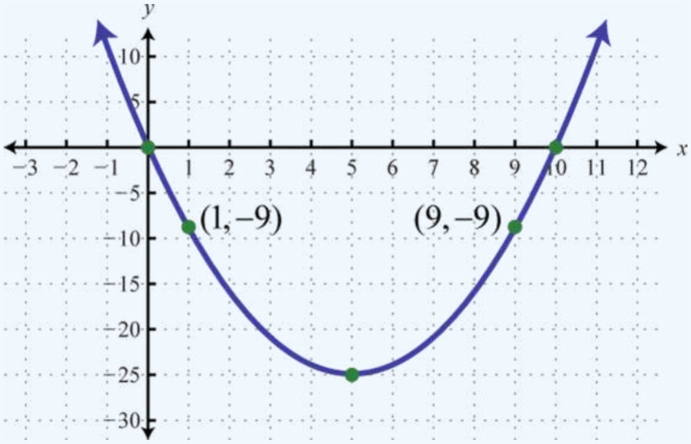

Figure 6.4.13

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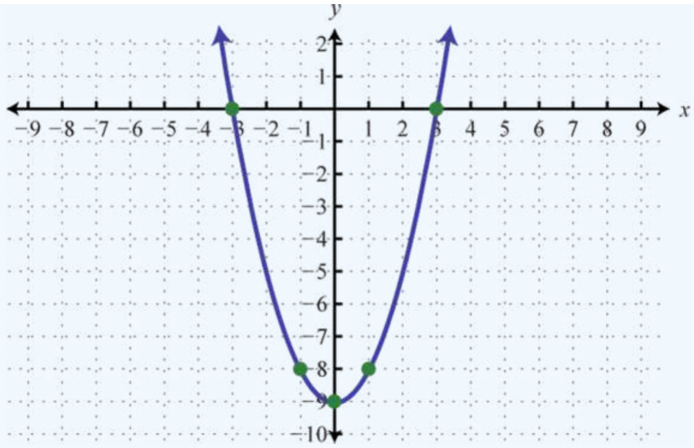

Figure 6.4.14

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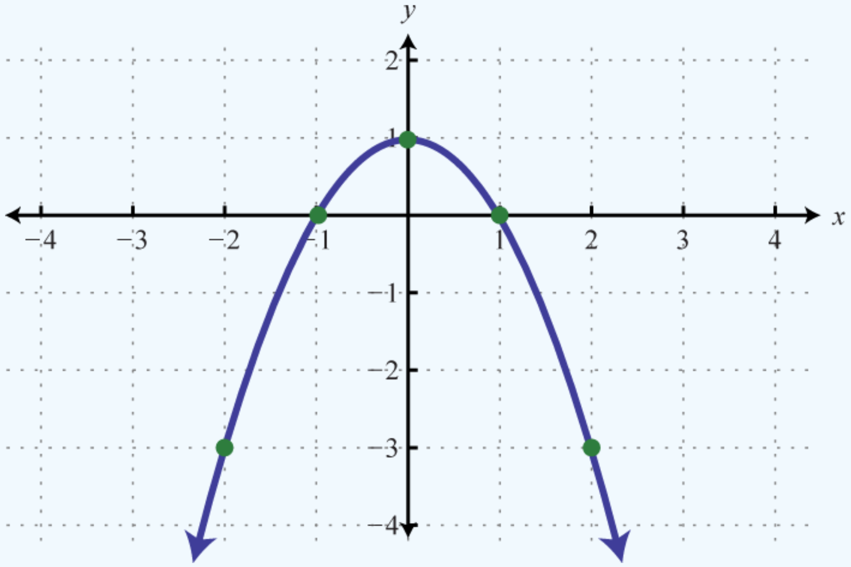

Figure 6.4.15

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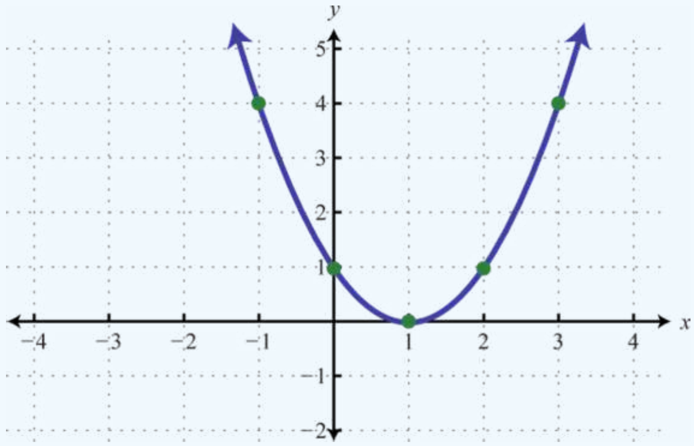

Figure 6.4.16

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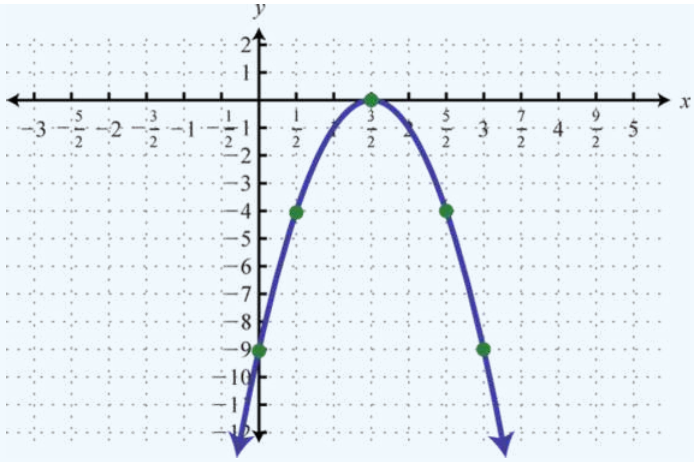

Figure 6.4.17

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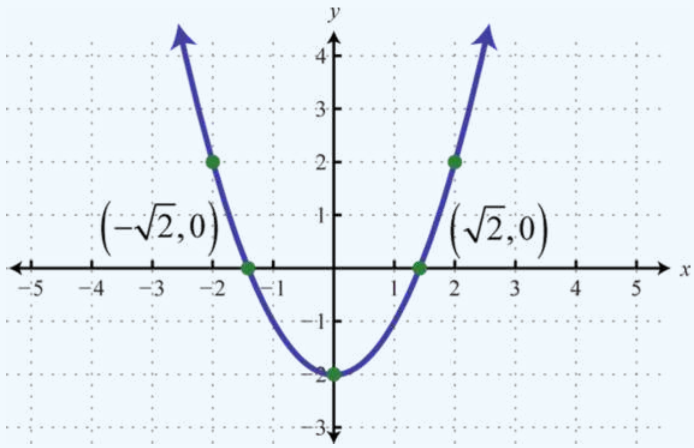

Figure 6.4.18

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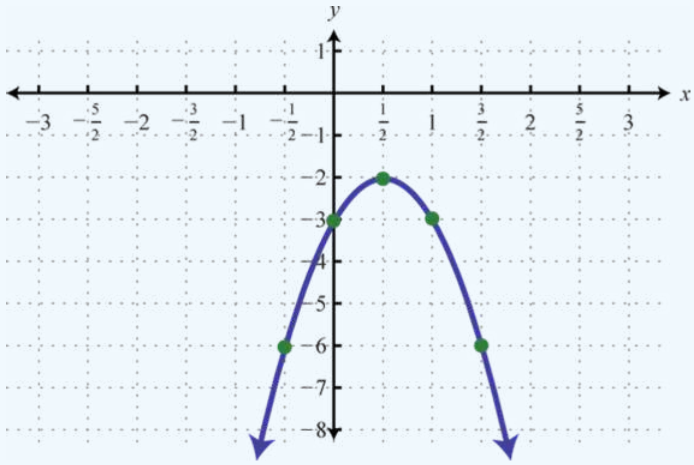

Figure 6.4.19

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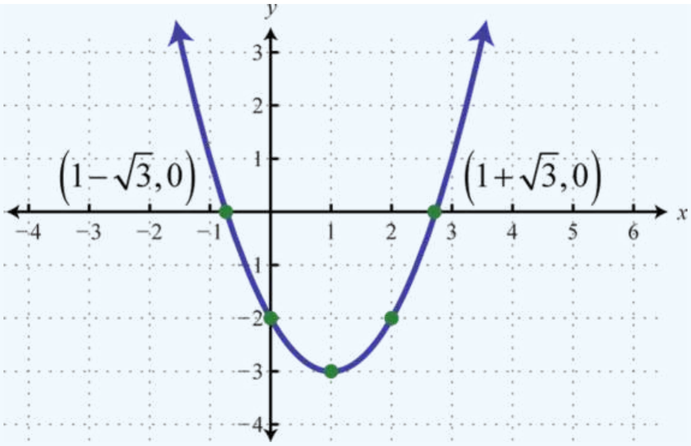

Figure 6.4.20

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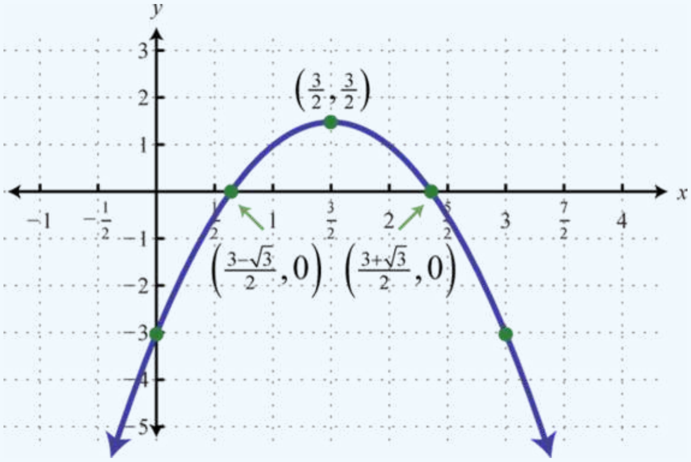

Figure 6.4.21

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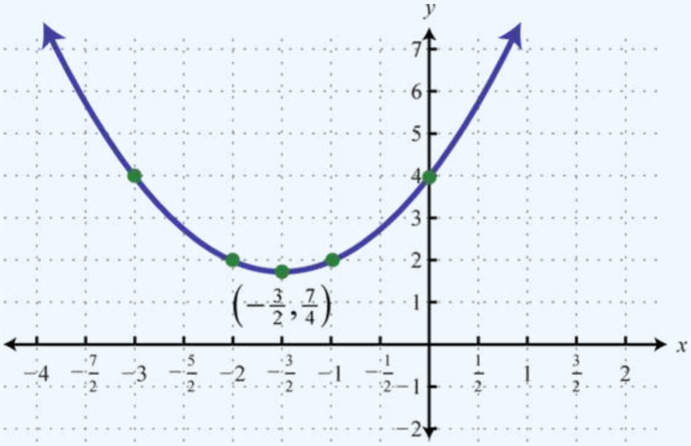

Figure 6.4.22

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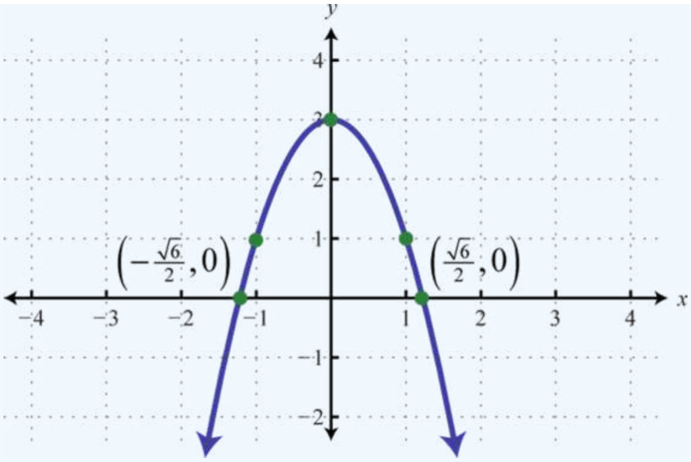

Figure $6.4.23$

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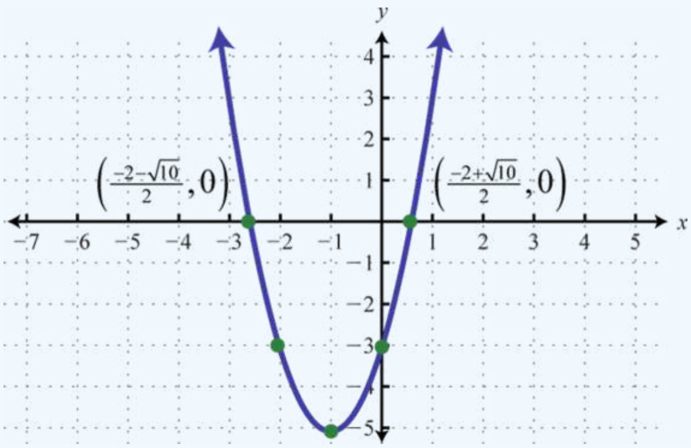

Figure $6.4.24$

Exercise $6.4.7$

Determine the maximum or minimum y-value.

1. $y=-x^2-6x+1$
2. $y=-x^2-4x+8$
3. $y=25x^2-10x+5$
4. $y=16x^2-24x+7$
5. $y=-x^2$
6. $y=1-9x^2$
7. $y=20x-10x^2$
8. $y=12x+4x^2$
9. $y=3x^2-4x-2$
10. $y=6x^2-8x+5$
11. $y=x^2-5x+1$
12. $y=1-x-x^2$

Answer

1. Maximum: $y=10$

3. Minimum: $y=\frac{4}{5}$

5. Maximum: $y=0$

7. Maximum: $y = 10$

9. Minimum: $y = -\frac{10}{3}$

11. Minimum: $y = -\frac{21}{4}$

Exercise 6.4.8

Given the following quadratic functions, determine the domain and range.

1. $f(x) = 3x^2 + 30x + 50$
2. $f(x) = 5x^2 - 10x + 1$
3. $g(x) = -2x^2 + 4x + 1$
4. $g(x) = -7x^2 - 14x - 9$
5. $f(x) = x^2 + x - 1$
6. $f(x) = -x^2 + 3x - 2$
7. The height in feet reached by a baseball tossed upward at a speed of 48 feet per second from the ground is given by the function $h(t) = -16t^2 + 48t$, where t represents the time in seconds after the ball is thrown. What is the baseball's maximum height and how long does it take to attain that height?
8. The height in feet of a projectile launched straight up from a mound is given by the function $h(t) = -16t^2 + 96t + 4$, where t represents seconds after launch. What is the maximum height?
9. The profit in dollars generated by producing and selling x custom lamps is given by the function $P(x) = -10x^2 + 800x - 12,000$. What is the maximum profit?
10. The profit in dollars generated from producing and selling a particular item is modeled by the formula $P(x) = 100x - 0.0025x^2$, where x represents the number of units produced and sold. What number of units must be produced and sold to maximize revenue?
11. The average number of hits to a radio station Web site is modeled by the formula $f(x) = 450t^2 - 3,600t + 8,000$, where t represents the number of hours since $8:00$ a.m. At what hour of the day is the number of hits to the Web site at a minimum?
12. The value in dollars of a new car is modeled by the formula $V(t) = 125t^2 - 3,000t + 22,000$, where t represents the number of years since it was purchased. Determine the minimum value of the car.
13. The daily production cost in dollars of a textile manufacturing company producing custom uniforms is modeled by the formula $C(x) = 0.02x^2 - 20x + 10,000$, where x represents the number of uniforms produced.
    1. How many uniforms should be produced to minimize the daily production costs?
    2. What is the minimum daily production cost?
14. The area in square feet of a certain rectangular pen is given by the formula $A = 14w - w^2$, where w represents the width in feet. Determine the width that produces the maximum area.

Answer

1. Domain: $(-\infty, \infty)$; range:$[-25, \infty)$

3. Domain: $(-\infty, \infty)$; range: $(-\infty, 3]$

5. Domain: $(-\infty, \infty)$; range: $[-\frac{5}{4}, \infty)$

7. The maximum height of 36 feet occurs after 1.5 seconds.

9. $\$4,000$

11. $12:00$ p.m.

13. (1) 500 uniforms (2) $\$5,000$

Exercise 6.4.9

Determine the vertex.

1. $y = -(x-5)^2 + 3$

2. $y=-2(x-1)^{2}+7$
3. $y=5(x+1)^{2}+6$
4. $y=3(x+4)^{2}+10$
5. $y=-5(x+8)^{2}-1$
6. $y=(x+2)^{2}-5$

Answer

1. $(5,3)$

3. $(-1,6)$

5. $(-8,-1)$

Exercise 6.4.10

Rewrite in vertex form $y=a(x-h)^{2}+k$ and determine the vertex.

1. $y=x^{2}-14x+24$
2. $y=x^{2}-12x+40$
3. $y=x^{2}+4x-12$
4. $y=x^{2}+6x-1$
5. $y=2x^{2}-12x-3$
6. $y=3x^{2}-6x+5$
7. $y=-x^{2}+16x+17$
8. $y=-x^{2}+10x$

Answer

1. $y=(x-7)^{2}-25$; vertex: $(7,-25)$

3. $y=(x+2)^{2}-16$; vertex: $(-2,-16)$

5. $y=2(x-3)^{2}-21$; vertex: $(3,-21)$

7. $y=-(x-8)^{2}+81$; vertex: $(8,81)$

Exercise 6.4.11

Graph. Find the vertex and the y-intercept. In addition, find the x-intercepts if they exist.

1. $f(x)=x^{2}-1$
2. $f(x)=x^{2}+1$
3. $f(x)=(x-1)^{2}$
4. $f(x)=(x+1)^{2}$
5. $f(x)=(x-4)^{2}-9$
6. $f(x)=(x-1)^{2}-4$
7. $f(x)=-2(x+1)^{2}+8$
8. $f(x)=-3(x+2)^{2}+12$
9. $f(x)=-5(x-1)^{2}$
10. $f(x)=-(x+2)^{2}$
11. $f(x)=-4(x-1)^{2}-2$
12. $f(x)=9(x+1)^{2}+2$
13. $f(x)=(x+5)^{2}-15$
14. $f(x)=2(x-5)^{2}-3$
15. $f(x)=-2(x-4)^{2}+22$
16. $f(x)=2(x+3)^{2}-13$

Answer

1.

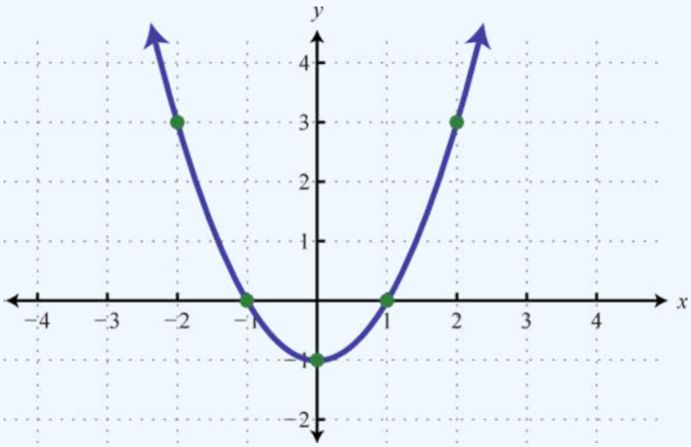

Figure $6.4.25$

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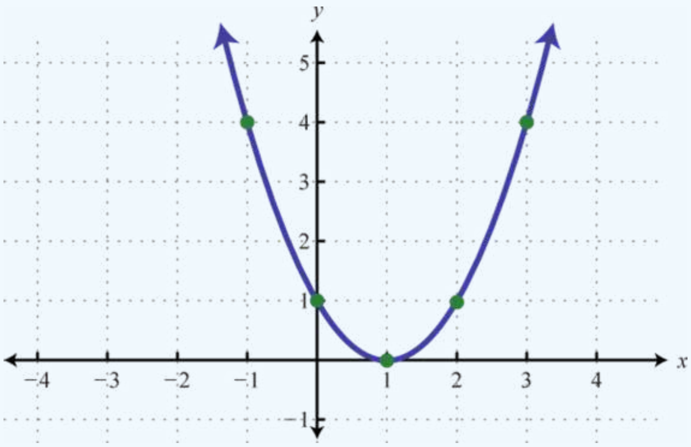

Figure $6.4.26$

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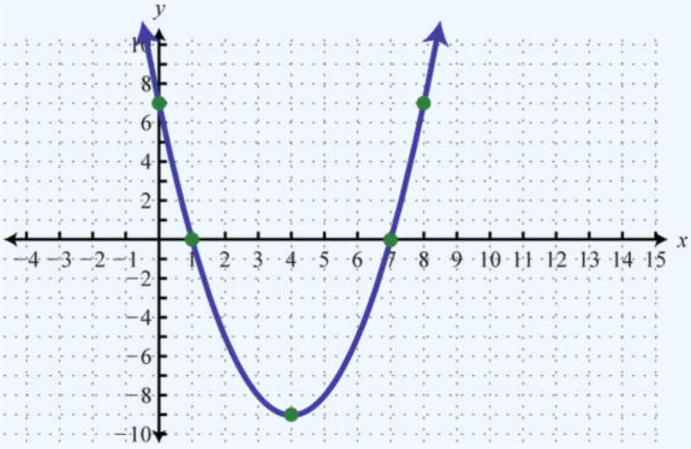

Figure $6.4.27$

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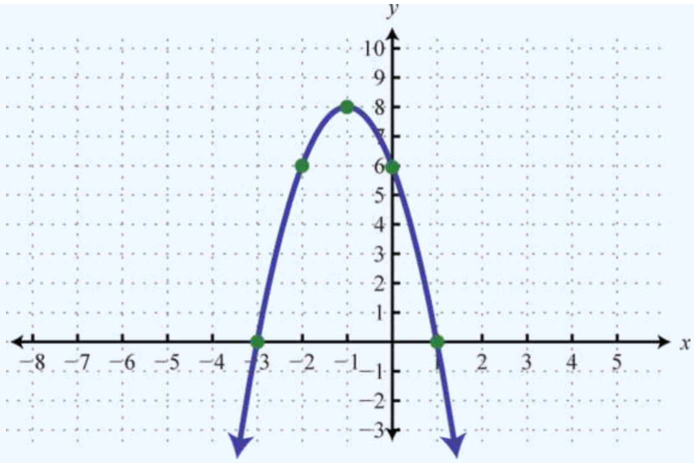

Figure 6.4.28

9.

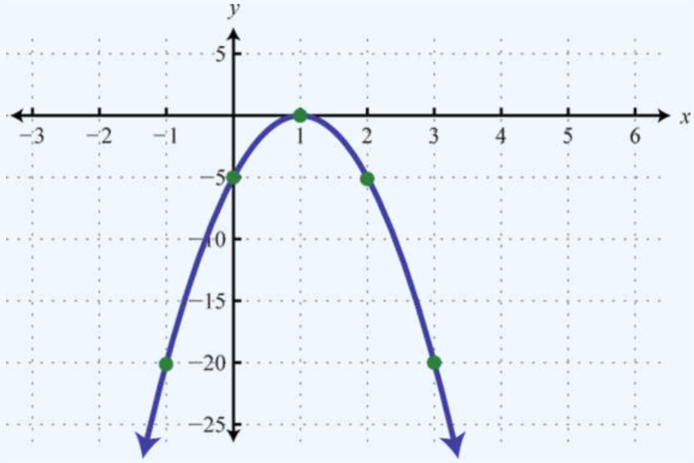

Figure 6.4.29

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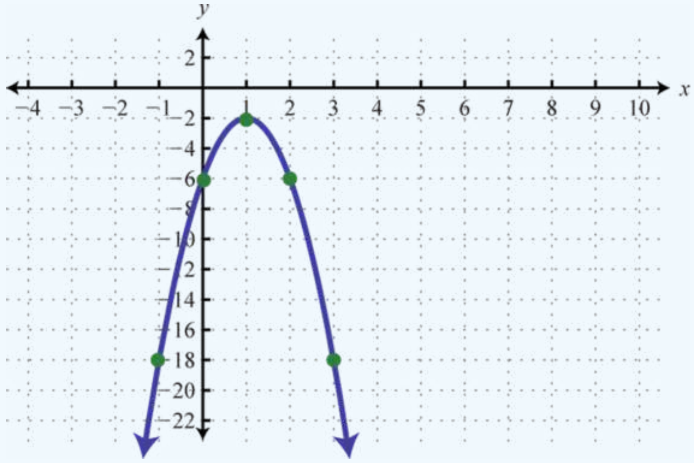

Figure 6.4.30

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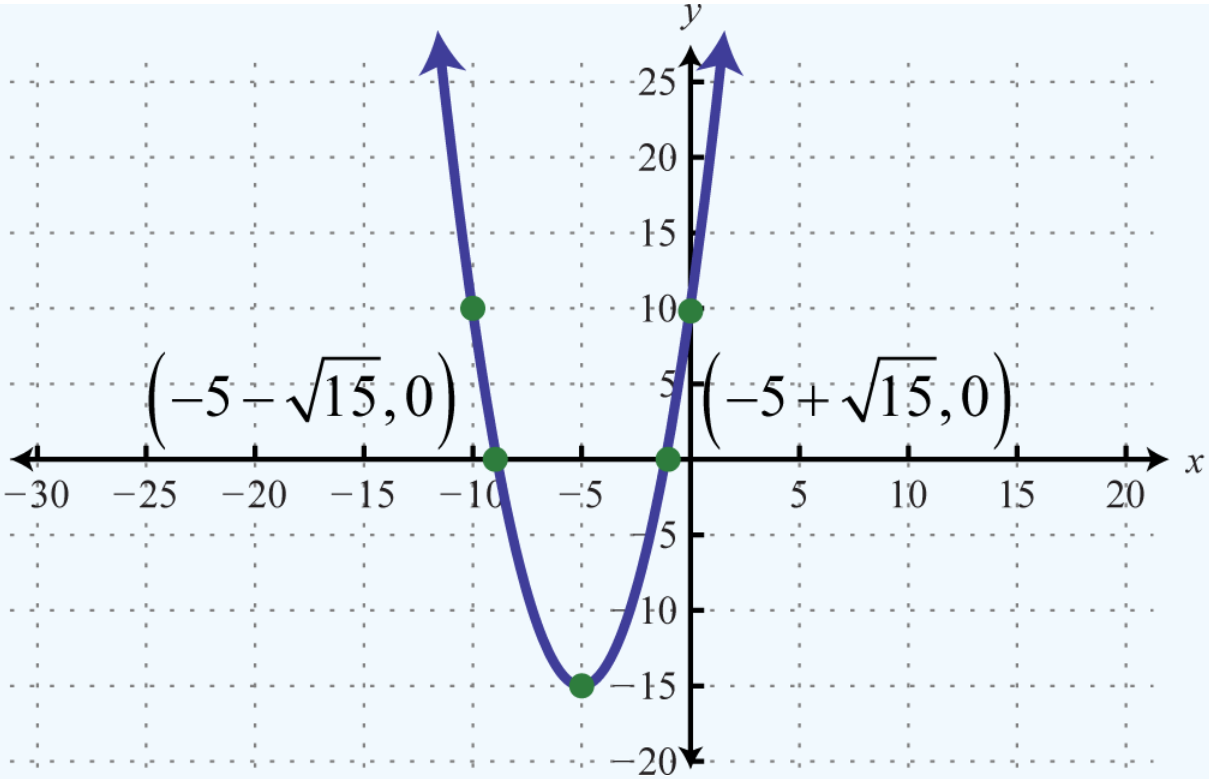

Figure 6.4.31

15.

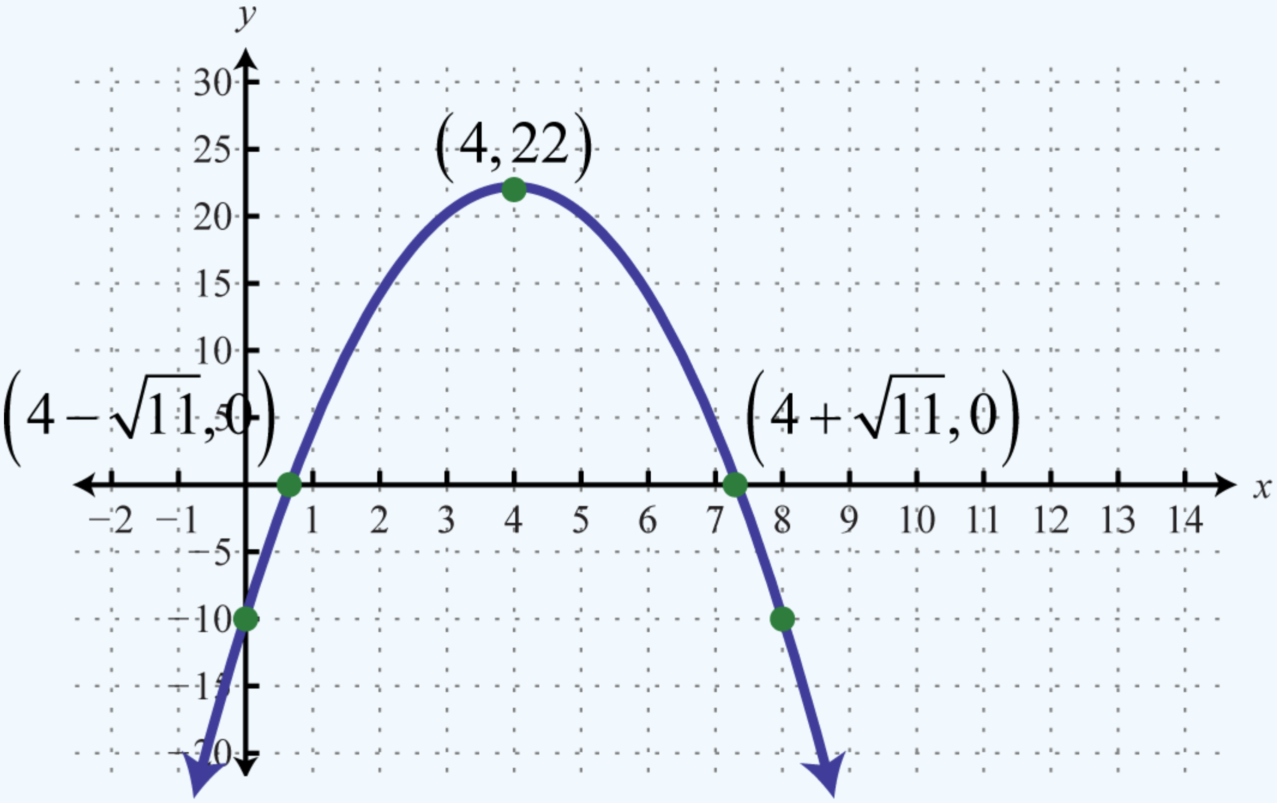

Figure 6.4.32

Exercise 6.4.12

1. Write down your plan for graphing a parabola on an exam. What will you be looking for and how will you present your answer? Share your plan on the discussion board.
2. Why is any parabola that opens upward or downward a function? Explain to a classmate how to determine the domain and range.
3. Research and discuss ways of finding a quadratic function that has a graph passing through any three given points. Share a list of steps as well as an example of how to do this.

Answer

1. Answer may vary

3. Answer may vary

Footnotes

[10] The U-shaped graph of any quadratic function defined by $f(x)=ax^2+bx+c$, where a, b, and c are real numbers and $a\neq 0$.

[11] The point that defines the minimum or maximum of a parabola.

[12] The vertical line through the vertex, $x=-\frac{b}{2a}$, about which the parabola is symmetric.

[13] A term used when referencing the line of symmetry.

[14] A quadratic function written in the form $f(x)=a(x-h)^2+k$.

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6.5: Solving Quadratic Inequalities

Learning Objectives

- Check solutions to quadratic inequalities with one variable.
- Understand the geometric relationship between solutions to quadratic inequalities and their graphs.
- Solve quadratic inequalities.

Solutions to Quadratic Inequalities

A **quadratic inequality**[15] is a mathematical statement that relates a quadratic expression as either less than or greater than another. Some examples of quadratic inequalities solved in this section follow.

Table $6.5.1$

$x^2 - 2x - 11 \leq 0$	$2x^2 - 7x + 3 > 0$	$9 - x^2 > 0$

A solution to a quadratic inequality is a real number that will produce a true statement when substituted for the variable.

Example $6.5.1$:

Are -3, -2, and -1 solutions to $x^2 - x - 6 \leq 0$?

Solution

Substitute the given value in for x and simplify.

$$\begin{array}{r|r|r} x^2 - x - 6 \leq 0 & x^2 - x - 6 \leq 0 & x^2 - x - 6 \leq 0 \\ (-3)^2 - (-3) - 6 \leq 0 & (-2)^2 - (-2) - 6 \leq 0 & (-1)^2 - (-1) - 6 \leq 0 \\ 9 + 3 - 6 \leq 0 & 4 + 2 - 6 \leq 0 & 1 + 1 - 6 \leq 0 \\ 6 \leq 0 \ \text{✗} & 0 \leq 0 \ \checkmark & -4 \leq 0 \ \checkmark \end{array}$$

Answer:

-2 and -1 are solutions and -3 is not.

Quadratic inequalities can have infinitely many solutions, one solution, or no solution. If there are infinitely many solutions, graph the solution set on a number line and/or express the solution using interval notation. Graphing the function defined by $f(x) = x^2 - x - 6$ found in the previous example we have

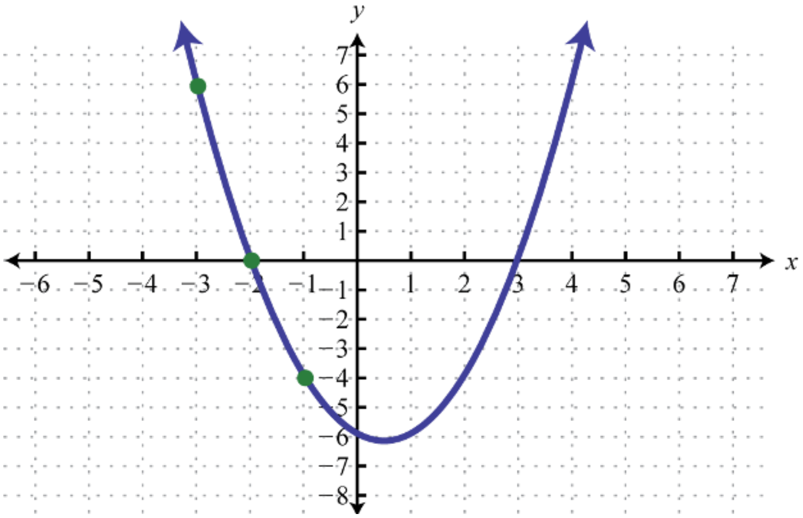

Figure $6.5.1$

The result of evaluating for any x-value will be negative, zero, or positive.

$$\begin{aligned} f(-3) &= 6 && \textit{Positive } f(x) > 0 \\ f(-2) &= 0 && \textit{Zero } f(x) = 0 \\ f(-1) &= -4 && \textit{Negative } f(x) < 0 \end{aligned}$$

The values in the domain of a function that separate regions that produce positive or negative results are called **critical numbers**[16]. In the case of a quadratic function, the critical numbers are the roots, sometimes called the zeros. For example, $f(x) = x^2 - x - 6 = (x+2)(x-3)$ has roots -2 and 3. These values bound the regions where the function is positive (above the x-axis) or negative (below the x-axis).

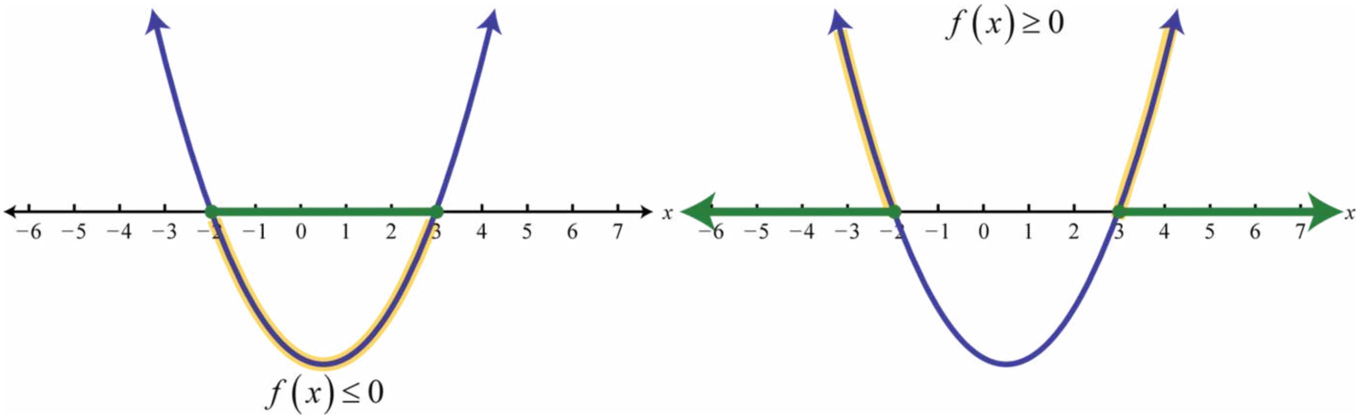

Figure $6.5.2$

Therefore $x^2 - x - 6 \leq 0$ has solutions where $-2 \leq x \leq 3$, using interval notation $[-2, 3]$. Furthermore, $x^2 - x - 6 \geq 0$ has solutions where $x \leq -2$ or $x \geq 3$, using interval notation $(-\infty, -2] \cup [-3, \infty)$.

Example $6.5.2$:

Given the graph of f determine the solutions to $f(x) > 0$:

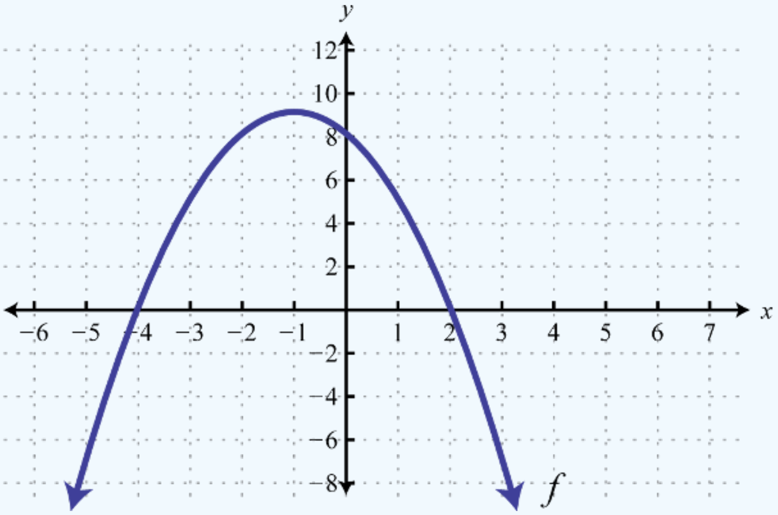

Figure $6.5.4$

Because of the strict inequality, the solution set is shaded with an open dot on each of the boundaries. This indicates that these critical numbers are not actually included in the solution set. This solution set can be expressed two ways,

$$\begin{aligned} \{x \mid -4 < x < 2\} && \textit{Set Notation} \\ (-4, 2) && \textit{Interval Notation} \end{aligned}$$

In this textbook, we will continue to present answers in interval notation.

Answer:

$(-4, 2)$

Exercise 6.5.1

Given the graph of f determine the solutions to $f(x) < 0$:

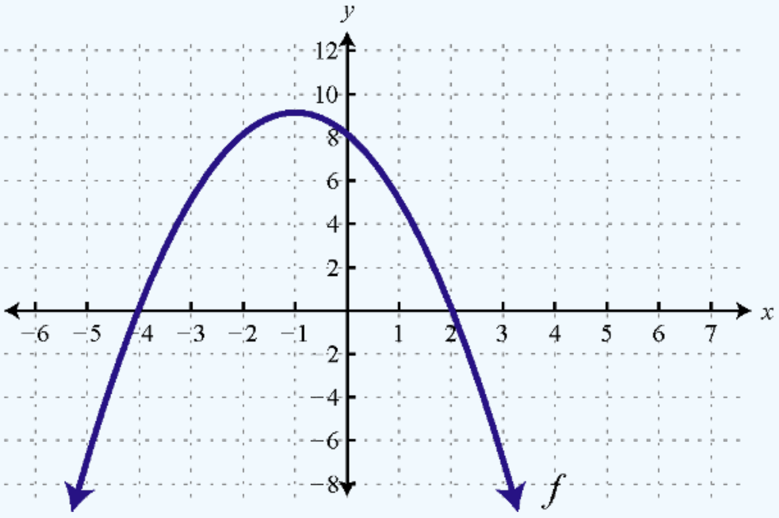

Figure 6.5.5

Answer

$(-\infty, -4) \cup (2, \infty)$

www.youtube.com/v/kkBaoM8buBo

Solving Quadratic Inequalities

Next we outline a technique used to solve quadratic inequalities without graphing the parabola. To do this we make use of a **sign chart**[17] which models a function using a number line that represents the x-axis and signs ($+$ or $-$) to indicate where the function is positive or negative. For example,

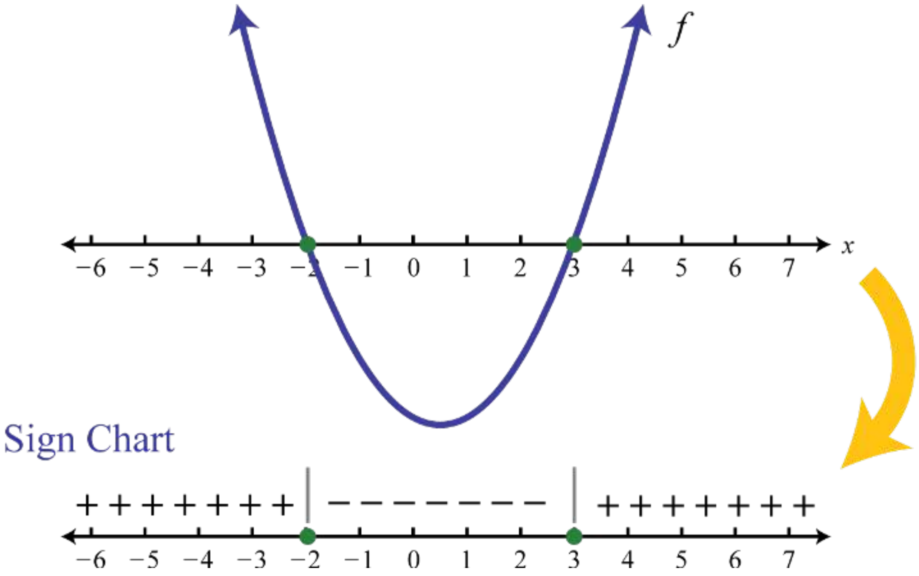

Figure 6.5.6

The plus signs indicate that the function is positive on the region. The negative signs indicate that the function is negative on the region. The boundaries are the critical numbers, -2 and 3 in this case. Sign charts are useful when a detailed picture of the graph is not needed and are used extensively in higher level mathematics. The steps for solving a quadratic inequality with one variable are outlined in the following example.

Example 6.5.3:

Solve: $-x^2+6x+7 \geq 0$.

Solution

It is important to note that this quadratic inequality is in standard form, with zero on one side of the inequality.

Step 1: Determine the critical numbers. For a quadratic inequality in standard form, the critical numbers are the roots. Therefore, set the function equal to zero and solve.

$$\begin{aligned} -x^2+6x+7&=0 \\ -\left(x^2-6x-7\right)&=0 \\ -(x+1)(x-7)&=0 \end{aligned}$$

$$\begin{array}{ccc} x+1=0 & \text{or} & x-7=0 \\ x=-1 & & x=7 \end{array}$$

The critical numbers are -1 and 7.

Step 2: Create a sign chart. Since the critical numbers bound the regions where the function is positive or negative, we need only test a single value in each region. In this case the critical numbers partition the number line into three regions and we choose test values $x=-3, x=0$, and $x=10$.

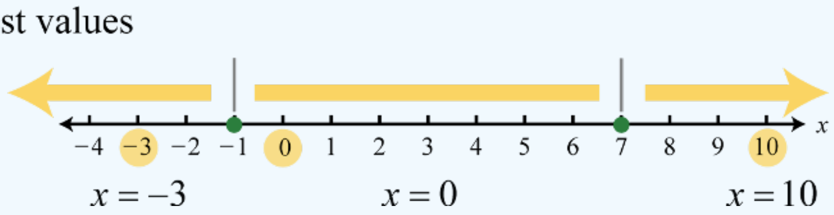

Figure 6.5.7

Test values may vary. In fact, we need only determine the sign $(+$ or $-)$ of the result when evaluating $f(x)=-x^2+6x+7$ $=-(x+1)(x-7)$. Here we evaluate using the factored form.

$$\begin{array}{ll} f(-3)=-(-3+1)(-3-7)=-(-2)(-10) & =-\textit{Negative} \\ f(0)=-(0+1)(0-7)=-(1)(-7) & =+\textit{Positive} \\ f(10)=-(10+1)(10-7)=-(11)(3) & =-\textit{Negative} \end{array}$$

Since the result of evaluating for -3 was negative, we place negative signs above the first region. The result of evaluating for 0 was positive, so we place positive signs above the middle region. Finally, the result of evaluating for 10 was negative, so we place negative signs above the last region, and the sign chart is complete.

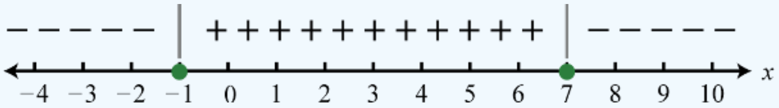

Figure 6.5.8

Step 3: Use the sign chart to answer the question. In this case, we are asked to determine where $f(x) \geq 0$, or where the function is positive or zero. From the sign chart we see this occurs when x-values are inclusively between -1 and 7.

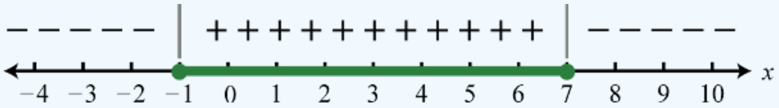

Figure 6.5.9

Using interval notation, the shaded region is expressed as $[-1,7]$. The graph is not required; however, for the sake of completeness it is provided below.

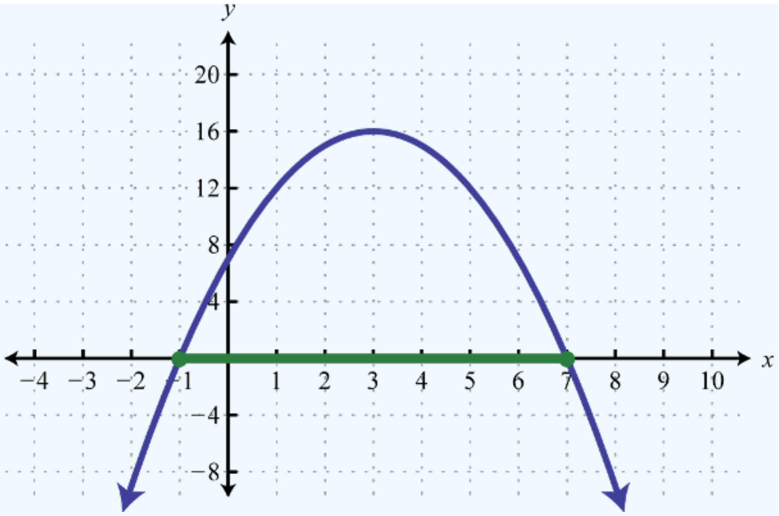

Figure 6.5.10

Indeed the function is greater than or equal to zero, above or on the x-axis, for x-values in the specified interval.

Answer:

$[-1, 7]$

Example 6.5.4:

Solve: $2x^2 - 7x + 3 > 0$.

Solution

Begin by finding the critical numbers, in this case, the roots of $f(x) = 2x^2 - 7x + 3$.

$$\begin{aligned} 2x^2 - 7x + 3 &= 0 \\ (2x-1)(x-3) &= 0 \end{aligned}$$

$$\begin{array}{rcr} 2x - 1 = 0 & \text{or} & x - 3 = 0 \\ 2x = 1 & & x = 3 \\ x = \dfrac{1}{2} & & \end{array}$$

The critical numbers are $\frac{1}{2}$ and 3. Because of the strict inequality $>$ we will use open dots.

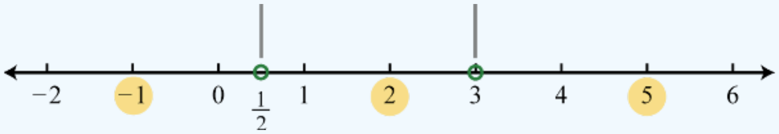

Figure 6.5.11

Next choose a test value in each region and determine the sign after evaluating $f(x) = 2x^2 - 7x + 3 = (2x-1)(x-3)$. Here we choose test values $-1, 2$, and 5.

$$\begin{aligned} f(-1) &= [2(-1)-1](-1-3) && = (-)(-) = + \\ f(2) &= [2(2)-1](2-3) && = (+)(-) = - \\ f(5) &= [2(5)-1](5-3) && = (+)(+) = + \end{aligned}$$

And we can complete the sign chart.

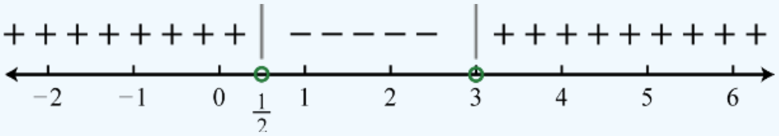

Figure 6.5.12

The question asks us to find the x-values that produce positive results (greater than zero). Therefore, shade in the regions with a $+$ over them. This is the solution set.

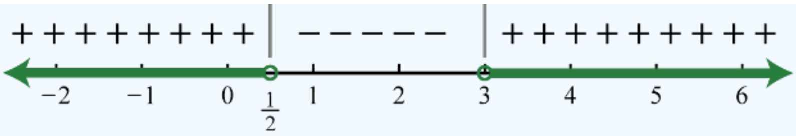

Figure 6.5.13

Answer:

$(-\infty, \frac{1}{2}) \cup (3, \infty)$

Sometimes the quadratic function does not factor. In this case we can make use of the quadratic formula.

Example 6.5.5:

Solve: $x^2 - 2x - 11 \leq 0$

Solution

Find the critical numbers.

$$x^2 - 2x - 11 = 0$$

Identify a, b, and c for use in the quadratic formula. Here $a = 1, b = -2$, and $c = -11$. Substitute the appropriate values into the quadratic formula and then simplify.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-11)}}{2(1)} \\ &= \frac{2 \pm \sqrt{48}}{2} \\ &= \frac{2 \pm 4\sqrt{3}}{2} \\ &= 1 \pm 2\sqrt{3} \end{aligned}$$

Therefore the critical numbers are $1 - 2\sqrt{3} \approx -2.5$ and $1 + 2\sqrt{3} \approx 4.5$. Use a closed dot on the number to indicate that these values will be included in the solution set.

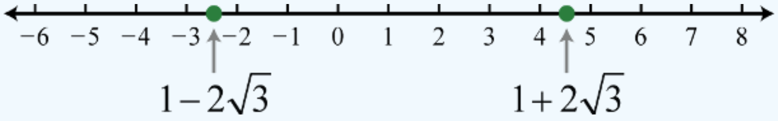

Figure 6.5.14

Here we will use test values $-5, 0$, and 7.

$$\begin{aligned} f(-5) &= (-5)^2 - 2(-5) - 11 = 25 + 10 - 11 && = + \\ f(0) &= (0)^2 - 2(0) - 11 = 0 + 0 - 11 && = - \\ f(7) &= (7)^2 - 2(7) - 11 = 49 - 14 - 11 && = + \end{aligned}$$

After completing the sign chart shade in the values where the function is negative as indicated by the question $(f(x) \leq 0)$.

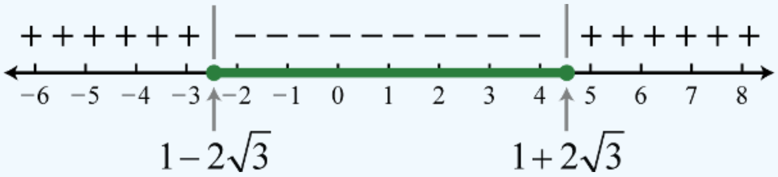

Figure 6.5.15

Answer:

$[1 - 2\sqrt{3}, 1 + 2\sqrt{3}]$

Exercise 6.5.2

Solve: $9-x^2>0$.

Answer

$(-3,3)$

www.youtube.com/v/7gMJ8gUvASw

It may be the case that there are no critical numbers.

Example 6.5.6:

Solve: $x^2-2x+3>0$.

Solution

To find the critical numbers solve,

$$x^2-2x+3=0$$

Substitute $a=1, b=-2$, and $c=3$ into the quadratic formula and then simplify.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{-(-2) \pm \sqrt{(-2)^2-4(1)(3)}}{2(1)} \\ &= \frac{2 \pm \sqrt{-8}}{2} \\ &= \frac{2 \pm 2i\sqrt{2}}{2} \\ &= 1+i\sqrt{2} \end{aligned}$$

Because the solutions are not real, we conclude there are no real roots; hence there are no critical numbers. When this is the case, the graph has no x-intercepts and is completely above or below the x-axis. We can test any value to create a sign chart. Here we choose $x=0$.

$$f(0)=(0)^2-2(0)+3=+$$

Because the test value produced a positive result the sign chart looks as follows:

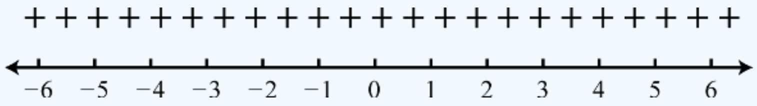

Figure 6.5.16

We are looking for the values where $f(x)>0$; the sign chart implies that any real number for x will satisfy this condition.

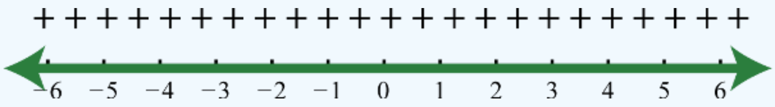

Figure 6.5.17

Answer:

$(-\infty, \infty)$

The function in the previous example is graphed below.

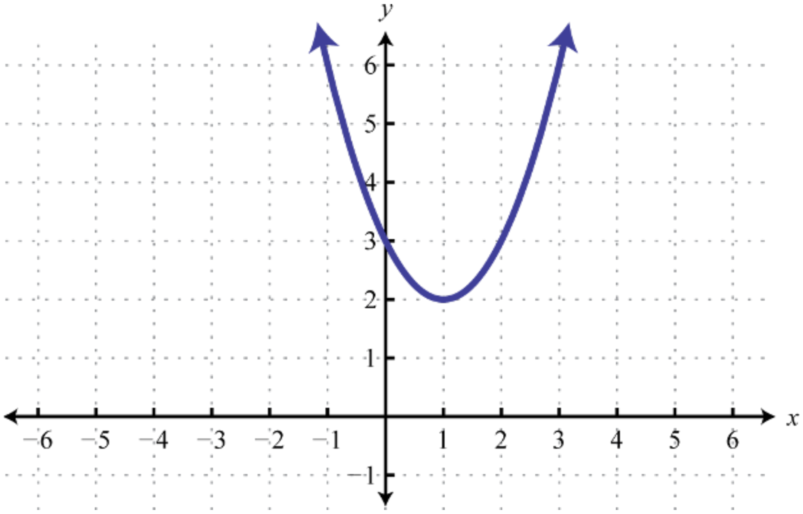

Figure 6.5.18

We can see that it has no x-intercepts and is always above the x-axis (positive). If the question was to solve $x^2 - 2x + 3 < 0$, then the answer would have been no solution. The function is never negative.

Exercise 6.5.3

Solve: $9x^2 - 12x + 4 \leq 0$

Answer

One solution, $\frac{2}{3}$.

www.youtube.com/v/E7VcOYVV_Ds

Example 6.5.7:

Find the domain: $f(x) = \sqrt{x^2 - 4}$.

Solution

Recall that the argument of a square root function must be nonnegative. Therefore, the domain consists of all real numbers for x such that $x^2 - 4$ is greater than or equal to zero.

$$x^2 - 4 \geq 0$$

It should be clear that $x^2 - 4 = 0$ has two solutions $x = \pm 2$; these are the critical values. Choose test values in each interval and evaluate $f(x) = x^2 - 4$.

$$\begin{aligned} f(-3) &= (-3)^2 - 4 = 9 - 4 = + \\ f(0) &= (0)^2 - 4 = 0 - 4 = - \\ f(3) &= (3)^2 - 4 = 9 - 4 = + \end{aligned}$$

Shade in the x-values that produce positive results.

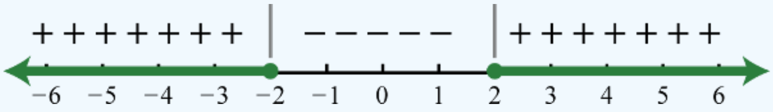

Figure 6.5.19

Answer:

Domain: $(-\infty, -2] \cup [2, \infty)$

Key Takeaways

- Quadratic inequalities can have infinitely many solutions, one solution or no solution.
- We can solve quadratic inequalities graphically by first rewriting the inequality in standard form, with zero on one side. Graph the quadratic function and determine where it is above or below the x-axis. If the inequality involves "less than," then determine the x-values where the function is below the x-axis. If the inequality involves "greater than," then determine the x-values where the function is above the x-axis.
- We can streamline the process of solving quadratic inequalities by making use of a sign chart. A sign chart gives us a visual reference that indicates where the function is above the x-axis using positive signs or below the x-axis using negative signs. Shade in the appropriate x-values depending on the original inequality.
- To make a sign chart, use the function and test values in each region bounded by the roots. We are only concerned if the function is positive or negative and thus a complete calculation is not necessary.

Exercise 6.5.4

Determine whether or not the given value is a solution.

1. $x^2-x+1<0; x=-1$
2. $x^2+x-1>0; x=-2$
3. $4x^2-12x+9\leq 0; x=\frac{3}{2}$
4. $5x^2-8x-4<0; x=-\frac{2}{5}$
5. $3x^2-x-2\geq 0; x=0$
6. $4x^2-x+3\leq 0; x=-1$
7. $2-4x-x^2<0; x=\frac{1}{2}$
8. $5-2x-x^2>0; x=0$
9. $-x^2-x-9<0; x=-3$
10. $-x^2+x-6\geq 0; x=6$

Answer

1. No

3. Yes

5. No

7. Yes

9. Yes

Exercise 6.5.5

Given the graph of f determine the solution set.

1. $f(x)\leq 0$;

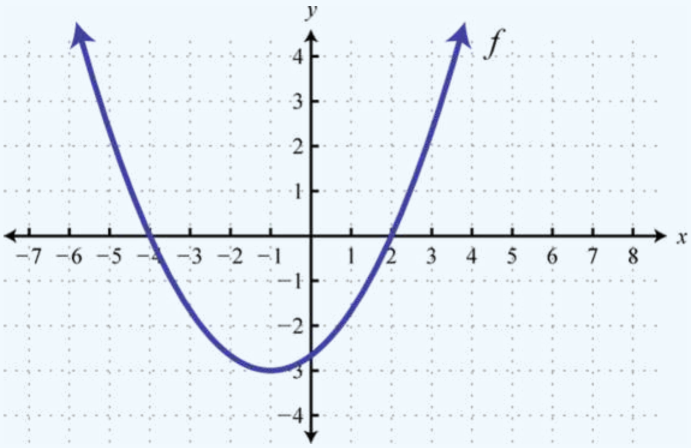

Figure 6.5.20

2. $f(x) \geq 0$;

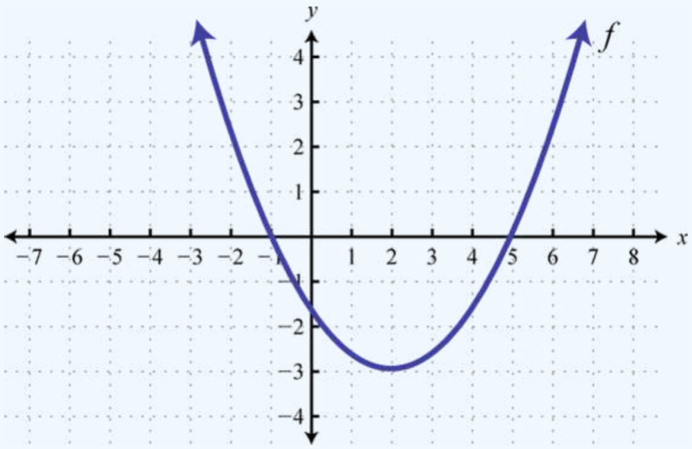

Figure 6.5.21

3. $f(x) \geq 0$;

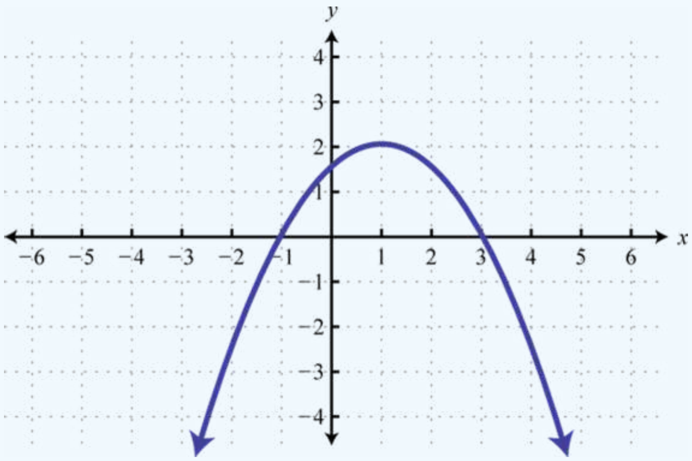

Figure 6.5.22

4. $f(x) \leq 0$;

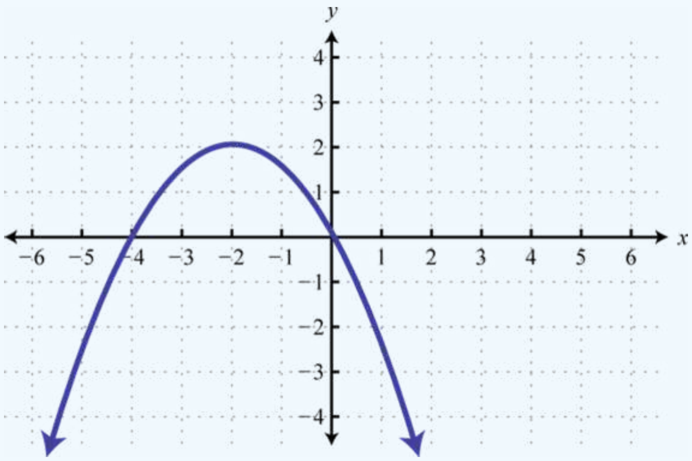

Figure 6.5.23

5. $f(x) > 0$;

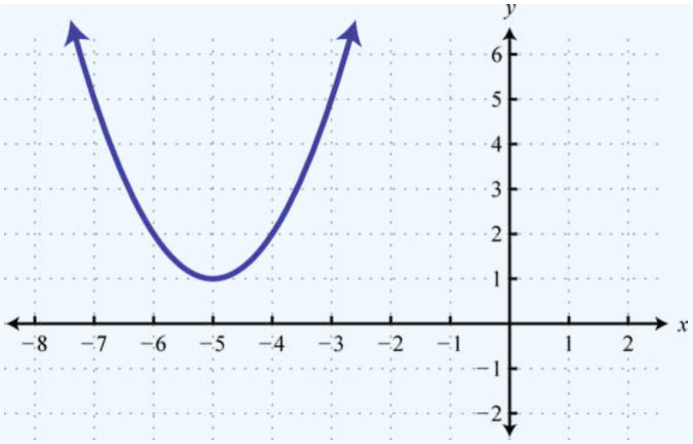

Figure 6.5.24

6. $f(x) < 0$;

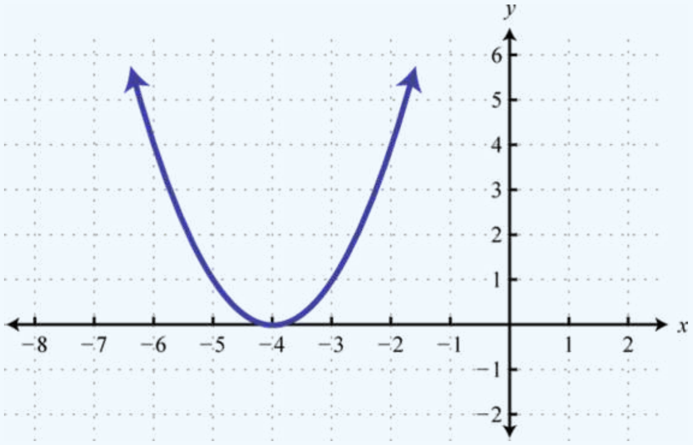

Figure 6.5.25

7. $f(x) > 0$;

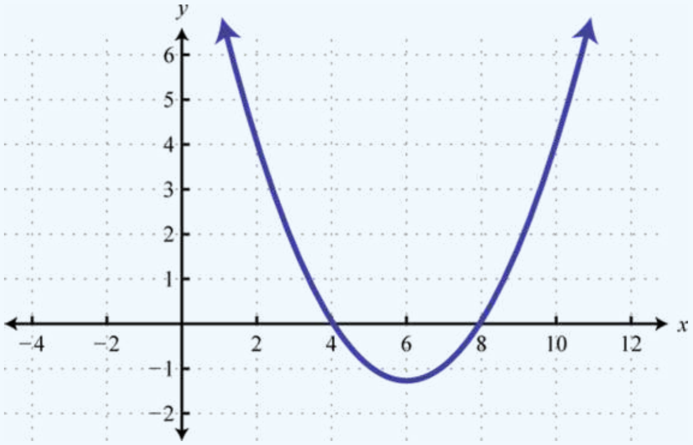

Figure 6.5.26

8. $f(x) < 0$;

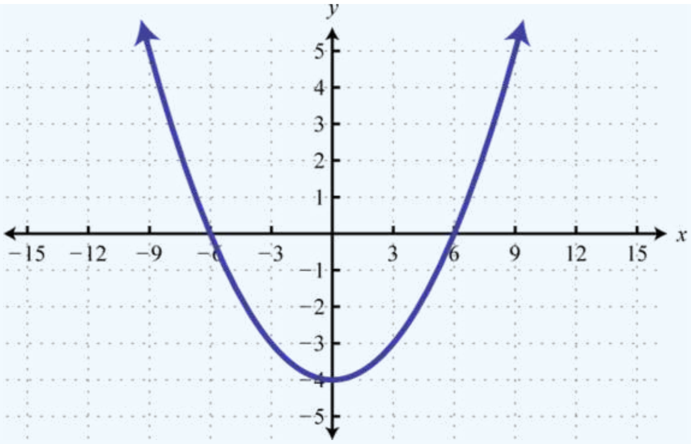

Figure 6.5.27

9. $f(x) \geq 0$;

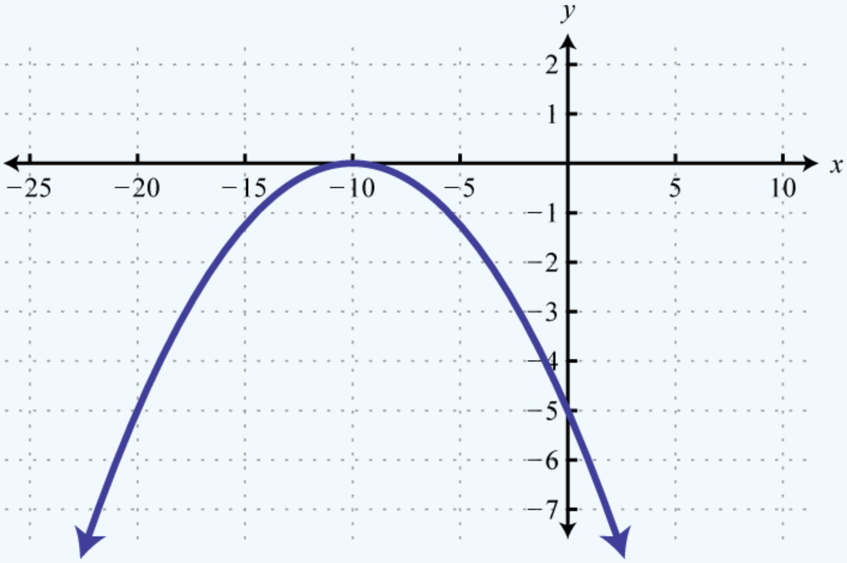

Figure 6.5.28

10. $f(x) < 0$;

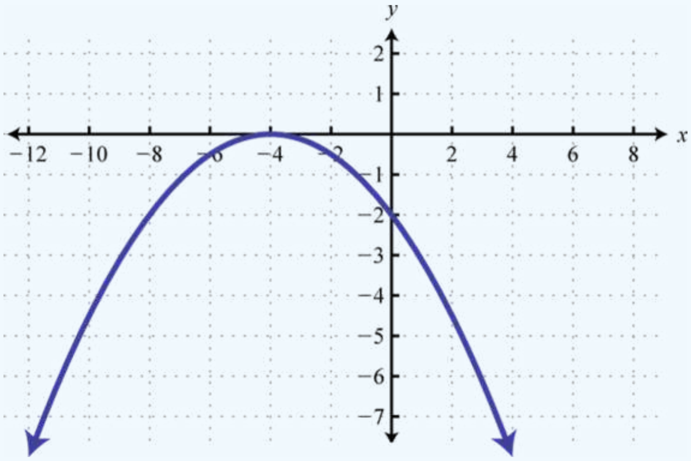

Figure 6.5.29

Answer

1. $[-4, 2]$

3. $[-1,3]$
5. $(-\infty,\infty)$
7. $(-\infty,4)\cup(8,\infty)$
9. $\{-10\}$

Exercise 6.5.6

Use the transformations to graph the following and then determine the solution set.

1. $x^2-1>0$
2. $x^2+2>0$
3. $(x-1)^2>0$
4. $(x+2)^2\leq 0$
5. $(x+2)^2-1\leq 0$
6. $(x+3)^2-4>0$
7. $-x^2+4\geq 0$
8. $-(x+2)^2>0$
9. $-(x+3)^2+1<0$
10. $-(x-4)^2+9>0$

Answer

1. $(-\infty,-1)\cup(1,\infty)$

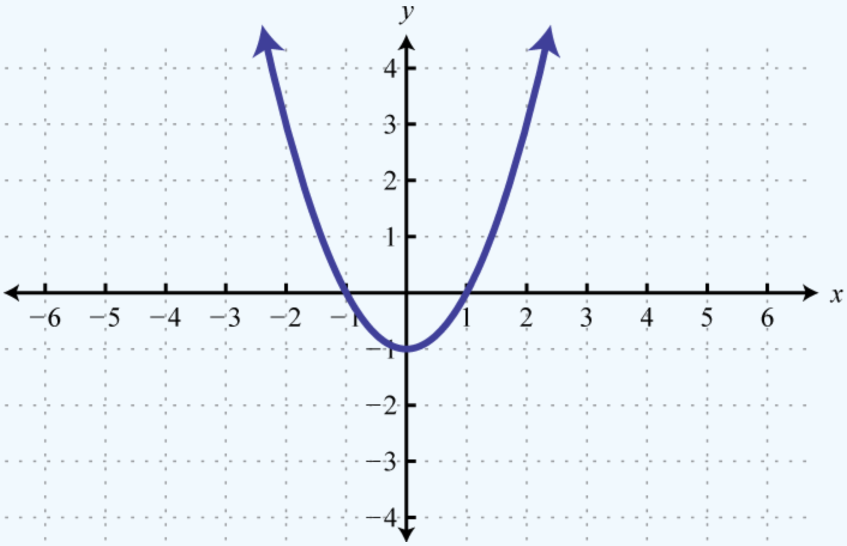

Figure 6.5.30

3. $(-\infty,1)\cup(1,\infty)$

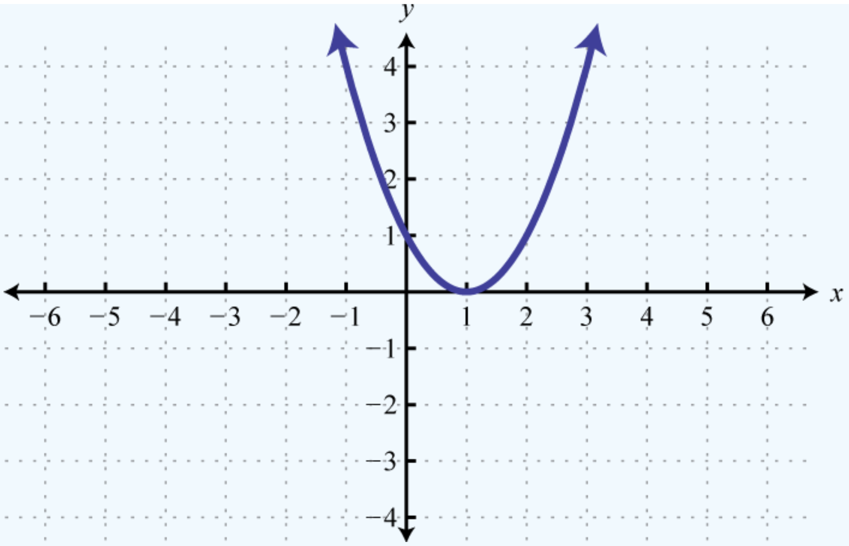

Figure 6.5.31

5. $[-3,-1]$

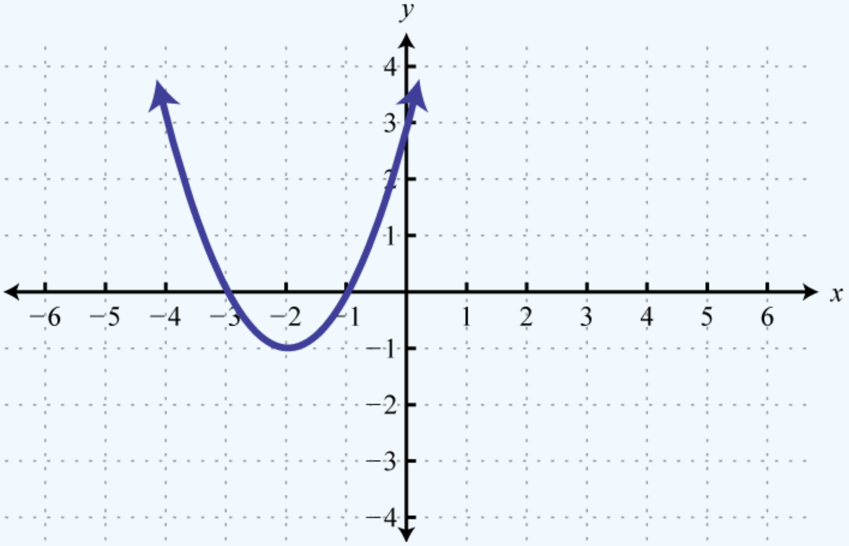

Figure 6.5.32

7. $[-2,2]$

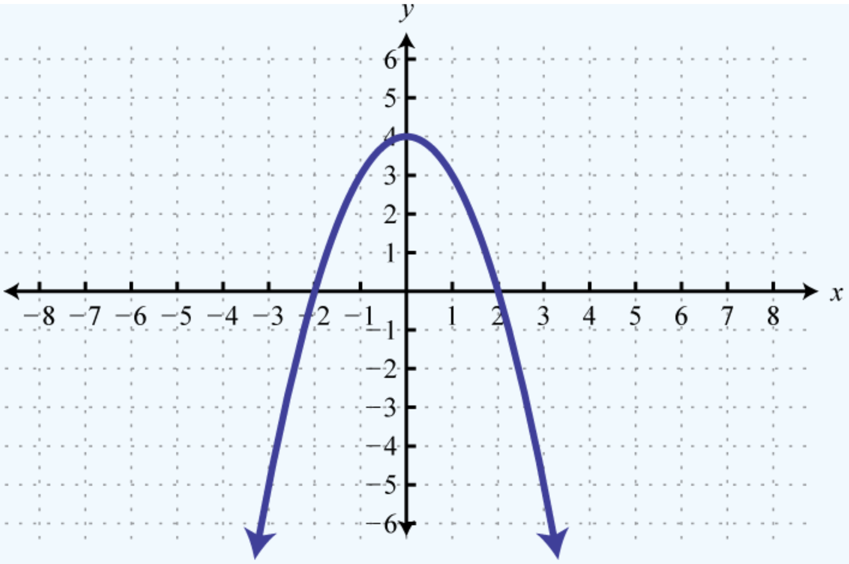

Figure 6.5.33

9. $(-\infty, -4) \cup (-2, \infty)$

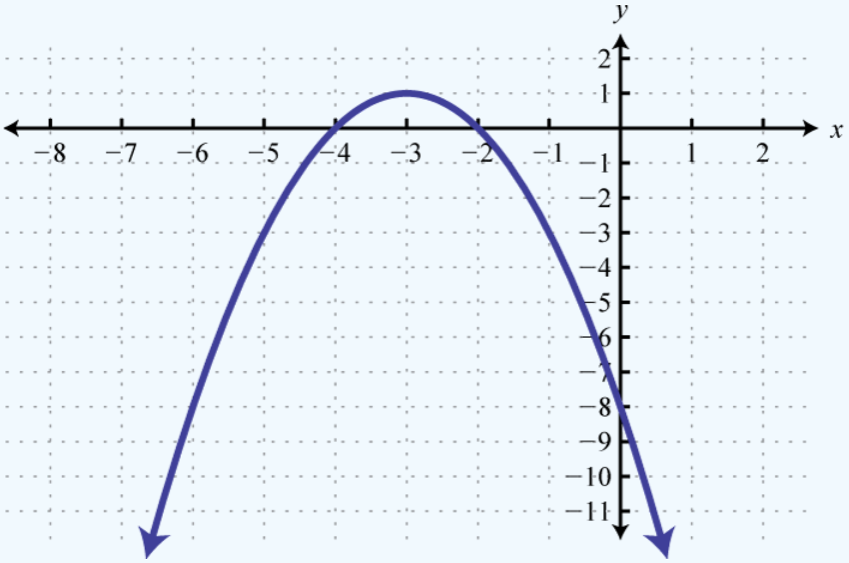

Figure 6.5.34

Exercise 6.5.7

Use a sign chart to solve and graph the solution set. Present answers using interval notation.

1. $x^2 - x - 12 > 0$
2. $x^2 - 10x + 16 > 0$
3. $x^2 + 2x - 24 < 0$
4. $x^2 + 15x + 54 < 0$
5. $x^2 - 23x - 24 \leq 0$
6. $x^2 - 12x + 20 \leq 0$
7. $2x^2 - 11x - 6 \geq 0$
8. $3x^2 + 17x - 6 \geq 0$
9. $8x^2 - 18x - 5 < 0$
10. $10x^2 + 17x + 6 > 0$
11. $9x^2 + 30x + 25 \leq 0$

12. $16x^2-40x+25\le 0$
13. $4x^2-4x+1>0$
14. $9x^2+12x+4>0$
15. $-x^2-x+30\ge 0$
16. $-x^2-6x+27\le 0$
17. $x^2-64<0$
18. $x^2-81\ge 0$
19. $4x^2-9\ge 0$
20. $16x^2-25<0$
21. $25-4x^2\ge 0$
22. $1-49x^2<0$
23. $x^2-8>0$
24. $x^2-75\le 0$
25. $2x^2+1>0$
26. $4x^2+3<0$
27. $x-x^2>0$
28. $3x-x^2\le 0$
29. $x^2-x+1<0$
30. $x^2+x-1>0$
31. $4x^2-12x+9\le 0$
32. $5x^2-8x-4<0$
33. $3x^2-x-2\ge 0$
34. $4x^2-x+3\le 0$
35. $2-4x-x^2<0$
36. $5-2x-x^2>0$
37. $-x^2-x-9<0$
38. $-x^2+x-6\ge 0$
39. $-2x^2+4x-1\ge 0$
40. $-3x^2-x+1\le 0$

Answer

1. $(-\infty,-3)\cup(4,\infty)$

3. $(-6,4)$

5. $[-1,24]$

7. $\left(-\infty,-\frac{1}{2}\right]\cup[6,\infty)$

9. $\left(-\frac{1}{4},\frac{5}{2}\right)$

11. $-\frac{5}{3}$

13. $\left(-\infty,\frac{1}{2}\right)\cup\left(\frac{1}{2},\infty\right)$

15. $[-6,5]$

17. $(-8,8)$

19. $\left(-\infty,-\frac{3}{2}\right]\cup\left[\frac{3}{2},\infty\right)$

21. $\left[-\frac{5}{2},\frac{5}{2}\right]$

23. $(-\infty,-2\sqrt{2})\cup(2\sqrt{2},\infty)$

25. $(-\infty,\infty)$

27. $(0,1)$

29. $\emptyset$

31. $\frac{3}{2}$

33. $\left(-\infty,-\frac{2}{3}\right] \cup [1,\infty)$

35. $(-\infty,-2-\sqrt{6}) \cup (-2+\sqrt{6},\infty)$

37. $(-\infty,\infty)$

39. $\left[\frac{2-\sqrt{2}}{2},\frac{2+\sqrt{2}}{2}\right]$

Exercise 6.5.8

Find the domain of the function.

1. $f(x)=\sqrt{x^2-25}$
2. $f(x)=\sqrt{x^2+3x}$
3. $g(x)=\sqrt{3x^2-x-2}$
4. $g(x)=\sqrt{12x^2-9x-3}$
5. $h(x)=\sqrt{16-x^2}$
6. $h(x)=\sqrt{3-2x-x^2}$
7. $f(x)=\sqrt{x^2+10}$
8. $f(x)=\sqrt{9+x^2}$
9. A robotics manufacturing company has determined that its weekly profit in thousands of dollars is modeled by $P(n)=-n^2+30n-200$ where n represents the number of units it produces and sells. How many units must the company produce and sell to maintain profitability. (Hint: Profitability occurs when profit is greater than zero.)
10. The height in feet of a projectile shot straight into the air is given by $h(t)=-16t^2+400t$ where t represents the time in seconds after it is fired. In what time intervals is the projectile under $1,000$ feet? Round to the nearest tenth of a second.

Answer

1. $(-\infty,-5] \cup [5,\infty)$

3. $\left(-\infty,-\frac{2}{3}\right] \cup [1,\infty)$

5. $[-4,4]$

7. $(-\infty,\infty)$

9. The company must produce and sell more than 10 units and fewer than 20 units each week.

Exercise 6.5.9

1. Does the sign chart for any given quadratic function always alternate? Explain and illustrate your answer with some examples.
2. Research and discuss other methods for solving a quadratic inequality.
3. Explain the difference between a quadratic equation and a quadratic inequality. How can we identify and solve each? What is the geometric interpretation of each?

Answer

1. Answer may vary

3. Answer may vary

Footnotes

[15] A mathematical statement that relates a quadratic expression as either less than or greater than another.

[16] The values in the domain of a function that separate regions that produce positive or negative results.

[17] A model of a function using a number line and signs ($+$ or $-$) to indicate regions in the domain where the function is positive or negative.

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6.6: Solving Polynomial and Rational Inequalities

Learning Objectives

- Solve polynomial inequalities.
- Solve rational inequalities.

Solving Polynomial Inequalities

A **polynomial inequality**[18] is a mathematical statement that relates a polynomial expression as either less than or greater than another. We can use sign charts to solve polynomial inequalities with one variable.

Example 6.6.1

Solve $x(x+3)^2(x-4)<0$.

Solution

Begin by finding the critical numbers. For a polynomial inequality in standard form, with zero on one side, the critical numbers are the roots. Because $f(x)=x(x+3)^2(x-4)$ is given in its factored form the roots are apparent. Here the roots are: $0, -3$, and 4. Because of the strict inequality, plot them using open dots on a number line.

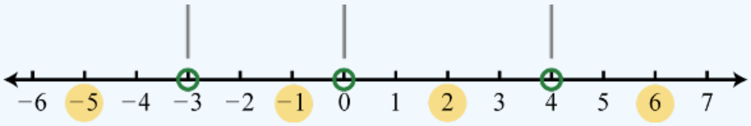

Figure 6.6.1

In this case, the critical numbers partition the number line into four regions. Test values in each region to determine if f is positive or negative. Here we choose test values $-5, -1, 2$, and 6. Remember that we are only concerned with the sign $(+$ or $-)$ of the result.

$$\begin{aligned} f(-5)&=(-5)(-5+3)^2(-5-4)=(-)(-)^2(-) &&=+\textit{Positive}\\ f(-1)&=(-1)(-1+3)^2(-1-4)=(-)(+)^2(-) &&=+\textit{Positive}\\ f(2)&=(2)(2+3)^2(2-4)=(+)(+)^2(-) &&=-\textit{Negative}\\ f(6)&=(6)(6+3)^2(6-4)=(+)(+)^2(+) &&=+\textit{Positive}\end{aligned}$$

After testing values we can complete a sign chart.

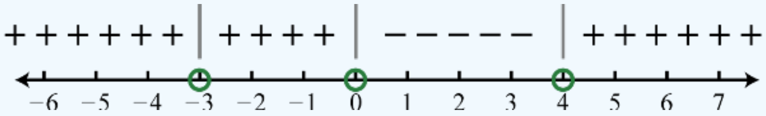

Figure 6.6.2

The question asks us to find the values where $f(x)<0$, or where the function is negative. From the sign chart we can see that the function is negative for x-values in between 0 and 4.

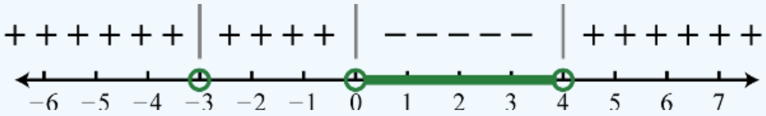

Figure 6.6.3

We can express this solution set in two ways:

$$\begin{array}{rr} \{x|0<x<4\} & \textit{Set notation}\\ (0,4) & \textit{Interval notation}\end{array}$$

In this textbook we will continue to present solution sets using interval notation.

Answer:

$(0,4)$

Graphing polynomials such as the one in the previous example is beyond the scope of this textbook. However, the graph of this function is provided below. Compare the graph to its corresponding sign chart.

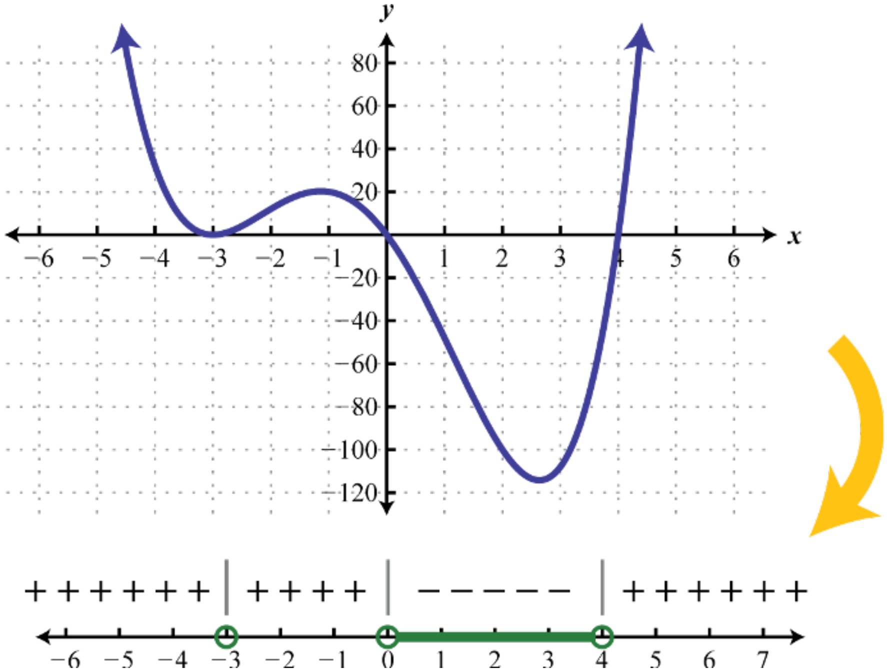

Figure 6.6.4

Certainly it may not be the case that the polynomial is factored nor that it has zero on one side of the inequality. To model a function using a sign chart, all of the terms should be on one side and zero on the other. The general steps for solving a polynomial inequality are listed in the following example.

Example 6.6.2:

Solve: $2x^4 > 3x^3 + 9x^2$.

Solution

Step 1: Obtain zero on one side of the inequality. In this case, subtract to obtain a polynomial on the left side in standard from.

$$\begin{aligned} 2x^4 &> 3x^3 + 9x^2 \\ 2x^4 - 3x^3 - 9x^2 &> 0 \end{aligned}$$

Step 2: Find the critical numbers. Here we can find the zeros by factoring.

$$\begin{aligned} 2x^4 - 3x^3 - 9x^2 &= 0 \\ x^2\left(2x^2 - 3x - 9\right) &= 0 \\ x^2(2x+3)(x-3) &= 0 \end{aligned}$$

There are three solutions, hence, three critical numbers $-\frac{3}{2}, 0,$ and 3. The strict inequality indicates that we should use open dots.

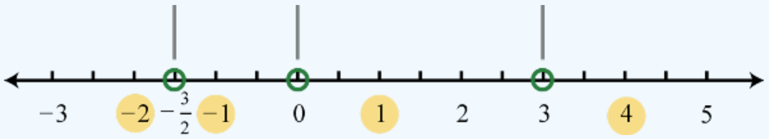

Figure 6.6.5

Step 3: Create a sign chart. In this case use $f(x) = x^2(2x+3)(x-3)$ and test values $-2, -1, 1,$ and 4 to determine the sign of the function in each interval.

$$\begin{aligned} f(-2) &= (-2)^2[2(-2)+3](-2-3) & &= (-)^2(-)(-) = + \\ f(-1) &= (-1)^2[2(-1)+3](-1-3) & &= (-)^2(+)(-) = - \\ f(1) &= (1)^2[2(1)+3](1-3) & &= (+)^2(+)(-) = - \\ f(4) &= (4)^2[2(4)+3](4-3) & &= (+)^2(+)(+) = + \end{aligned}$$

With this information we can complete the sign chart.

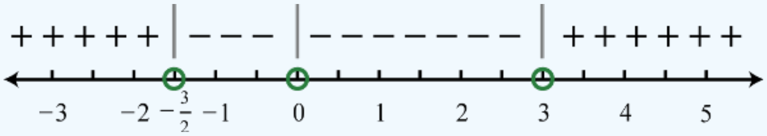

Figure 6.6.6

Step 4: Use the sign chart to answer the question. Here the solution consists of all values for which $f(x) > 0$. Shade in the values that produce positive results and then express this set in interval notation.

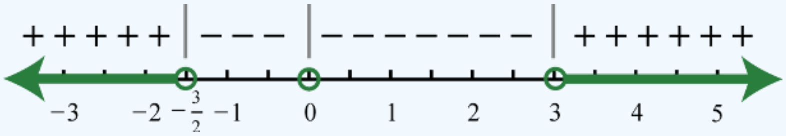

Figure 6.6.7

Answer:

$(-\infty, -\frac{3}{2}) \cup (3, \infty)$

Example 6.6.3

Solve: $x^3 + x^2 \leq 4(x+1)$.

Solution

Begin by rewriting the inequality in standard form, with zero on one side.

$$\begin{aligned} x^3 + x^2 &\leq 4(x+1) \\ x^3 + x^2 &\leq 4x + 4 \\ x^3 + x^2 - 4x - 4 &\leq 0 \end{aligned}$$

Next find the critical numbers of $f(x) = x^3 + x^2 - 4x - 4$:

$$\begin{aligned} x^3 + x^2 - 4x - 4 &= 0 \quad \color{#1a9fd9}{\textit{Factor by grouping.}} \\ x^2(x+1) - 4(x+1) &= 0 \\ (x+1)\left(x^2-4\right) &= 0 \\ (x+1)(x+2)(x-2) &= 0 \end{aligned}$$

The critical numbers are $-2, -1$, and 2. Because of the inclusive inequality $(\leq)$ we will plot them using closed dots.

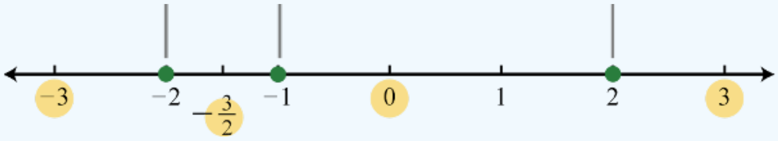

Figure 6.6.8

Use test values, $-3, -\frac{3}{2}, 0$, and 3 to create a sign chart.

$$\begin{aligned} f(-3) &= (-3+1)(-3+2)(-3-2) &&= (-)(-)(-) = - \\ f(-\frac{3}{2}) &= (-\frac{3}{2}+1)(-\frac{3}{2}+2)(-\frac{3}{2}-2) &&= (-)(+)(-) = + \\ f(0) &= (0+1)(0+2)(0-2) &&= (+)(+)(-) = - \\ f(3) &= (3+1)(3+2)(3-2) &&= (+)(+)(+) = + \end{aligned}$$

And we have

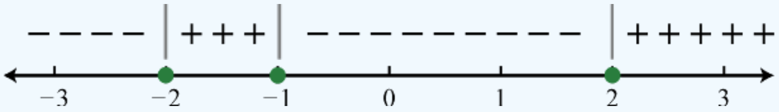

Figure 6.6.9

Use the sign chart to shade in the values that have negative results $(f(x) \leq 0)$.

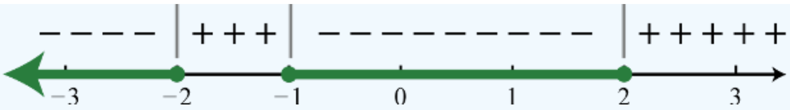

Figure 6.6.10

Answer

$(-\infty,-2]\cup[-1,2]$

Exercise 6.6.1

Solve $-3x^4+12x^3-9x^2>0$.

Answer

$(1,3)$

www.youtube.com/v/EXpe_0LzbSY

Solving Rational Inequalities

A **rational inequality**[19] is a mathematical statement that relates a rational expression as either less than or greater than another. Because rational functions have restrictions to the domain we must take care when solving rational inequalities. In addition to the zeros, we will include the restrictions to the domain of the function in the set of critical numbers.

Example 6.6.4

Solve: $\frac{(x-4)(x+2)}{(x-1)}\geq 0$

Solution

The zeros of a rational function occur when the numerator is zero and the values that produce zero in the denominator are the restrictions. In this case,

$$\begin{array}{c|c}\text{Roots (Numerator)} & \text{Restriction(Denominator)}\\ x-4=0 \text{ or } x+2=0 & x-1=0\\ x=4 \qquad x=-2 & x=1\end{array}$$

Therefore the critical numbers are $-2,1$, and 4. Because of the inclusive inequality $(\geq)$ use a closed dot for the roots $-2,4$ and always use an open dot for restrictions 1. Restrictions are never included in the solution set.

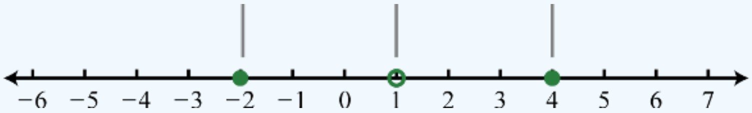

Figure 6.6.11

Use test values $x=-4,0,2,6$.

$$\begin{aligned} f(-4)&=\frac{(-4-4)(-4+2)}{(-4-1)} &&=\frac{(-)(-)}{(-)}=-\\ f(0)&=\frac{(0-4)(0+2)}{(0-1)} &&=\frac{(-)(+)}{(-)}=+\\ f(2)&=\frac{(2-4)(2+2)}{(2-1)} &&=\frac{(-)(+)}{(+)}=-\\ f(6)&=\frac{(6-4)(6+2)}{(6-1)} &&=\frac{(+)(+)}{(+)}=+\end{aligned}$$

And then complete the sign chart.

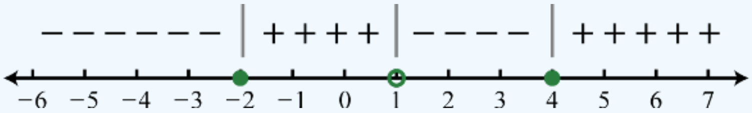

Figure 6.6.12

The question asks us to find the values for which $f(x) \geq 0$, in other words, positive or zero. Shade in the appropriate regions and present the solution set in interval notation.

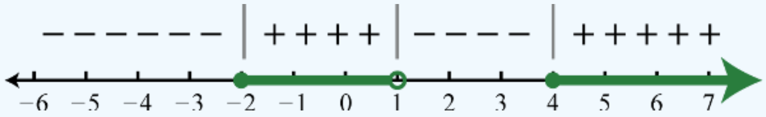

Figure 6.6.13

Answer

$[-2, 1) \cup [4, \infty)$

Graphing such rational functions like the one in the previous example is beyond the scope of this textbook. However, the graph of this function is provided below. Compare the graph to its corresponding sign chart.

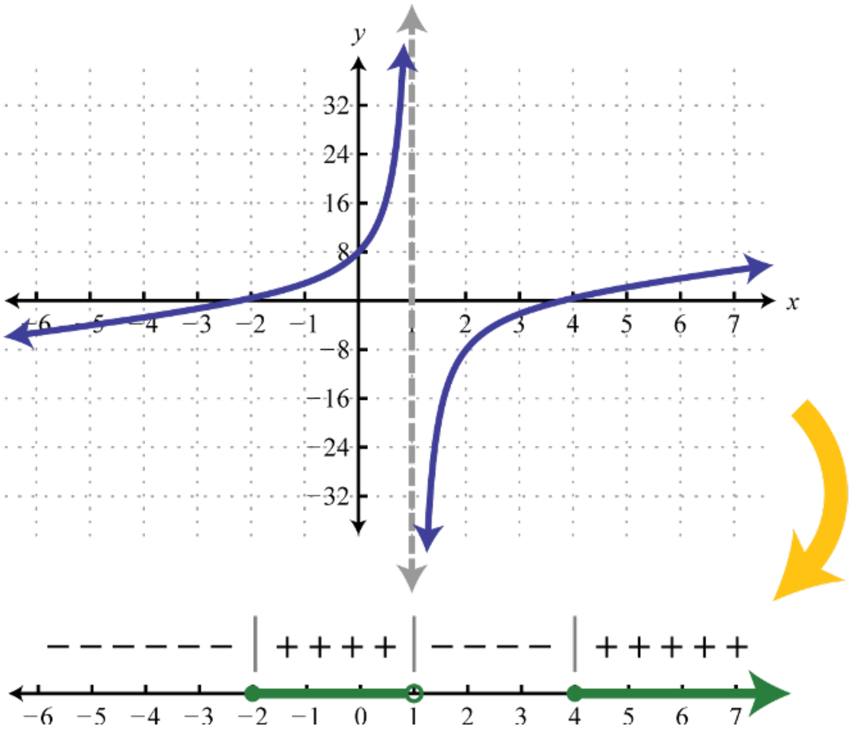

Figure 6.6.14

Notice that the restriction $x = 1$ corresponds to a vertical asymptote which bounds regions where the function changes from positive to negative. While not included in the solution set, the restriction is a critical number. Before creating a sign chart we must ensure the inequality has a zero on one side. The general steps for solving a rational inequality are outlined in the following example.

Example 6.6.5:

Solve $\frac{7}{x+3} < 2$.

Solution

Step 1: Begin by obtaining zero on the right side.

$$\frac{7}{x+3} < 2$$
$$\frac{7}{x+3} - 2 < 0$$

Step 2: Determine the critical numbers. The critical numbers are the zeros and restrictions. Begin by simplifying to a single algebraic fraction.

$$\begin{aligned} \frac{7}{x+3}-\frac{2}{1} &< 0 \\ \frac{7-2(x+3)}{x+3} &< 0 \\ \frac{7-2x-6}{x+3} &< 0 \\ \frac{-2x+1}{x+3} &< 0 \end{aligned}$$

Next find the critical numbers. Set the numerator and denominator equal to zero and solve.

Root	Restriction
$-2x+1=0$	
$-2x=-1$	$x+3=0$
$x=\frac{1}{2}$	$x=-3$

In this case, the strict inequality indicates that we should use an open dot for the root.

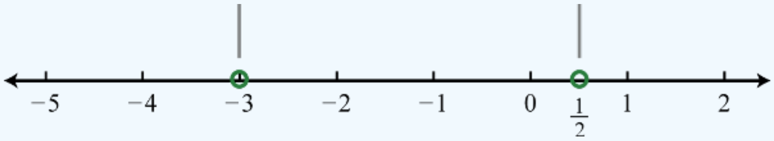

Figure 6.6.15

Step 3: Create a sign chart. Choose test values $-4, 0$, and 1.

$$\begin{aligned} f(-4) &= \frac{-2(-4)+1}{-4+3} && = \frac{+}{-} = - \\ f(0) &= \frac{-2(0)+1}{0+3} && = \frac{+}{+} = + \\ f(1) &= \frac{-2(1)+1}{1+3} && = \frac{-}{+} = - \end{aligned}$$

And we have

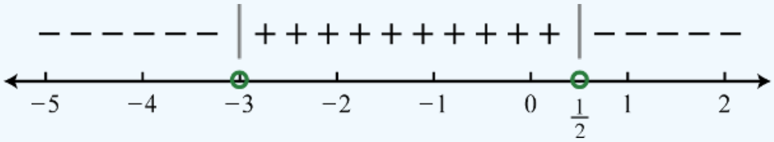

Figure 6.6.16

Step 4: Use the sign chart to answer the question. In this example we are looking for the values for which the function is negative, $f(x)<0$. Shade the appropriate values and then present your answer using interval notation.

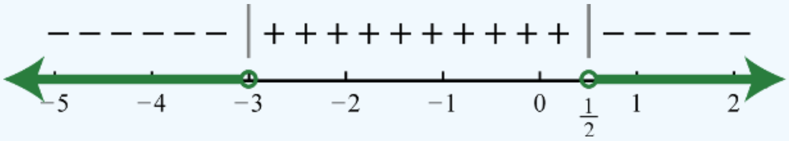

Figure 6.6.17

Answer:

$(-\infty,-3)\cup\left(\frac{1}{2},\infty\right)$

Example $6.6.6$:

Solve: $\frac{1}{x^2-4}\leq\frac{1}{2-x}$.

Solution

Begin by obtaining zero on the right side.

$$\begin{aligned} \frac{1}{x^2-4} &\leq \frac{1}{2-x} \\ \frac{1}{x^2-4}-\frac{1}{2-x} &\leq 0 \end{aligned}$$

Next simplify the left side to a single algebraic fraction.

$$\begin{aligned} \frac{1}{x^2-4}-\frac{1}{2-x} &\leq 0 \\ \frac{1}{(x+2)(x-2)}-\frac{1}{-(x-2)} &\leq 0 \\ \frac{1}{(x+2)(x-2)}+\frac{1{\color{cyan}(x+2)}}{(x-2){\color{cyan}(x+2)}} &\leq 0 \\ \frac{1+x+2}{(x+2)(x-2)} &\leq 0 \\ \frac{x+3}{(x+2)(x-2)} &\leq 0 \end{aligned}$$

The critical numbers are $-3, -2$, and 2. Note that ± 2 are restrictions and thus we will use open dots when plotting them on a number line. Because of the inclusive inequality we will use a closed dot at the root -3.

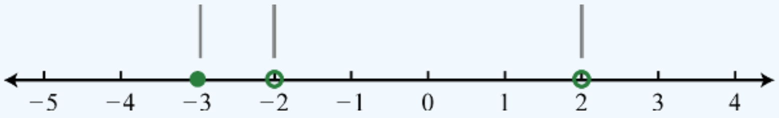

Figure $6.6.18$

Choose test values $-4, -2\frac{1}{2}=-\frac{5}{2}, 0$, and 3.

$$\begin{aligned} f({\color{green}-4}) &= \frac{{\color{green}-4}+3}{({\color{green}-4}+2)({\color{green}-4}-2)} && = \frac{(-)}{(-)(-)} = - \\ f(-\tfrac{{\color{green}5}}{{\color{green}2}}) &= \frac{-\frac{{\color{green}5}}{{\color{green}2}}+3}{(-\frac{{\color{green}5}}{{\color{green}2}}+2)(-\frac{{\color{green}5}}{{\color{green}2}}-2)} && = \frac{(+)}{(-)(-)} = + \\ f({\color{green}0}) &= \frac{{\color{green}0}+3}{({\color{green}0}+2)({\color{green}0}-2)} && = \frac{(+)}{(+)(-)} = - \\ f({\color{green}3}) &= \frac{{\color{green}3}+3}{({\color{green}3}+2)({\color{green}3}-2)} && = \frac{(+)}{(+)(+)} = + \end{aligned}$$

Construct a sign chart.

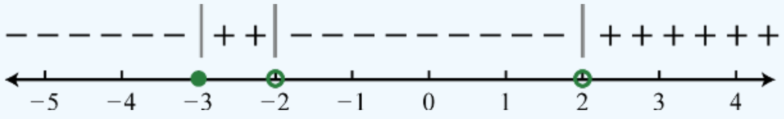

Figure $6.6.19$

Answer the question; in this case, find x where $f(x) \leq 0$.

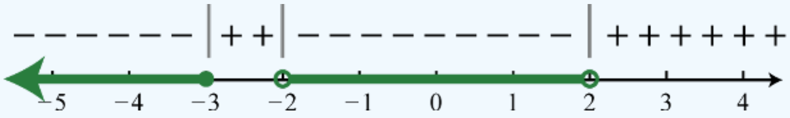

Figure $6.6.20$

Answer

$(-\infty, -3] \cup (-2, 2)$

Exercise $6.6.2$

Solve: $\frac{2x^2}{2x^2+7x-4} \geq \frac{x}{x+4}$.

Answer

$(-4, 0] \cup (\frac{1}{2}, \infty)$

www.youtube.com/v/3ljcHJqExBY

Key Takeaways

- When a polynomial inequality is in standard form, with zero on one side, the roots of the polynomial are the critical numbers. Create a sign chart that models the function and then use it to answer the question.
- When a rational inequality is written as a single algebraic fraction, with zero on one side, the roots as well as the restrictions are the critical numbers. The values that produce zero in the numerator are the roots, and the values that produce zero in the

denominator are the restrictions. Always use open dots for restrictions, regardless of the given inequality, because restrictions are not part of the domain. Create a sign chart that models the function and then use it to answer the question.

Exercise 6.6.3

Solve. Present answers using interval notation.

1. $x(x+1)(x-3)>0$
2. $x(x-1)(x+4)<0$
3. $(x+2)(x-5)^2<0$
4. $(x-4)(x+1)^2\geq 0$
5. $(2x-1)(x+3)(x+2)\leq 0$
6. $(3x+2)(x-4)(x-5)\geq 0$
7. $x(x+2)(x-5)^2<0$
8. $x(2x-5)(x-1)^2>0$
9. $x(4x+3)(x-1)^2\geq 0$
10. $(x-1)(x+1)(x-4)^2<0$
11. $(x+5)(x-10)(x-5)^2\geq 0$
12. $(3x-1)(x-2)(x+2)^2\leq 0$
13. $-4x(4x+9)(x-8)^2>0$
14. $-x(x-10)(x+7)^2>0$

Answer

1. $(-1,0)\cup(3,\infty)$
3. $(-\infty,-2)$
5. $(-\infty,-3]\cup\left[-2,\frac{1}{2}\right]$
7. $(-2,0)$
9. $\left(-\infty,-\frac{3}{4}\right]\cup[0,\infty)$
11. $(-\infty,-5]\cup[5,5]\cup[10,\infty)$
13. $\left(-\frac{9}{4},0\right)$

Exercise 6.6.4

Solve.

1. $x^3+2x^2-24x\geq 0$
2. $x^3-3x^2-18x\leq 0$
3. $4x^3-22x^2-12x<0$
4. $9x^3+30x^2-24x>0$
5. $12x^4+44x^3>80x^2$
6. $6x^4+12x^3<48x^2$
7. $x\left(x^2+25\right)<10x^2$
8. $x^3>12x(x-3)$
9. $x^4-5x^2+4\leq 0$
10. $x^4-13x^2+36\geq 0$
11. $x^4>3x^2+4$
12. $4x^4<3-11x^2$
13. $9x^3-3x^2-81x+27\leq 0$
14. $2x^3+x^2-50x-25\geq 0$
15. $x^3-3x^2+9x-27>0$
16. $3x^3+5x^2+12x+20<0$

Answer

1. $[-6,0] \cup [4,\infty)$
3. $(-\infty,-\frac{1}{2}) \cup (0,6)$
5. $(-\infty,-5) \cup (\frac{4}{3},\infty)$
7. $(-\infty,0)$
9. $[-2,-1] \cup [1,2]$
11. $(-\infty,-2) \cup (2,\infty)$
13. $(-\infty,-3] \cup [\frac{1}{3},3]$
15. $(3,\infty)$

Exercise 6.6.5

Solve.

1. $\frac{x}{x-3} > 0$
2. $\frac{x-5}{x} > 0$
3. $\frac{(x-3)(x+1)}{x} < 0$
4. $\frac{(x+5)(x+4)}{(x-2)} < 0$
5. $\frac{(2x+1)(x+5)}{(x-3)(x-5)} \leq 0$
6. $\frac{(3x-1)(x+6)}{(x-1)(x+9)} \geq 0$
7. $\frac{(x-8)(x+8)}{-2x(x-2)} \geq 0$
8. $\frac{(2x+7)(x+4)}{x(x+5)} \leq 0$
9. $\frac{x^2}{(2x+3)(2x-3)} \leq 0$
10. $\frac{(x-4)^2}{-x(x+1)} > 0$
11. $\frac{-5x(x-2)^-}{(x+5)(x-6)} \geq 0$
12. $\frac{(3x-4)(x+5)}{x(x-4)^2} \geq 0$
13. $\frac{1}{(x-5)^4} > 0$
14. $\frac{1}{(x-5)^4} < 0$

Answer

1. $(-\infty,-0) \cup (3,\infty)$
3. $(-\infty,-1) \cup (0,3)$
5. $[-5,-\frac{1}{2}] \cup (3,5)$
7. $[-8,0) \cup (2,8]$
9. $(-\frac{3}{2},\frac{3}{2})$
11. $(-\infty,-5) \cup [0,6)$
13. $(-\infty,5) \cup (5,\infty)$

Exercise 6.6.6

Solve.

1. $\frac{x^2-11x-12}{x+4} < 0$
2. $\frac{x^2-10x+24}{x-2} > 0$
3. $\frac{x^2+x-30}{2x+1} \geq 0$
4. $\frac{2x^2+x-3}{x-3} \leq 0$
5. $\frac{3x^2-4x+1}{x^2-9} \leq 0$
6. $\frac{x^2-16}{2x^2-3x-2} \geq 0$
7. $\frac{x^2-12x+20}{x^2-10x+25} > 0$

8. $\frac{x^2+15x+36}{x^2-8x+16} < 0$
9. $\frac{8x^2-2x-1}{2x^2-3x-14} \leq 0$
10. $\frac{4x^2-4x-15}{x^2+4x-5} \geq 0$
11. $\frac{1}{x+5} + \frac{5}{x-1} > 0$
12. $\frac{5}{x+4} - \frac{1}{x-4} < 0$
13. $\frac{1}{x+7} > 1$
14. $\frac{1}{x-1} < -5$
15. $x \geq \frac{30}{x-1}$
16. $x \leq \frac{1-2x}{x-2}$
17. $\frac{1}{x-1} \leq \frac{2}{x}$
18. $\frac{3}{x+1} > -\frac{1}{x}$
19. $\frac{4}{x-3} \leq \frac{1}{x+3}$
20. $\frac{2x-9}{x} + \frac{49}{x-8} < 0$
21. $\frac{x}{2(x+2)} - \frac{1}{x+2} \leq \frac{12}{x(x+2)}$
22. $\frac{1}{2x+1} - \frac{9}{2x-1} > 2$
23. $\frac{3x}{x^2-4} - \frac{2}{x-2} < 0$
24. $\frac{x}{2x+1} + \frac{4}{2x^2-7x-4} < 0$
25. $\frac{x+1}{2x^2+5x-3} \geq \frac{x}{4x^2-1}$
26. $\frac{x^2-14}{2x^2-7x-4} \leq \frac{5}{1+2x}$

Answer

1. $(-\infty, -4) \cup (-1, 12)$

3. $\left[-6, -\frac{1}{2}\right) \cup [5, \infty)$

5. $\left(-3, \frac{1}{3}\right] \cup [1, 3)$

7. $(-\infty, 2) \cup (10, \infty)$

9. $\left(-2, -\frac{1}{4}\right] \cup \left[\frac{1}{2}, \frac{7}{2}\right)$

11. $(-5, -4) \cup (1, \infty)$

13. $(-7, -6)$

15. $[-5, 1) \cup [6, \infty)$

17. $(0, 1) \cup [2, \infty)$

19. $(-\infty, 5] \cup (-3, 3)$

21. $[-4, -2) \cup (0, 6]$

23. $(-\infty, -2) \cup (2, 4)$

25. $\left(-3, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

Exercise 6.6.7

1. Does the sign chart for any given polynomial or rational function always alternate? Explain and illustrate your answer with some examples.
2. Write down your own steps for solving a rational inequality and illustrate them with an example. Do your steps also work for a polynomial inequality? Explain.

Answer

1. Answer may vary

Footnotes

[18] A mathematical statement that relates a polynomial expression as either less than or greater than another.

[19] A mathematical statement that relates a rational expression as either less than or greater than another.

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6.E: Solving Equations and Inequalities (Exercises)

Exercise $6.E.1$

Solve by extracting the roots.

1. $x^2-81=0$
2. $y^2-\frac{1}{4}=0$
3. $9x^2-8=0$
4. $5x^2-12=0$
5. $2y^2-7=0$
6. $3y^2-6=0$
7. $(2x-3)^2-16=0$
8. $4(x-1)^2-5=0$
9. $9(x-3)^2+4=0$
10. $5(2x+1)^2+1=0$
11. $2x^2+10=0$
12. $x^2+64=0$
13. The height in feet of an object dropped from a 20-foot stepladder is given by $h(t)=-16t^2+20$ where t represents the time in seconds after the object has been dropped. How long does it take the object to hit the ground after it has been dropped? Round to the nearest tenth of a second.
14. A 20-foot ladder, leaning against a building, reaches a height of 19 feet. How far is the base of the ladder from the wall? Round to the nearest tenth of a foot.

Answer

1. ± 9

3. $\pm\frac{2\sqrt{2}}{3}$

5. $\pm\frac{\sqrt{14}}{2}$

7. $-\frac{1}{2},\frac{7}{2}$

9. $3\pm\frac{2}{3}i$

11. $\pm i\sqrt{5}$

13. 1.1 seconds

Exercise $6.E.2$

Solve by completing the square.

1. $x^2+4x-5=0$
2. $x^2+2x-17=0$
3. $x^2-4x+1=0$
4. $x^2-6x-2=0$
5. $x^2-3x-1=0$
6. $x^2+5x-6=0$
7. $x^2+x-2=0$
8. $x^2-x-4=0$
9. $5x^2-10x+1=0$
10. $4x^2+8x-3=0$
11. $2x^2-6x+1=0$
12. $3x^2+10x+6=0$
13. $x^2-x+3=0$
14. $2x^2+6x+5=0$
15. $x(x+9)+10=5x+2$
16. $(2x+5)(x+2)=8x+7$

Answer

1. $-5, 1$
3. $2 \pm \sqrt{3}$
5. $\frac{3 \pm \sqrt{13}}{2}$
7. $-2, 1$
9. $\frac{5 \pm 2\sqrt{5}}{5}$
11. $\frac{3 \pm \sqrt{7}}{2}$
13. $\frac{1}{2} \pm \frac{\sqrt{11}}{2} i$
15. $-2 \pm 2i$

Exercise $6.E.3$

Solve using the quadratic formula.

1. $2x^2 - x - 6 = 0$
2. $3x^2 + x - 4 = 0$
3. $9x^2 + 12x + 2 = 0$
4. $25x^2 - 10x - 1 = 0$
5. $-x^2 + 8x - 2 = 0$
6. $-x^2 - x + 1 = 0$
7. $5 - 2x - x^2 = 0$
8. $2 + 4x - 3x^2 = 0$
9. $3x^2 - 2x + 4 = 0$
10. $7x^2 - x + 1 = 0$
11. $-x^2 + 2x - 6 = 0$
12. $-3x^2 + 4x - 2 = 0$
13. $36x^2 + 60x + 25 = 0$
14. $72x^2 + 54x - 35 = 0$
15. $1.3x^2 - 2.8x - 4.2 = 0$
16. $5.5x^2 - 4.1x + 2.2 = 0$
17. $(x + 2)^2 - 3x = 4$
18. $(3x + 1)^2 - 6 = 6x - 3$
19. The height in feet of a baseball tossed upward at a speed of 48 feet per second from the ground is given by the function, $h(t) = -16t^2 + 48t$, where t represents the time in seconds after the ball is tossed. At what time does the baseball reach a height of 18 feet? Round off to the nearest hundredth of a second.
20. The height in feet reached by a model rocket launched from a 3-foot platform is given by the function $h(t) = -16t^2 + 256t + 3$ where t represents time in seconds after launch. At what times will the rocket reach $1,000$ feet? Round off to the nearest tenth of a second.

Answer

1. $-\frac{3}{2}, 2$
3. $\frac{-2 \pm \sqrt{2}}{3}$
5. $4 \pm \sqrt{14}$
7. $-1 \pm \sqrt{6}$
9. $\frac{1}{3} \pm \frac{\sqrt{11}}{3} i$
11. $1 \pm i\sqrt{5}$
13. $-\frac{5}{6}$
15. $x \approx -1.0, x \approx 3.2$

17. $-1, 0$

19. The ball will reach 18 feet at 0.44 seconds and again at 2.56 seconds.

Exercise $6.E.4$

Use the discriminant to determine the number and type of solutions.

1. $-x^2 + 6x + 1 = 0$
2. $-x^2 + x - 3 = 0$
3. $4x^2 - 4x + 1 = 0$
4. $16x^2 - 9 = 0$

Answer

1. Two irrational solutions

3. One rational solution

Exercise $6.E.5$

Solve using any method.

1. $x^2 - 4x - 96 = 0$
2. $25x^2 + x = 0$
3. $25t^2 - 1 = 0$
4. $t^2 + 25 = 0$
5. $y^2 - y - 7 = 0$
6. $5y^2 - 25y = 0$
7. $2x^2 - 9 = 0$
8. $25x^2 - 10x + 1 = 0$
9. $(2x+5)^2 - 9 = 0$
10. $(x-2)(x-5) = 5$
11. The length of a rectangle is 3 inches less than twice the width. If the area of the rectangle measures 30 square inches, then find the dimensions of the rectangle. Round off to the nearest hundredth of an inch.
12. The value in dollars of a new car is modeled by the function $V(t) = 125t^2 - 2,500t + 18,000$ where t represents the number of years since it was purchased. Determine the age of the car when its value is $\$18,000$.

Answer

1. $-8, 12$

3. $\pm\frac{1}{5}$

5. $\frac{1 \pm \sqrt{29}}{2}$

7. $\pm\frac{3\sqrt{2}}{2}$

9. $-4, -1$

11. Length: 6.38 inches; width: 4.69 inches

Exercise $6.E.6$

Find all solutions.

1. $x^4 - 16x^2 + 48 = 0$
2. $x^{2/3} - x^{1/3} - 20 = 0$
3. $x^{-2} - 5x^{-1} - 50 = 0$
4. $\left(\frac{t+3}{t}\right)^2 + 11\left(\frac{t+3}{t}\right) - 12 = 0$
5. $x + 2\sqrt{x} - 24 = 0$
6. $2x^{1/2} - 3x^{1/4} + 1 = 0$

7. $4\left(\frac{1}{x+1}\right)^2 - 4\left(\frac{1}{x+1}\right) - 3 = 0$
8. $5t^{-2} - 27t^{-1} - 18 = 0$
9. $3x^{2/3} - 5x^{1/3} + 2 = 0$
10. $4x + 4\sqrt{x} + 1 = 0$
11. $16y^4 - 25 = 0$
12. $x^{-2} - 64 = 0$

Answer

1. $\pm 2, \pm 2\sqrt{3}$

3. $-\frac{1}{5}, \frac{1}{10}$

5. 16

7. $-3, -\frac{1}{3}$

9. $1, \frac{8}{27}$

11. $\pm\frac{\sqrt{5}}{2}, \pm\frac{\sqrt{5}}{2}i$

Exercise $6.E.7$

Find the set of all roots.

1. $f(x) = x^2 - 50$
2. $f(x) = x^3 - 64$
3. $f(x) = x^4 - 81$
4. $f(x) = x^4 + 8x$

Answer

1. $\{\pm 5\sqrt{2}\}$

3. $\{\pm 3, \pm 3i\}$

Exercise $6.E.8$

Find a quadratic equation with integer coefficients and the given set of solutions.

1. $\{\frac{4}{3}, -\frac{1}{2}\}$
2. $\{\pm\sqrt{5}\}$
3. $\{\pm 4\sqrt{2}\}$
4. $\{\pm 6i\}$
5. $\{2 \pm i\}$
6. $\{3 \pm \sqrt{5}\}$

Answer

1. $6x^2 - 5x - 4 = 0$

3. $x^2 - 32 = 0$

5. $x^2 - 4x + 5 = 0$

Exercise $6.E.9$

Determine the x- and y-intercepts.

1. $y = 2x^2 + 5x - 12$
2. $y = x^2 - 18$
3. $y = x^2 + 4x + 7$
4. $y = -9x^2 + 12x - 4$

Answer

1. x-intercepts: $(-4,0),(\frac{3}{2},0)$; y-intercept: $(0,-12)$

3. x-intercepts: none; y-intercept: $(0,7)$

Exercise $6.E.10$

Find the vertex and the line of symmetry.

1. $y=x^2-4x-12$
2. $y=-x^2+8x-1$
3. $y=x^2+3x-1$
4. $y=4x^2-1$

Answer

1. Vertex: $(2,-16)$; line of symmetry: $x=2$

3. Vertex: $(-\frac{3}{2},-\frac{13}{4})$; line of symmetry: $x=-\frac{3}{2}$

Exercise $6.E.11$

Graph. Find the vertex and the y-intercept. In addition, find the x-intercepts if they exist.

1. $y=x^2+8x+12$
2. $y=-x^2-6x+7$
3. $y=-2x^2-4$
4. $y=x^2+4x$
5. $y=4x^2-4x+1$
6. $y=-2x^2$
7. $y=-2x^2+8x-7$
8. $y=3x^2-1$

Answer

1.

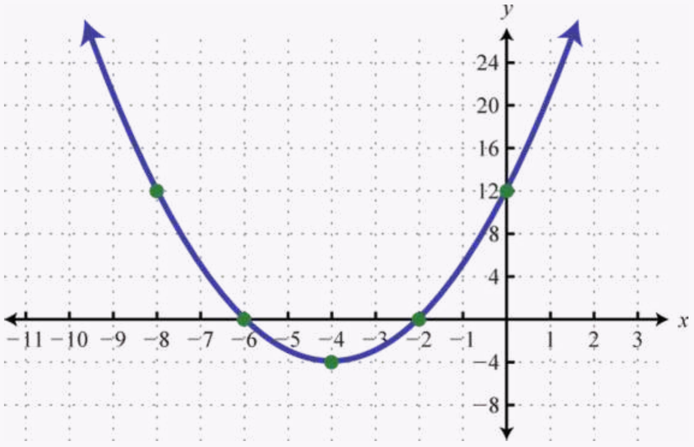

Figure 6.E.1

3.

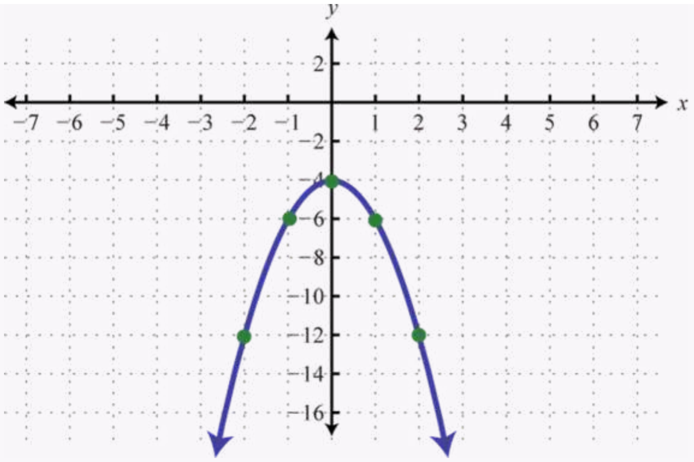

Figure 6.E.2

5.

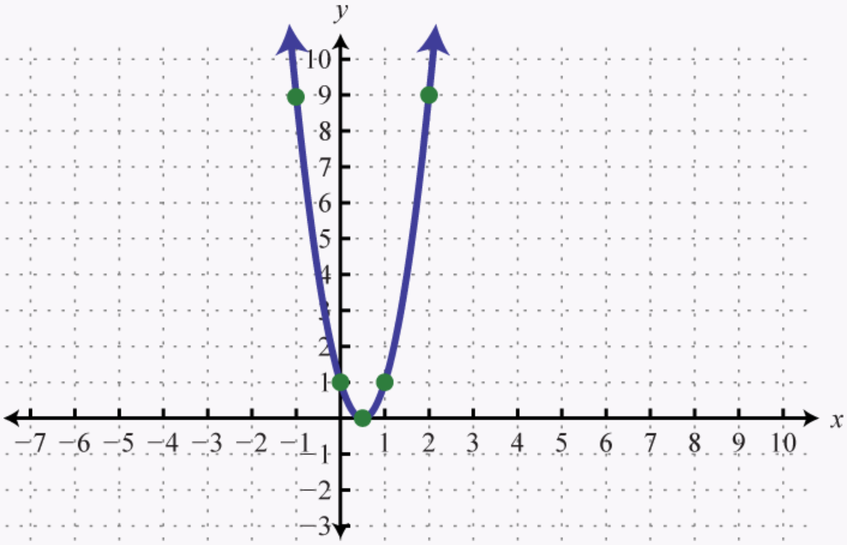

Figure 6.E.3

7.

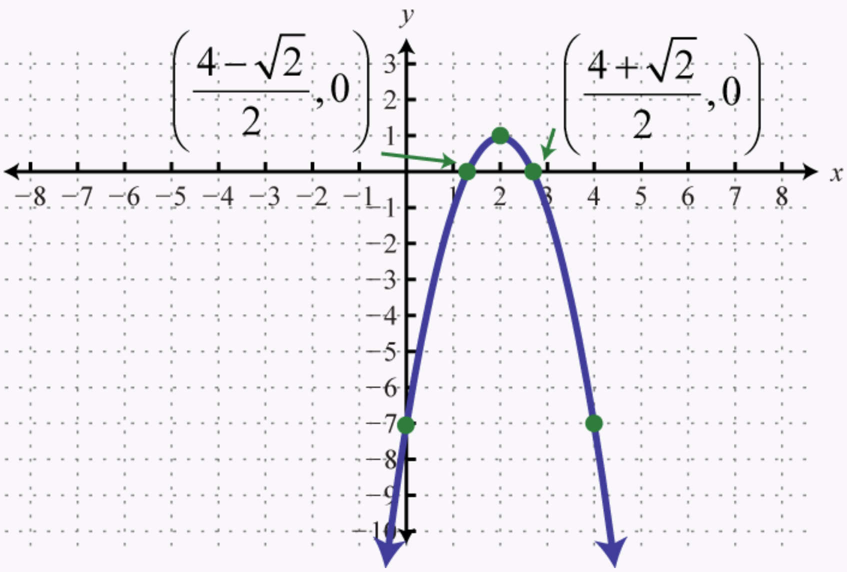

Figure 6.E.4

Exercise 6.E.12

Determine the maximum or minimum y-value.

1. $y=x^2-10x+1$
2. $y=-x^2+10x-1$
3. $y=-3x^2+2x-1$
4. $y=2x^2-x+2$

Answer

1. Minimum: $y=-24$

3. Maximum: $y=-\frac{2}{3}$

Exercise 6.E.13

Rewrite in vertex form $y=a(x-h)^2+k$ and determine the vertex.

1. $y=x^2-6x+13$
2. $y=x^2+10x+24$
3. $y=2x^2-4x-1$
4. $y=-x^2-8x-11$

Answer

1. $y=(x-3)^2+4$; vertex: $(3,4)$

3. $y=2(x-1)^2-3$; vertex: $(1,-3)$

Exercise 6.E.14

Graph. Find the vertex and the y-intercept. In addition, find the x-intercepts if they exist.

1. $f(x)=(x-4)^2-2$
2. $f(x)=-(x+6)^2+4$
3. $f(x)=-x^2+10$
4. $f(x)=(x+10)^2-20$
5. $f(x)=2(x-1)^2-3$
6. $f(x)=-3(x+1)^2-2$
7. The value in dollars of a new car is modeled by $V(t)=125t^2-3,000t+22,000$ where t represents the number of years since it was purchased. Determine the age of the car when its value is at a minimum.
8. The height in feet of a baseball tossed upward at a speed of 48 feet per second from the ground is given by the function, $h(t)=-16t^2+32t$, where t represents the time in seconds after it is tossed. What is the maximum height of the baseball?
9. The rectangular area in square feet that can be enclosed with 200 feet of fencing is given by $A(w)=w(100-w)$ where w represents the width of the rectangular area in feet. What dimensions will maximize the area that can be enclosed?
10. A manufacturing company has found that production costs in thousands of dollars are modeled by $C(x)=0.4x^2-72x+8,050$ where x represents the number of employees. Determine the number of employees that will minimize production costs.

Answer

1.

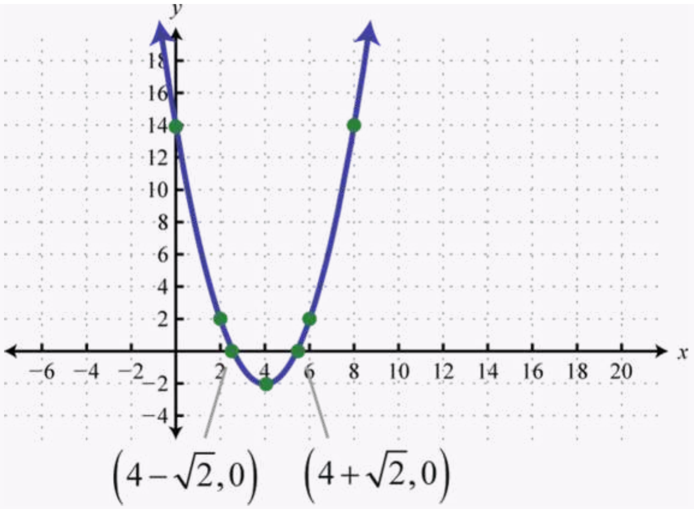

Figure 6.E.5

3.

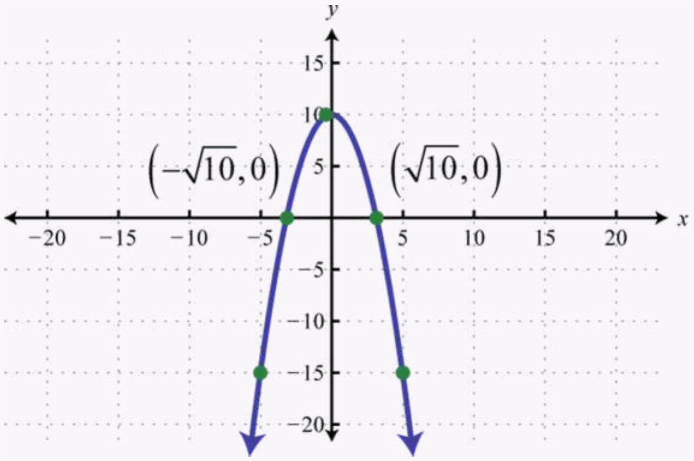

Figure 6.E.6

5.

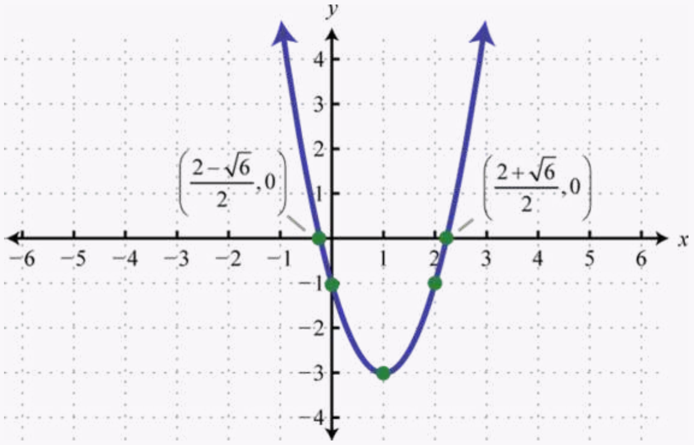

Figure 6.E.7

7. The car will have minimum value 12 years after it is purchased.

9. Length: 50 feet; width: 50 feet

Exercise $6.E.15$

Solve. Present answers using interval notation.

1. $-2(x-1)(x+3)<0$
2. $x^2+2x-35<0$
3. $x^2-6x-16\leq 0$
4. $x^2+14x+40\geq 0$
5. $x^2-10x-24>0$
6. $36-x^2>0$
7. $1-9x^2<0$
8. $8x-12x^2\leq 0$
9. $5x^2+3\leq 0$
10. $x^2-28\geq 0$
11. $9x^2-30x+25\leq 0$
12. $x^2-8x+18>0$
13. $x^2-2x-4<0$
14. $-x^2+3x+18>0$

Answer

1. $(-\infty,-3)\cup(1,\infty)$
3. $[-2,8]$
5. $(-\infty,-2)\cup(12,\infty)$
7. $\left(-\infty,-\frac{1}{3}\right)\cup\left(\frac{1}{3},\infty\right)$
9. $\varnothing$
11. $\frac{5}{3}$
13. $(1-\sqrt{5},1+\sqrt{5})$

Exercise $6.E.16$

Find the domain of the function.

1. $f(x)=\sqrt{x^2-100}$
2. $f(x)=\sqrt{3x-6x^2}$
3. $g(x)=\sqrt{3x^2+9}$
4. $g(x)=\sqrt{8+2x-x^2}$

Answer

1. $(-\infty,-10]\cup[10,\infty)$
3. $(-\infty,\infty)$

Exercise $6.E.17$

Solve. Present answers using interval notation.

1. $x(x-5)(x+2)>0$
2. $(x+4)^2(x-3)<0$
3. $x^2(x+3)\geq 0$
4. $x(x-1)^2\leq 0$
5. $x^3+4x^2-9x-36>0$
6. $2x(4x-1)\geq 3$
7. $4x^3-12x^2+9x<0$
8. $x^3-9x^2+20x\geq 0$

9. $x^3 - 2x^2 - x + 2 < 0$
10. $6x(x+1) + 5x \leq 35$
11. $\frac{(x-2)(2x+1)}{x(x-1)} \leq 0$
12. $\frac{x(x-3)^2}{x-4} \leq 0$
13. $\frac{x^2+4x+4}{4x^2-1} < 0$
14. $\frac{x^2-10x+24}{x^2+10x+25} > 0$
15. $\frac{1}{x-2} + \frac{3}{x} \geq 0$
16. $\frac{2}{x-1} - \frac{1}{x+1} \leq 0$
17. $\frac{3(x+1)}{x^2+2x-3} \leq \frac{2}{x-1}$
18. $\frac{x-4}{x+5} \geq \frac{x-2}{x-5}$

Answer

1. $(-2,0) \cup (5,\infty)$

3. $[-3,\infty)$

5. $(-4,-3) \cup (3,\infty)$

7. $(-\infty,0)$

9. $(-\infty,-1) \cup (1,2)$

11. $\left[-\frac{1}{2},0\right) \cup (1,2]$

13. $\left(-\frac{1}{2},\frac{1}{2}\right)$

15. $\left(0,\frac{3}{2}\right] \cup (2,\infty)$

17. $(-\infty,-3) \cup (1,3]$

Sample Exam

Exercise $6.E.18$

1. Solve by extracting the roots: $2x^2 - 5 = 0$.
2. Solve by completing the square: $x^2 - 16x + 1 = 0$.

Answer

1. $\pm\frac{\sqrt{10}}{2}$

Exercise $6.E.19$

Solve using the quadratic formula.

1. $x^2 + x + 1 = 0$
2. $2x^2 - x - 4 = 0$
3. $-4x^2 + 2x - 1 = 0$
4. $(x-4)(x-2) = 6$

Answer

1. $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

3. $\frac{1}{4} \pm \frac{\sqrt{3}}{4}i$

Exercise $6.E.20$

1. Find a quadratic equation with integer coefficients and solutions $\{\pm\sqrt{5}\}$.
2. The area of a rectangle is 22 square centimeters. If the length is 5 centimeters less than twice the width, then find the dimensions of the rectangle. Round off to the nearest tenth of a centimeter.
3. Assuming dry road conditions and average reaction times, the safe stopping distance in feet of a certain car is given by $d(x) = \frac{1}{20}x^2 + x$ where x represents the speed of the car in miles per hour. Determine the safe speed of the car if you

expect to stop in 100 feet. Round off to the nearest mile per hour.

Answer

1. $x^2 - 5 = 0$

9. 36 miles per hour

Exercise 6.E.21

Find all solutions.

1. $x^4 + x^2 - 12 = 0$
2. $3x^{-2} - 5x^{-1} - 2 = 0$
3. $2x^{2/3} + 3x^{1/3} - 2 = 0$
4. $x - 3\sqrt{x} - 4 = 0$
5. $\left(\frac{t}{t+1}\right)^2 + 4\left(\frac{t}{t+1}\right) - 12 = 0$

Answer

2. $-3, \frac{1}{2}$

4. 16

Exercise 6.E.22

Graph. Find the vertex and the y-intercept. In addition, find the x-intercepts if they exist.

1. $f(x) = x^2 + 4x - 12$
2. $f(x) = -x^2 + 2x + 3$

Answer

1.

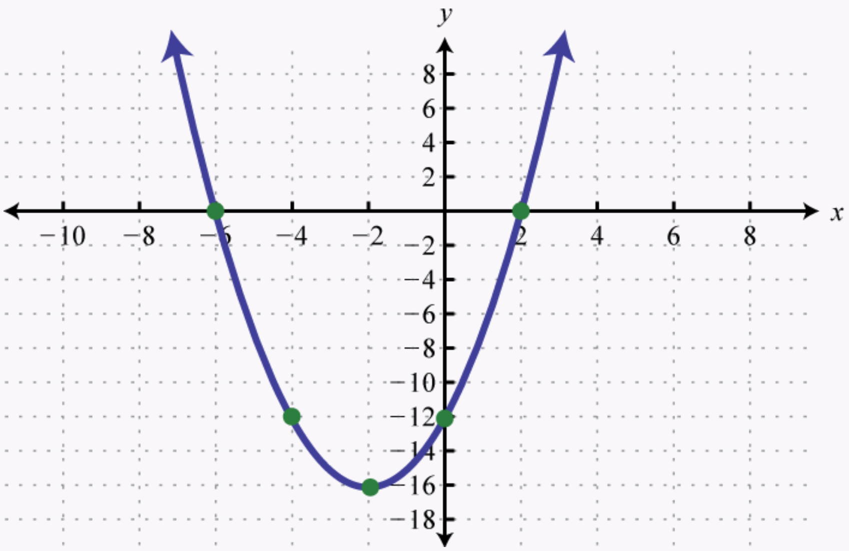

Figure 6.E.8

Exercise 6.E.23

1. Given the function defined by $y = 3x^2 - 6x - 5$:
    1. Does the function have a minimum or maximum? Explain.
    2. Find the minimum or maximum y-value.
2. The height in feet of a water rocket launched from the ground is given by the function $h(t) = -16t^2 + 96t$ where t represents the number of seconds after launch. What is the maximum height attained by the rocket?

Answer

1. (1) Minimum (2) $y=-8$

Exercise $6.E.24$

Sketch the graph and use it to solve the given inequality.

1. Graph $f(x)=(x+1)^2-4$ and find x where $f(x)\geq 0$.
2. Graph $f(x)=-x^2+4$ and find x where $f(x)\geq 0$.

Answer

1. $(-\infty,-3]\cup[1,\infty)$

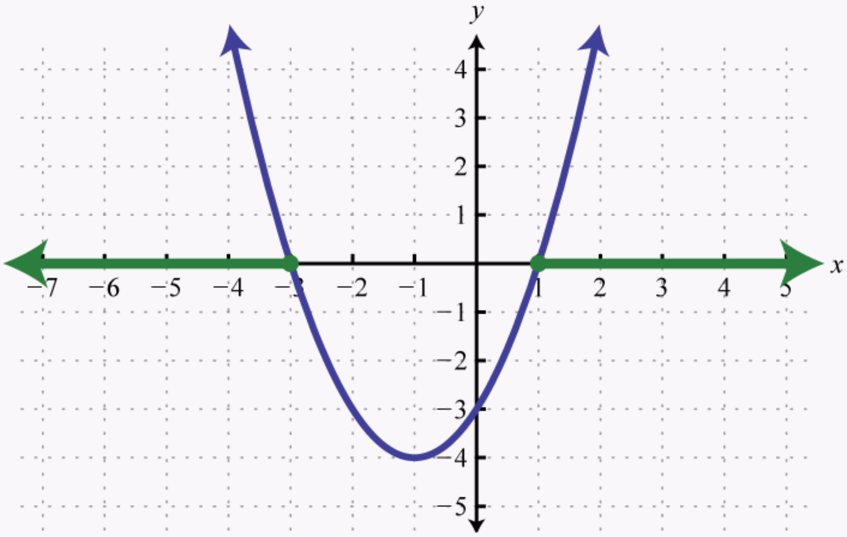

Figure 6.E.9

Exercise $6.E.25$

Solve. Present answers using interval notation.

1. $x^2-2x-15<0$
2. $x(2x-1)>10$
3. $x(x+3)(x-2)^2\leq 0$
4. $\frac{x^2-10x+25}{x+1}\geq 0$
5. $\frac{x^2-5x+4}{x^2+x}\leq 0$

Answer

1. $(-3,5)$

3. $[-3,0]\cup\{2\}$

5. $(-1,0)\cup[1,4]$

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CHAPTER OVERVIEW

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7: Exponential and Logarithmic Functions

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7.1: Composition and Inverse Functions

Learning Objectives

- Perform function composition.
- Determine whether or not given functions are inverses.
- Use the horizontal line test.
- Find the inverse of a one-to-one function algebraically.

Composition of Functions

In mathematics, it is often the case that the result of one function is evaluated by applying a second function. For example, consider the functions defined by $f(x)=x^2$ and $g(x)=2x+5$. First, g is evaluated where $x=-1$ and then the result is squared using the second function, f.

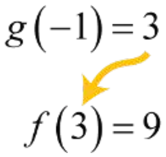


Figure 7.1.1

This sequential calculation results in 9. We can streamline this process by creating a new function defined by $f(g(x))$, which is explicitly obtained by substituting $g(x)$ into $f(x)$.

$$\begin{aligned} f(g(x)) &= f(2x+5) \\ &= (2x+5)^2 \\ &= 4x^2+20x+25 \end{aligned}$$

Therefore, $f(g(x))=4x^2+20x+25$ and we can verify that when $x=-1$ the result is 9.

$$\begin{aligned} f(g(-1)) &= 4(-1)^2+20(-1)+25 \\ &= 4-20+25 \\ &= 9 \end{aligned}$$

The calculation above describes **composition of functions**[1], which is indicated using the **composition operator** [2](○). If given functions f and g,

$$(f\circ g)(x)=f(g(x)) \quad \textit{Composition of Functions}$$

The notation $f○g$ is read, "f composed with g." This operation is only defined for values, \(x\), in the domain of g such that $g(x)$ is in the domain of f.

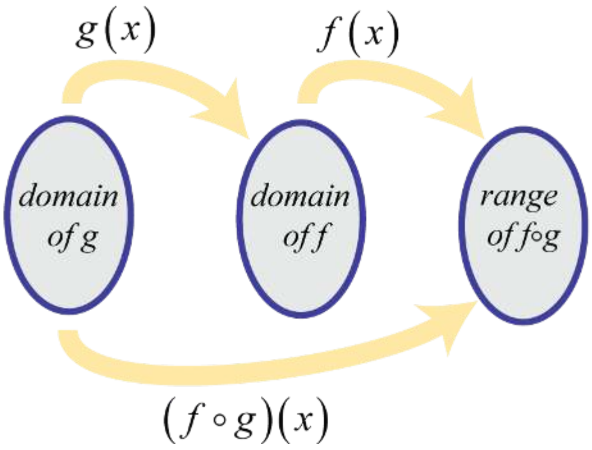


Figure 7.1.2

Example 7.1.1:

Given $f(x)=x^2-x+3$ and $g(x)=2x-1$ calculate:

1. $(f \bigcirc g)(x)$.
2. $(g \bigcirc f)(x)$.

Solution

1. Substitute g into f.

$$\begin{aligned}(f\circ g)(x) &= f(g(x))\\ &= f(2x-1)\\ &= (2x-1)^2-(2x-1)+3\\ &= 4x^2-4x+1-2x+1+3\\ &= 4x^2-6x+5\end{aligned}$$

2. Substitute f into g.

$$\begin{aligned}(g\circ f)(x) &= g(f(x))\\ &= g(x^2-x+3)\\ &= 2(x^2-x+3)-1\\ &= 2x^2-2x+6-1\\ &= 2x^2-2x+5\end{aligned}$$

Answer:

1. $(f \bigcirc g)(x)=4x^2-6x+5$
2. $(g \bigcirc f)(x)=2x^2-2x+5$

The previous example shows that composition of functions is not necessarily commutative.

Example 7.1.2

Given $f(x)=x^3+1$ and $g(x)=\sqrt[3]{3x-1}$ find $(f \bigcirc g)(4)$.

Solution

Begin by finding $(f \bigcirc g)(x)$.

$$\begin{aligned}(f\circ g)(x) &= f(g(x))\\ &= f(\sqrt[3]{3x-1})\\ &= (\sqrt[3]{3x-1})^3+1\\ &= 3x-1+1\\ &= 3x\end{aligned}$$

Next, substitute 4 in for x.

$$\begin{aligned}(f\circ g)(x) &= 3x\\ (f\circ g)(4) &= 3(4)\\ &= 12\end{aligned}$$

Answer:

$(f \bigcirc g)(4)=12$

Functions can be composed with themselves.

Example 7.1.3

Given $f(x)=x^2-2$ find $(f \bigcirc f)(x)$.

Solution

$$\begin{aligned}(f\circ f)(x)&=f(f(x))\\&=f\left(x^2-2\right)\\&=\left(x^2-2\right)^2-2\\&=x^4-4x^2+4-2\\&=x^4-4x^2+2\end{aligned}$$

Answer:

$(f\circ f)(x)=x^4-4x^2+2$

Exercise 7.1.1

Given $f(x)=2x+3$ and $g(x)=\sqrt{x-1}$ find $(f\bigcirc g)(5)$.

Answer

7

www.youtube.com/v/cwEGsJa5Tlc

Inverse Functions

Consider the function that converts degrees Fahrenheit to degrees Celsius: $C(x)=\frac{5}{9}(x-32)$. We can use this function to convert 77°F to degrees Celsius as follows.

$$\begin{aligned}C(77)&=\frac{5}{9}(77-32)\\&=\frac{5}{9}(45)\\&=25\end{aligned}$$

Therefore, 77°F is equivalent to 25°C. If we wish to convert 25°C back to degrees Fahrenheit we would use the formula: $F(x)=\frac{9}{5}x+32$.

$$\begin{aligned}F(25)&=\frac{9}{5}(25)+32\\&=45+32\\&=77\end{aligned}$$

Notice that the two functions C and F each reverse the effect of the other.

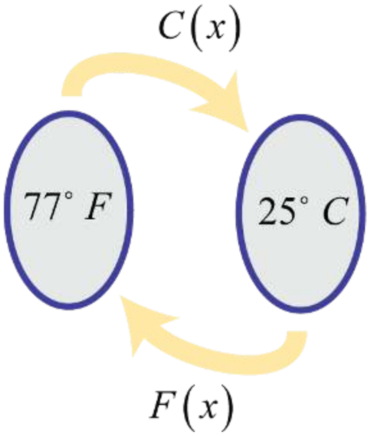


Figure 7.1.3

This describes an inverse relationship. In general, f and g are **inverse functions** if,

$$\begin{aligned}(f\circ g)(x)&=f(g(x))=x\quad \textit{for all } x \textit{ in the domain of } g \textit{ and}\\(gOf)(x)&=g(f(x))=x\quad \textit{for all } x \textit{ in the domain of } f.\end{aligned}$$

In this example,

$$\begin{aligned}C(F(25))&=C(77)=25\\F(C(77))&=F(25)=77\end{aligned}$$

Example 7.1.4

Verify algebraically that the functions defined by $f(x)=\frac{1}{2}x-5$ and $g(x)=2x+10$ are inverses.

Solution

Compose the functions both ways and verify that the result is x.

Table 7.1.1

$\begin{aligned}(f\circ g)(x) &= f(g(x))\\ &= f(2x+10)\\ &= \frac{1}{2}(2x+10)-5\\ &= x+5-5\\ &= x\ \checkmark\end{aligned}$	$\begin{aligned}(g\ \mathrm{Of}\ )(x) &= g(f(x))\\ &= g\left(\frac{1}{2}x-5\right)\\ &= 2\left(\frac{1}{2}x-5\right)+10\\ &= x-10+10\\ &= x\ \checkmark\end{aligned}$

Answer:

Both $(f\circ g)(x)=(g\circ f)(x)=x$; therefore, they are inverses.

Next we explore the geometry associated with inverse functions. The graphs of both functions in the previous example are provided on the same set of axes below.

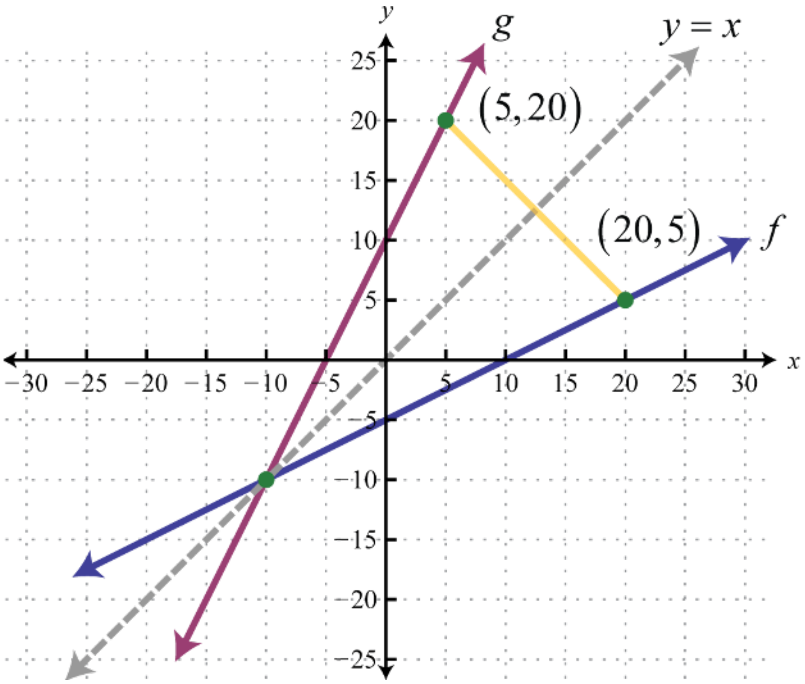


Figure 7.1.4

Note that there is symmetry about the line $y=x$; the graphs of f and g are mirror images about this line. Also notice that the point $(20,5)$ is on the graph of f and that $(5,20)$ is on the graph of g. Both of these observations are true in general and we have the following properties of inverse functions:

1. The graphs of inverse functions are symmetric about the line $y=x$.
2. If (a,b) is on the graph of a function, then (b,a) is on the graph of its inverse.

Furthermore, if g is the inverse of f we use the notation $g=f^{-1}$. Here f^{-1} is read, “f inverse,” and should not be confused with negative exponents. In other words, $f^{-1}(x)\neq\frac{1}{f(x)}$ and we have,

$$\begin{aligned}\left(f\circ f^{-1}\right)(x) &= f\left(f^{-1}(x)\right)=x \text{ and}\\ \left(f^{-1}\circ f\right)(x) &= f^{-1}(f(x))=x\end{aligned}$$

Example 7.1.5:

Verify algebraically that the functions defined by $f(x)=\frac{1}{x}-2$ and $f^{-1}(x)=\frac{1}{x+2}$ are inverses.

Solution

Compose the functions both ways to verify that the result is x.

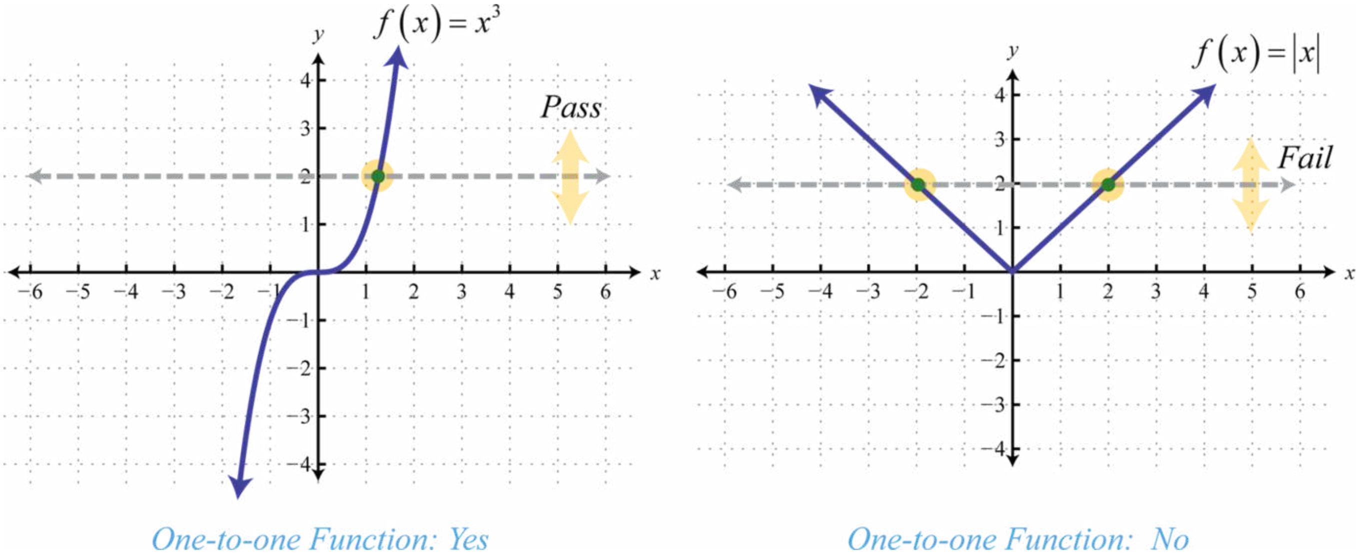
Table 7.1.2

$\begin{aligned}\left(f\circ f^{-1}\right)(x)&=f\left(f^{-1}(x)\right)\\&=f\left(\frac{1}{x+2}\right)\\&=\frac{1}{\left(\frac{1}{x+2}\right)}-2\\&=\frac{x+2}{1}-2\\&=x+2-2\\&=x\ \checkmark\end{aligned}$	$\begin{aligned}\left(f^{-1}\circ f\right)(x)&=f^{-1}(f(x))\\&=f^{-1}\left(\frac{1}{x}-2\right)\\&=\frac{1}{\left(\frac{1}{x}-2\right)+2}\\&=\frac{1}{\frac{1}{x}}\\&=x\ \checkmark\end{aligned}$

Answer:

Since $\left(f\circ f^{-1}\right)(x)=\left(f^{-1}\circ f\right)(x)=x$ they are inverses.

Recall that a function is a relation where each element in the domain corresponds to exactly one element in the range. We use the vertical line test to determine if a graph represents a function or not. Functions can be further classified using an inverse relationship. **One-to-one functions**[3] are functions where each value in the range corresponds to exactly one element in the domain. The **horizontal line test**[4] is used to determine whether or not a graph represents a one-to-one function. If a horizontal line intersects a graph more than once, then it does not represent a one-to-one function.



One-to-one Function: Yes

One-to-one Function: No

Figure 7.1.5

The horizontal line represents a value in the range and the number of intersections with the graph represents the number of values it corresponds to in the domain. The function defined by $f(x)=x^3$ is one-to-one and the function defined by $f(x)=|x|$ is not. Determining whether or not a function is one-to-one is important because a function has an inverse if and only if it is one-to-one. In other words, a function has an inverse if it passes the horizontal line test.

Note

In this text, when we say *"a function has an inverse,"* we mean that there is another function, f^{-1}, such that $(f\bigcirc f^{-1})(x)=(f^{-1}\bigcirc f)(x)=x$.

Example 7.1.6:

Determine whether or not the given function is one-to-one.

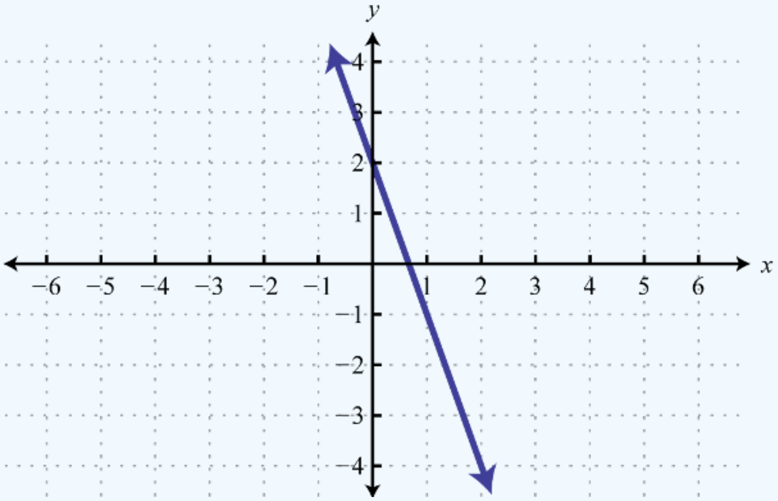


Figure 7.1.6

Solution

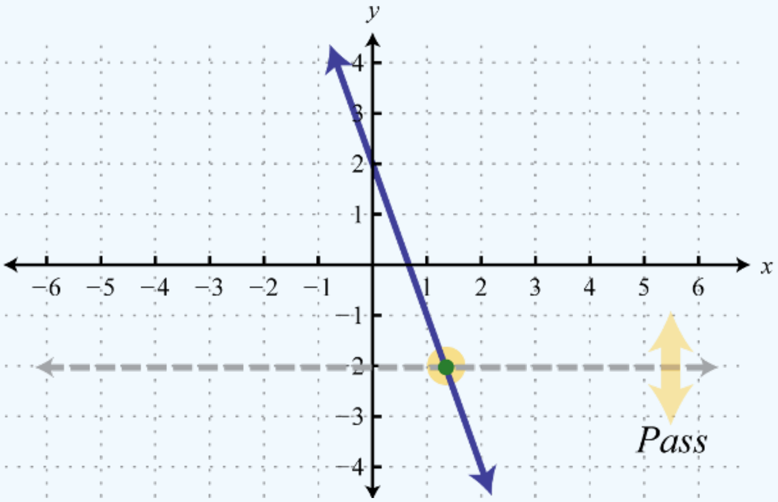


Figure 7.1.7

Answer:

The given function passes the horizontal line test and thus is one-to-one.

In fact, any linear function of the form $f(x)=mx+b$ where $m\neq 0$, is one-to-one and thus has an inverse. The steps for finding the inverse of a one-to-one function are outlined in the following example.

Example 7.1.7

Find the inverse of the function defined by $f(x)=\frac{3}{2}x-5$.

Solution

Before beginning this process, you should verify that the function is one-to-one. In this case, we have a linear function where $m\neq 0$ and thus it is one-to-one.

Step 1: Replace the function notation $f(x)$ with y.

$$\begin{aligned} f(x)&=\frac{3}{2}x-5 \\ y&=\frac{3}{2}x-5 \end{aligned}$$

Step 2: Interchange x and y. We use the fact that if (x, y) is a point on the graph of a function, then (y, x) is a point on the graph of its inverse.

$$x = \frac{3}{2}y - 5$$

Step 3: Solve for y.

$$\begin{aligned} x &= \frac{3}{2}y - 5 \\ x + 5 &= \frac{3}{2}y \\ \\ \frac{2}{3} \cdot (x + 5) &= \frac{2}{3} \cdot \frac{3}{2}y \\ \frac{2}{3}x + \frac{10}{3} &= y \end{aligned}$$

Step 4: The resulting function is the inverse of f. Replace y with $f^{-1}(x)$.

$$f^{-1}(x) = \frac{2}{3}x + \frac{10}{3}$$

Step 5: Check.

Table 7.1.3

$\left(f \circ f^{-1}\right)(x)$ $= f\left(f^{-1}(x)\right)$ $= f\left(\frac{2}{3}x + \frac{10}{3}\right)$ $= \frac{3}{2}\left(\frac{2}{3}x + \frac{10}{3}\right) - 5$ $= x + 5 - 5$ $= x\ \checkmark$	$\left(f^{-1} \circ f\right)(x)$ $= f^{-1}(f(x))$ $= f^{-1}\left(\frac{3}{2}x - 5\right)$ $= \frac{2}{3}\left(\frac{3}{2}x - 5\right) + \frac{10}{3}$ $= x - \frac{10}{3} + \frac{10}{3}$ $= x\ \checkmark$

Answer:

$f^{-1}(x) = \frac{2}{3}x + \frac{10}{3}$

If a function is not one-to-one, it is often the case that we can restrict the domain in such a way that the resulting graph is one-to-one. For example, consider the squaring function shifted up one unit, $g(x) = x^2 + 1$. Note that it does not pass the horizontal line test and thus is not one-to-one. However, if we restrict the domain to nonnegative values, $x \geq 0$, then the graph does pass the horizontal line test.

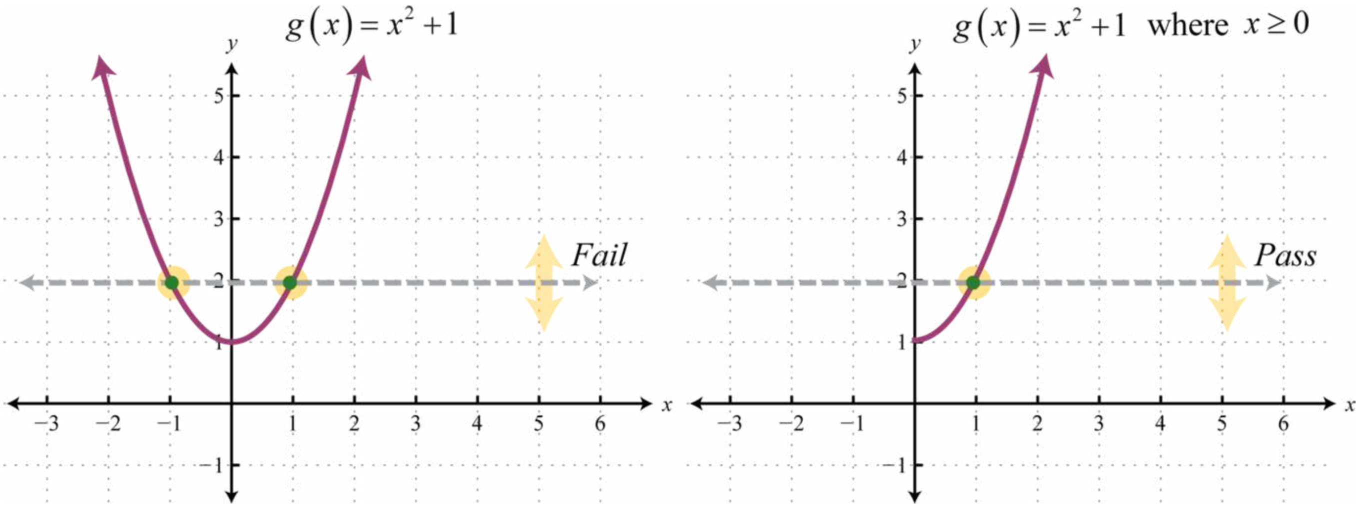


Figure 7.1.8

On the restricted domain, g is one-to-one and we can find its inverse.

Example 7.1.8:

Find the inverse of the function defined by $g(x) = x^2 + 1$ where $x \geq 0$.

Solution

Begin by replacing the function notation $g(x)$ with y.

$$\begin{aligned} g(x) &= x^2 + 1 \\ y &= x^2 + 1 \text{ where } x \geq 0 \end{aligned}$$

Interchange x and y.

$$x = y^2 + 1 \text{ where } y \geq 0$$

Solve for y.

$$\begin{aligned} x &= y^2 + 1 \\ x - 1 &= y^2 \\ \pm\sqrt{x-1} &= y \end{aligned}$$

Since $y \geq 0$ we only consider the positive result.

$$\begin{aligned} y &= \sqrt{x-1} \\ g^{-1}(x) &= \sqrt{x-1} \end{aligned}$$

Answer:

$g^{-1}(x) = \sqrt{x-1}$. The check is left to the reader.

The graphs in the previous example are shown on the same set of axes below. Take note of the symmetry about the line $y = x$.

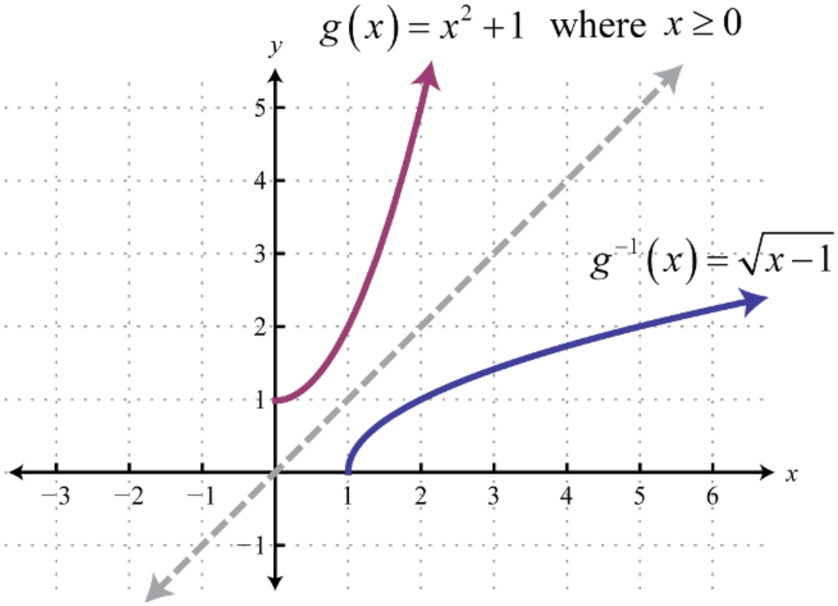


Figure 7.1.9

Example 7.1.9:

Find the inverse of the function defined by $f(x) = \frac{2x+1}{x-3}$.

Solution

Use a graphing utility to verify that this function is one-to-one. Begin by replacing the function notation $f(x)$ with y.

$$\begin{aligned} f(x) &= \frac{2x+1}{x-3} \\ y &= \frac{2x+1}{x-3} \end{aligned}$$

Interchange x and y.

$$x=\frac{2y+1}{y-3}$$

Solve for y.

$$\begin{aligned} x &= \frac{2y+1}{y-3} \\ x(y-3) &= 2y+1 \\ xy-3x &= 2y+1 \end{aligned}$$

Obtain all terms with the variable y on one side of the equation and everything else on the other. This will enable us to treat y as a GCF.

$$\begin{aligned} xy-3x &= 2y+1 \\ xy-2y &= 3x+1 \\ y(x-2) &= 3x+1 \\ y &= \frac{3x+1}{x-2} \end{aligned}$$

Answer:

$f^{-1}(x)=\frac{3x+1}{x-2}$. The check is left to the reader.

Exercise 7.1.2

Find the inverse of $f(x)=\sqrt[3]{x+1}-3$.

Answer

$f^{-1}(x)=(x+3)^3-1$

www.youtube.com/v/MwSB9WuYfCA

Key Takeaways

- The composition operator $(\bigcirc)$ indicates that we should substitute one function into another. In other words, $(f \bigcirc g)(x)=f(g(x))$ indicates that we substitute $g(x)$ into $f(x)$.
- If two functions are inverses, then each will reverse the effect of the other. Using notation, $(f \bigcirc g)(x)=f(g(x))=x$ and $(g \bigcirc f)(x)=g(f(x))=x$.
- Inverse functions have special notation. If g is the inverse of f, then we can write $g(x)=f^{-1}(x)$. This notation is often confused with negative exponents and does not equal one divided by $f(x)$.
- The graphs of inverses are symmetric about the line $y=x$. If (a,b) is a point on the graph of a function, then (b,a) is a point on the graph of its inverse.
- If each point in the range of a function corresponds to exactly one value in the domain then the function is one-to-one. Use the horizontal line test to determine whether or not a function is one-to-one.
- A one-to-one function has an inverse, which can often be found by interchanging x and y, and solving for y. This new function is the inverse of the original function.

Exercise 7.1.3

Given the functions defined by f and g find $(f \circ g)(x)$ and $(g \circ f)(x)$.

1. $f(x)=4x-1, g(x)=3x$
2. $f(x)=-2x+5, g(x)=2x$
3. $f(x)=3x-5, g(x)=x-4$
4. $f(x)=5x+1, g(x)=2x-3$
5. $f(x)=x^2-x+1, g(x)=2x-1$
6. $f(x)=x^2-3x-2, g(x)=x-2$
7. $f(x)=x^2+3, g(x)=x^2-5$
8. $f(x)=2x^2, g(x)=x^2-x$
9. $f(x)=8x^3+5, g(x)=\sqrt[3]{x-5}$
10. $f(x)=27x^3-1, g(x)=\sqrt[3]{x+1}$
11. $f(x)=\frac{1}{x+5}, g(x)=\frac{1}{x}$

12. $f(x)=\frac{1}{x}-3, g(x)=\frac{3}{x+3}$
13. $f(x)=5\sqrt{x}, g(x)=3x-2$
14. $f(x)=\sqrt{2x}, g(x)=4x+1$
15. $f(x)=\frac{1}{2x}, g(x)=x^2+8$
16. $f(x)=2x-1, g(x)=\frac{1}{x+1}$
17. $f(x)=\frac{1-x}{2x}, g(x)=\frac{1}{2x+1}$
18. $f(x)=\frac{2x}{x+1}, g(x)=\frac{x+1}{x}$

Answer

1. $(f\circ g)(x)=12x-1; (g\circ f)(x)=12x-3$

3. $(f\circ g)(x)=3x-17; (g\circ f)(x)=3x-9$

5. $(f\circ g)(x)=4x^2-6x+3; (g\circ f)(x)=2x^2-2x+1$

7. $(f\circ g)(x)=x^4-10x^2+28; (g\circ f)(x)=x^4+6x^2+4$

9. $(f\circ g)(x)=8x-35; (g\circ f)(x)=2x$

11. $(f\circ g)(x)=\frac{x}{5x+1}; (g\circ f)(x)=x+5$

13. $(f\circ g)(x)=5\sqrt{3x-2}; (g\circ f)(x)=15\sqrt{x}-2$

15. $(f\circ g)(x)=\frac{1}{2x^2+16}; (g\circ f)(x)=\frac{1+32x^2}{4x^2}$

17. $(f\circ g)(x)=x; (g\circ f)(x)=x$

Exercise 7.1.4

Given the functions defined by $f(x)=3x^2-2, g(x)=5x+1$, and $h(x)=\sqrt{x}$, calculate the following.

1. $(f\circ g)(2)$
2. $(g\circ f)(-1)$
3. $(g\circ f)(0)$
4. $(f\circ g)(0)$
5. $(f\circ h)(3)$
6. $(g\circ h)(16)$
7. $(h\circ g)\left(\frac{3}{5}\right)$
8. $(h\circ f)(-3)$

Answer

1. 361

3. -9

5. 7

7. 2

Exercise 7.1.5

Given the functions defined by $f(x)=\sqrt[3]{x+3}, g(x)=8x^3-3$, and $h(x)=2x-1$, calculate the following.

1. $(f\circ g)(1)$
2. $(g\circ f)(-2)$
3. $(g\circ f)(0)$
4. $(f\circ g)(-2)$
5. $(f\circ h)(-1)$
6. $(h\circ g)\left(-\frac{1}{2}\right)$
7. $(h\circ f)(24)$

8. $(g \circ h)(0)$

Answer

1. 2
3. 21
5. 0
7. 5

Exercise 7.1.6

Given the function, determine $(f \circ f)(x)$.

1. $f(x) = 3x - 1$
2. $f(x) = \frac{2}{5}x + 1$
3. $f(x) = x^2 + 5$
4. $f(x) = x^2 - x + 6$
5. $f(x) = x^3 + 2$
6. $f(x) = x^3 - x$
7. $f(x) = \frac{1}{x+1}$
8. $f(x) = \frac{x+1}{2x}$

Answer

1. $(f \circ f)(x) = 9x - 4$
3. $(f \circ f)(x) = x^4 + 10x^2 + 30$
5. $(f \circ f)(x) = x^9 + 6x^6 + 12x^3 + 10$
7. $(f \circ f)(x) = \frac{x+1}{x+2}$

Exercise 7.1.7

Are the given functions one-to-one? Explain.

1.

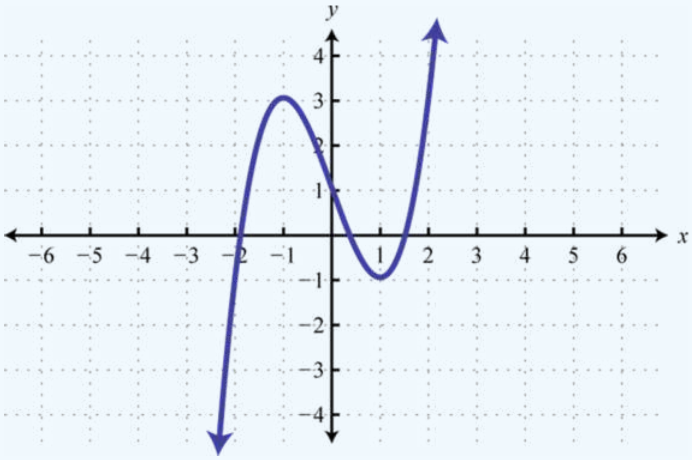


Figure 7.1.10

2.

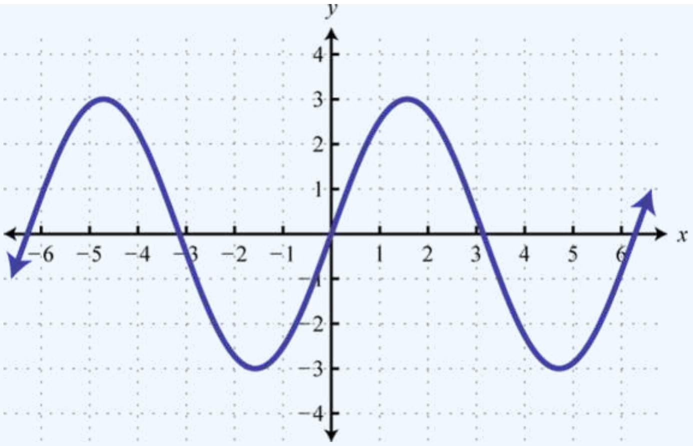


Figure 7.1.11

3.

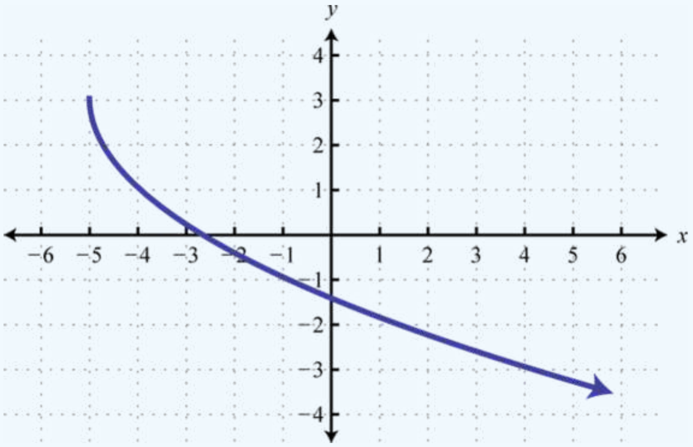


Figure 7.1.12

4.

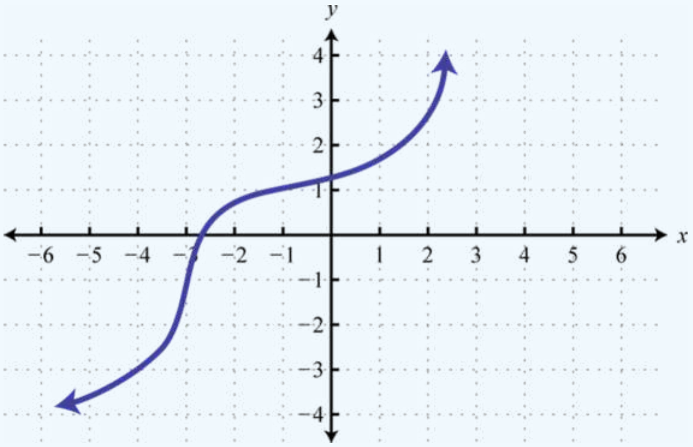


Figure 7.1.13

5. $f(x)=x+1$

6. $g(x)=x^2+1$

7. $h(x)=|x|+1$

8. $r(x)=x^3+1$

9. $f(x)=\sqrt{x+1}$

10. $g(x)=3$

Answer

1. No, fails the HLT
3. Yes, passes the HLT
5. Yes, its graph passes the HLT.
7. No, its graph fails the HLT.
9. Yes, its graph passes the HLT.

Exercise 7.1.8

Given the graph of a one-to-one function, graph its inverse.

1.

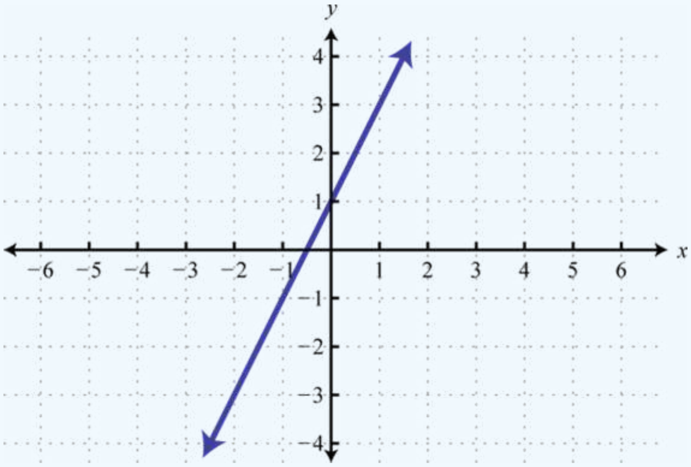


Figure 7.1.14

2.

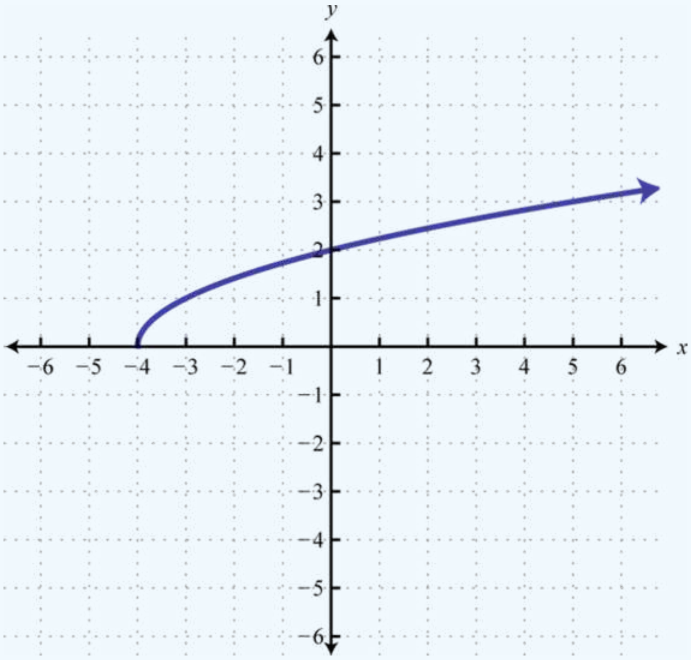


Figure 7.1.15

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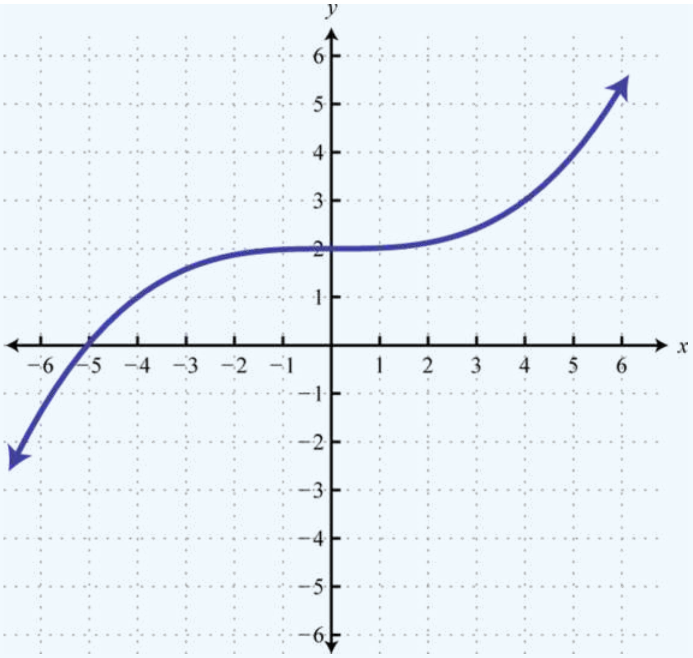

Figure 7.1.16

4.

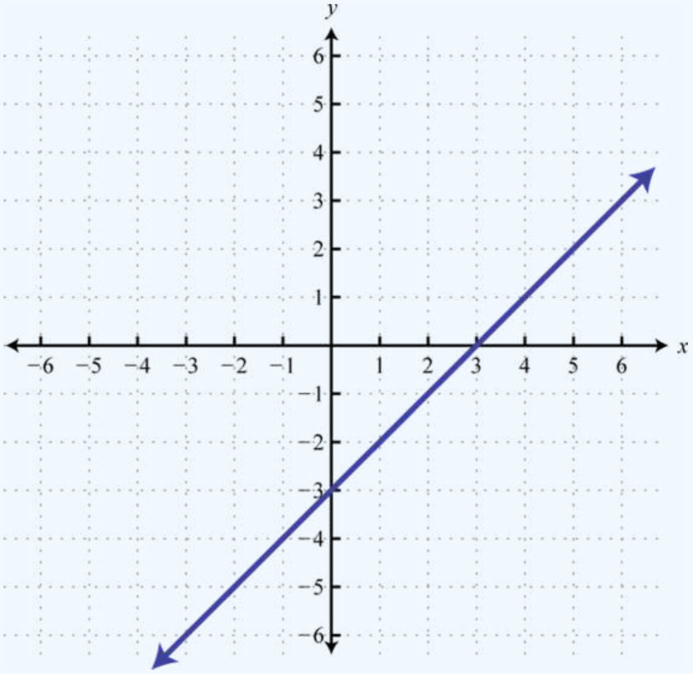

Figure 7.1.17

5.

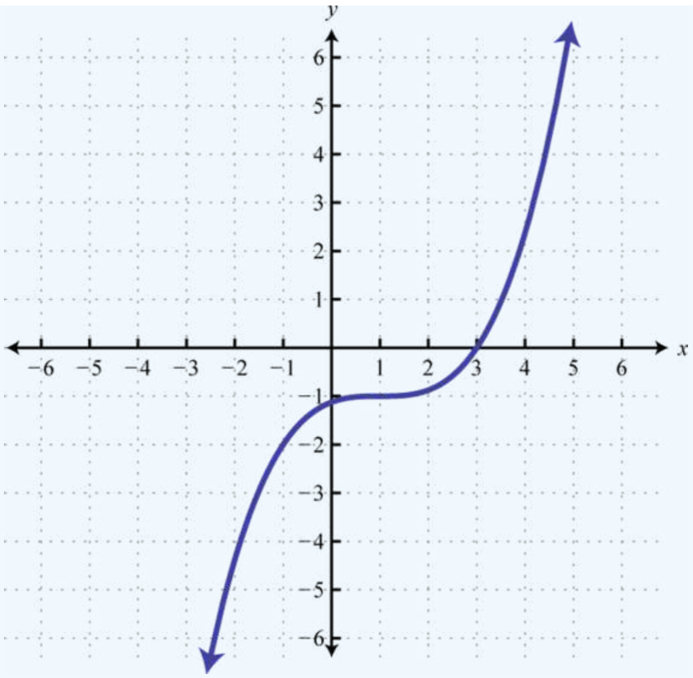

Figure 7.1.18

6.

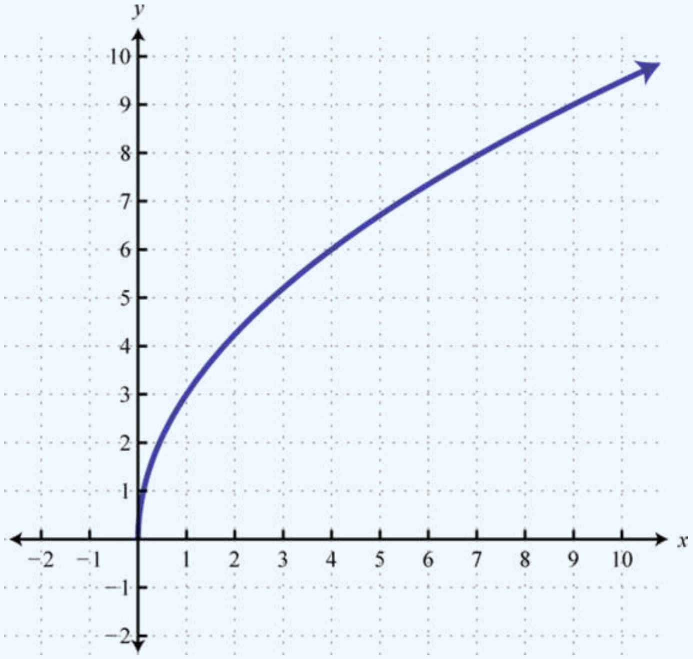

Figure 7.1.19

Answer

1.

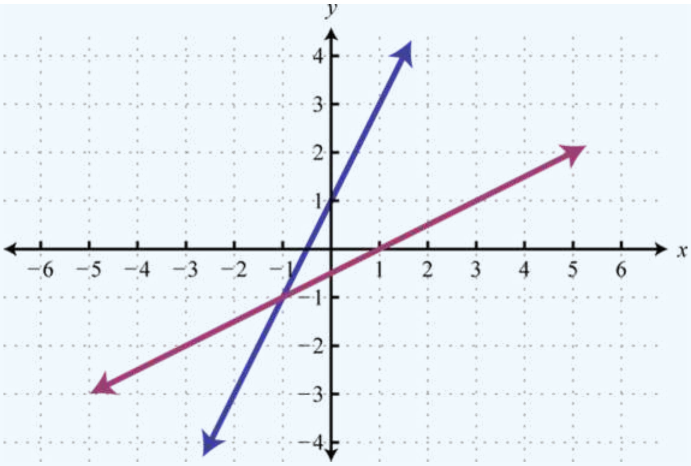

Figure 7.1.20

3.

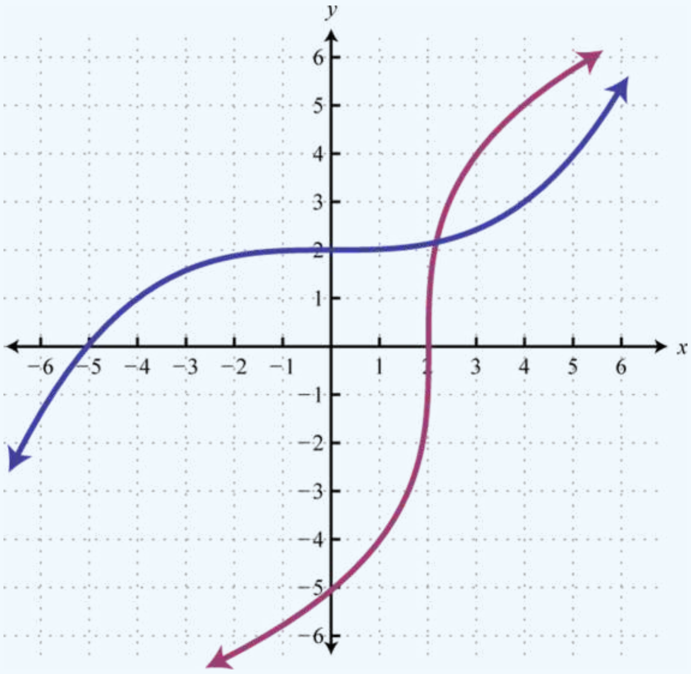

Figure 7.1.21

5.

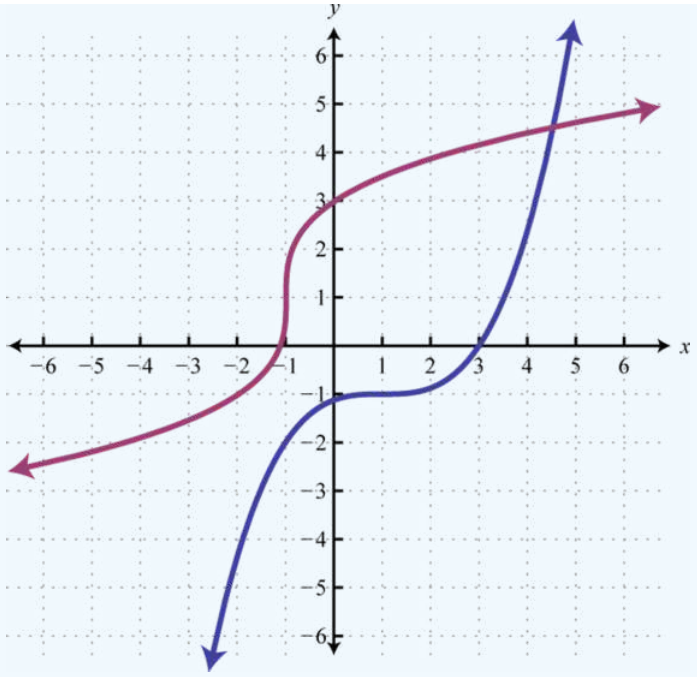

Figure 7.1.22

Exercise 7.1.9

Verify algebraically that the two given functions are inverses. In other words, show that $\left(f \circ f^{-1}\right)(x)=x$ and $\left(f^{-1} \circ f\right)(x)=x$.

1. $f(x)=3x-4, f^{-1}(x)=\frac{x+4}{3}$
2. $f(x)=-5x+1, f^{-1}(x)=\frac{1-x}{5}$
3. $f(x)=-\frac{2}{3}x+1, f^{-1}(x)=-\frac{3}{2}x+\frac{3}{2}$
4. $f(x)=4x-\frac{1}{3}, f^{-1}(x)=\frac{1}{4}x+\frac{1}{12}$
5. $f(x)=\sqrt{x-8}, f^{-1}(x)=x^{2}+8, x \geq 0$
6. $f(x)=\sqrt[3]{6x}-3, f^{-1}(x)=\frac{(x+3)^{3}}{6}$
7. $f(x)=\frac{x}{x+1}, f^{-1}(x)=\frac{x}{1-x}$
8. $f(x)=\frac{x-3}{3x}, f^{-1}(x)=\frac{3}{1-3x}$
9. $f(x)=2(x-1)^{3}+3, f^{-1}(x)=1+\sqrt[3]{\frac{x-3}{2}}$
10. $f(x)=\sqrt[3]{5x-1}+4, f^{-1}(x)=\frac{(x-4)^{3}+1}{5}$

Answer

1. Proof

3. Proof

5. Proof

7. Proof

9. Proof

Exercise 7.1.10

Find the inverses of the following functions.

1. $f(x)=5x$
2. $f(x)=\frac{1}{2}x$
3. $f(x)=2x+5$
4. $f(x)=-4x+3$

5. $f(x)=-\frac{2}{3}x+\frac{1}{3}$
6. $f(x)=-\frac{1}{2}x+\frac{3}{4}$
7. $g(x)=x^2+5, x\geq 0$
8. $g(x)=x^2-7, x\geq 0$
9. $f(x)=(x-5)^2, x\geq 5$
10. $f(x)=(x+1)^2, x\geq -1$
11. $h(x)=3x^3+5$
12. $h(x)=2x^3-1$
13. $f(x)=(2x-3)^3$
14. $f(x)=(x+4)^3-1$
15. $g(x)=\frac{2}{x^3+1}$
16. $g(x)=\frac{1}{x^3}-2$
17. $f(x)=\frac{5}{x+1}$
18. $f(x)=\frac{1}{2x-9}$
19. $f(x)=\frac{x+5}{x-5}$
20. $f(x)=\frac{3x-4}{2x-1}$
21. $h(x)=\frac{x-5}{10x}$
22. $h(x)=\frac{9x+1}{3x}$
23. $g(x)=\sqrt[3]{5x+2}$
24. $g(x)=\sqrt[3]{4x-3}$
25. $f(x)=\sqrt[3]{x-6}-4$
26. $f(x)=2\sqrt[3]{x+2}+5$
27. $h(x)=\sqrt[5]{x+1}-3$
28. $h(x)=\sqrt[5]{x-8}+1$
29. $f(x)=mx+b, m\neq 0$
30. $f(x)=ax^2+c, x\geq 0$
31. $f(x)=ax^3+d$
32. $f(x)=a(x-h)^2+k, x\geq h$

Answer

1. $f^{-1}(x)=\frac{x}{5}$
3. $f^{-1}(x)=\frac{1}{2}x-\frac{5}{2}$
5. $f^{-1}(x)=-\frac{3}{2}x+\frac{1}{2}$
7. $g^{-1}(x)=\sqrt{x-5}$
9. $f^{-1}(x)=\sqrt{x}+5$
11. $h^{-1}(x)=\sqrt[3]{\frac{x-5}{3}}$
13. $f^{-1}(x)=\frac{\sqrt[3]{x}+3}{2}$
15. $g^{-1}(x)=\sqrt[3]{\frac{2-x}{x}}$
17. $f^{-1}(x)=\frac{5-x}{x}$
19. $f^{-1}(x)=\frac{5(x+1)}{x-1}$
21. $h^{-1}(x)=-\frac{5}{10x-1}$
23. $g^{-1}(x)=\frac{x^3-2}{5}$
25. $f^{-1}(x)=(x+4)^3+6$
27. $h^{-1}(x)=(x+3)^5-1$
29. $f^{-1}(x)=\frac{x-b}{m}$
31. $f^{-1}(x)=\sqrt[3]{\frac{x-d}{a}}$

Exercise 7.1.11

Graph the function and its inverse on the same set of axes.

1. $f(x)=x+2$
2. $f(x)=\frac{2}{3}x-4$
3. $f(x)=-2x+2$
4. $f(x)=-\frac{1}{3}x+4$
5. $g(x)=x^2-2, x\geq 0$
6. $g(x)=(x-2)^2, x\geq 2$
7. $h(x)=x^3+1$
8. $h(x)=(x+2)^3-2$
9. $f(x)=2-\sqrt{x}$
10. $f(x)=\sqrt{-x}+1$

Answer

1.

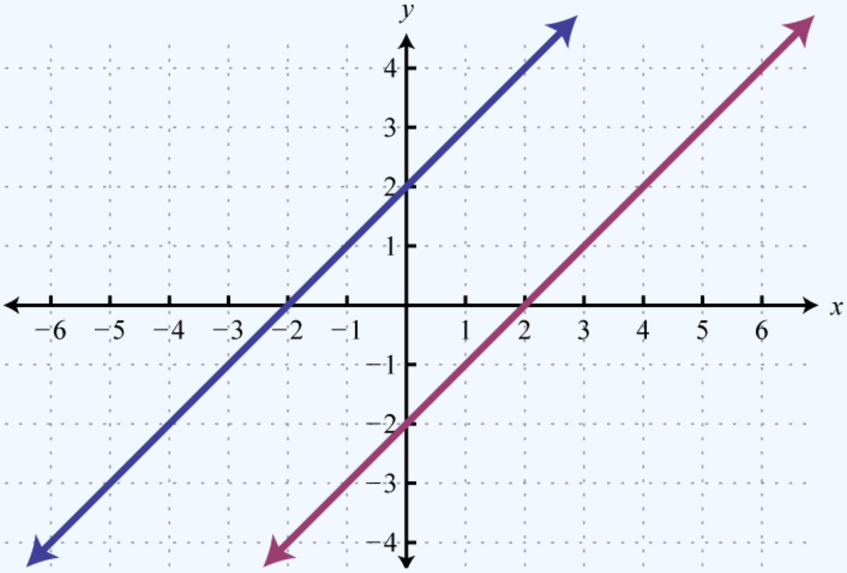

Figure 7.1.23

3.

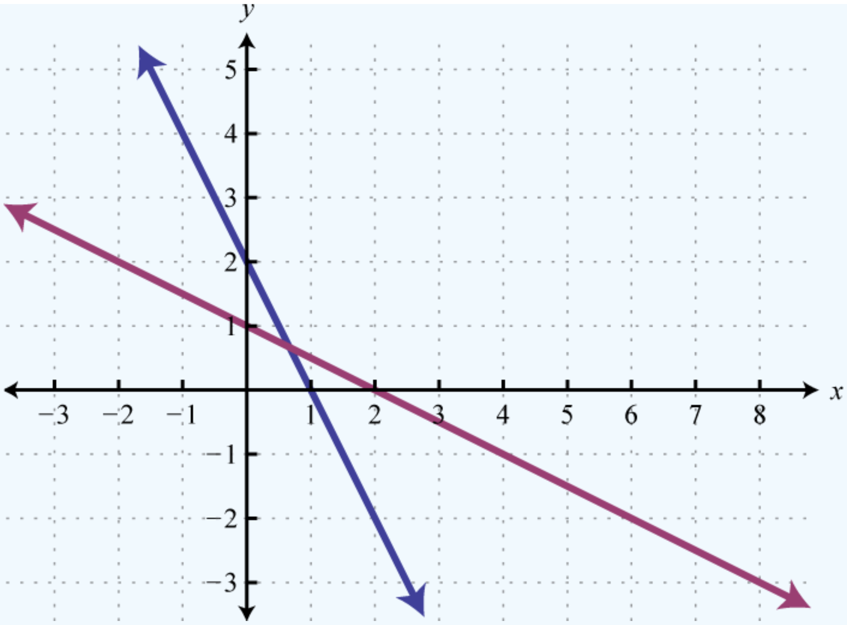

Figure 7.1.24

5.

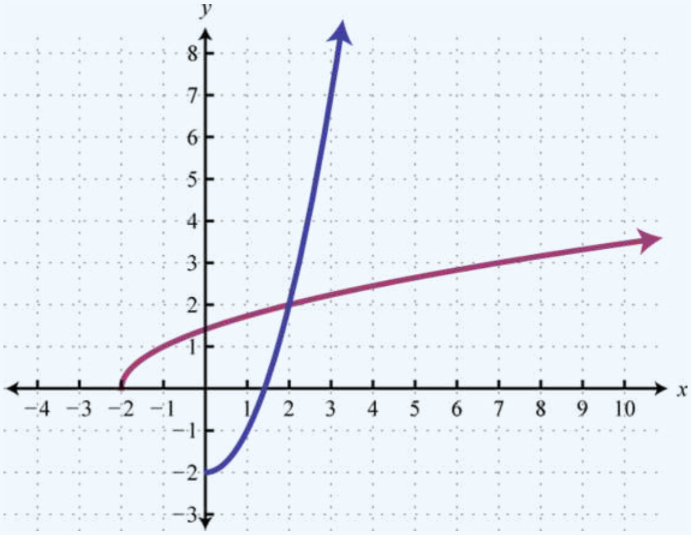

Figure 7.1.25

7.

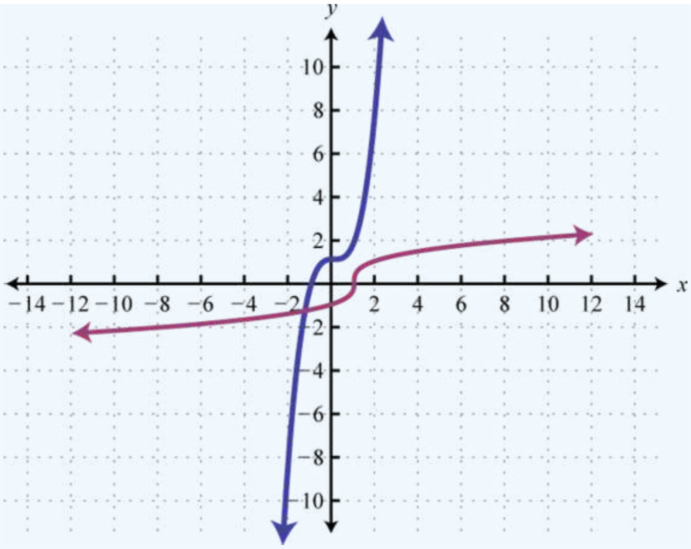

Figure 7.1.26

9.

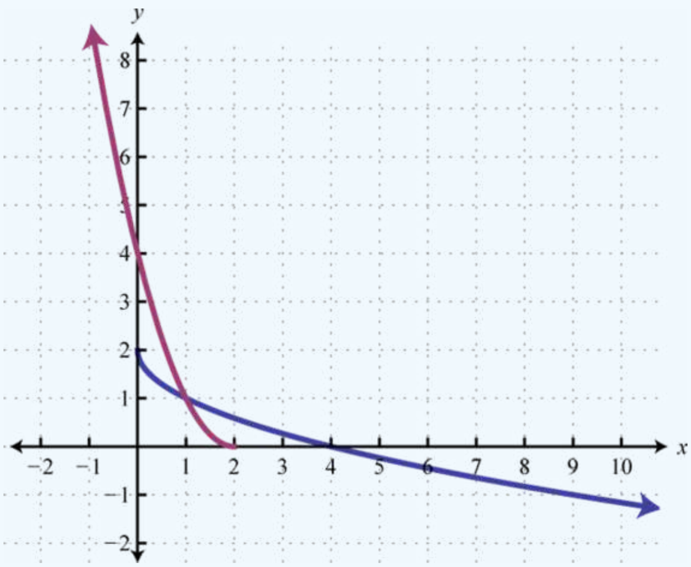

Figure 7.1.27

Exercise 7.1.12

1. Is composition of functions associative? Explain.
2. Explain why $C(x)=\frac{5}{9}(x-32)$ and $F(x)=\frac{9}{5}x+32$ define inverse functions. Prove it algebraically.
3. Do the graphs of all straight lines represent one-to-one functions? Explain.
4. If the graphs of inverse functions intersect, then how can we find the point of intersection? Explain.

Answer

1. Answer may vary

3. Answer may vary

Footnotes

[1] Applying a function to the results of another function.

[2] The open dot used to indicate the function composition $(f \bigcirc g)(x)=f(g(x))$

[3] Functions where each value in the range corresponds to exactly one value in the domain.

[4] If a horizontal line intersects the graph of a function more than once, then it is not one-to-one.

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7.2: Exponential Functions and Their Graphs

Learning Objectives

- Identify and evaluate exponential functions.
- Sketch the graph of exponential functions and determine the domain and range.
- Identify and graph the natural exponential function.
- Apply the formulas for compound interest.

Exponential Functions

At this point in our study of algebra we begin to look at transcendental functions or functions that seem to "transcend" algebra. We have studied functions with variable bases and constant exponents such as x^2 or y^{-3}. In this section we explore functions with a constant base and variable exponents. Given a real number $b > 0$ where $b \neq 1$ an **exponential function**[5] has the form,

$$f(x) = b^x \quad \textit{Exponential Function}$$

For example, if the base b is equal to 2, then we have the exponential function defined by $f(x) = 2^x$. Here we can see the exponent is the variable. Up to this point, rational exponents have been defined but irrational exponents have not. Consider $2^{\sqrt{7}}$, where the exponent is an irrational number in the range,

$$2.64 < \sqrt{7} < 2.65$$

We can use these bounds to estimate $2^{\sqrt{7}}$,

$$2^{2.64} < 2^{\sqrt{7}} < 2^{2.65}$$
$$6.23 < 2^{\sqrt{7}} < 6.28$$

Using rational exponents in this manner, an approximation of $2^{\sqrt{7}}$ can be obtained to any level of accuracy. On a calculator,

$$2^\wedge\sqrt{7} \approx 6.26$$

Therefore the domain of any exponential function consists of all real numbers $(-\infty, \infty)$. Choose some values for x and then determine the corresponding y-values.

Table 7.2.1

x	y	$f(x) = 2^x$	*Solutions*
-2	$\frac{1}{4}$	$y = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$(-2, \frac{1}{4})$
-1	$\frac{1}{2}$	$y = 2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$(-1, \frac{1}{2})$
0	1	$y = 2^0 = 1$	$(0, 1)$
1	2	$y = 2^1 = 2$	$(1, 2)$
2	4	$y = 2^2 = 4$	$(2, 4)$
$\sqrt{7}$	6.26	$y = 2^{\sqrt{7}} \approx 6.26$	$(2.65, 6.26)$

Because exponents are defined for any real number we can sketch the graph using a continuous curve through these given points:

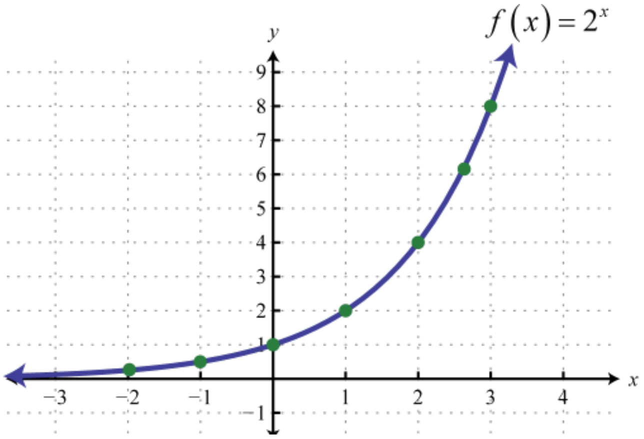

Figure 7.2.1

It is important to point out that as x approaches negative infinity, the results become very small but never actually attain zero. For example,

$$\begin{gathered}
f(-5)=2^{-5}=\frac{1}{2^{5}} \approx 0.03125 \\
f(-10)=2^{-10}=\frac{1}{2^{10}} \approx 0.0009766 \\
f(-15)=2^{-15}=\frac{1}{2^{-15}} \approx .00003052
\end{gathered}$$

This describes a horizontal asymptote at $y=0$, the x-axis, and defines a lower bound for the range of the function: $(0, \infty)$.

The base b of an exponential function affects the rate at which it grows. Below we have graphed $y=2^{x}, y=3^{x}$, and $y=10^{x}$ on the same set of axes.

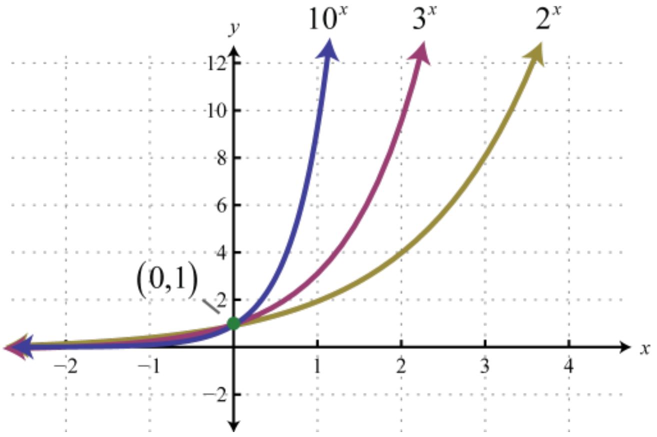

Figure 7.2.2

Note that all of these exponential functions have the same y-intercept, namely $(0,1)$. This is because $f(0)=b^{0}=1$ for any function defined using the form $f(x)=b^{x}$. As the functions are read from left to right, they are interpreted as increasing or growing exponentially. Furthermore, any exponential function of this form will have a domain that consists of all real numbers $(-\infty, \infty)$ and a range that consists of positive values $(0, \infty)$ bounded by a horizontal asymptote at $y=0$.

Example 7.2.1:

Sketch the graph and determine the domain and range: $f(x)=10^{x}+5$.

Solution

The base 10 is used often, most notably with scientific notation. Hence, 10 is called the common base. In fact, the exponential function $y=10^{x}$ is so important that you will find a button 10^{x} dedicated to it on most modern scientific calculators. In this example, we will sketch the basic graph $y=10^{x}$ and then shift it up 5 units.

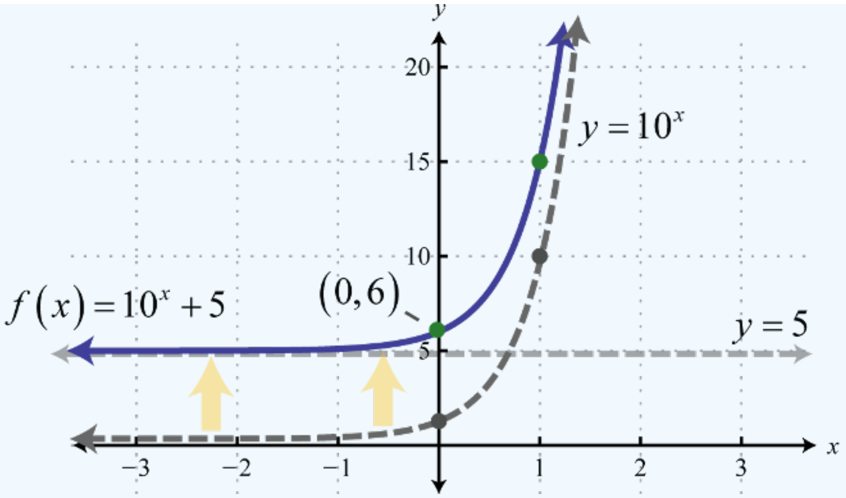

Figure 7.2.3

Note that the horizontal asymptote of the basic graph $y = 10^x$ was shifted up 5 units to $y = 5$ (shown dashed). Take a minute to evaluate a few values of x with your calculator and convince yourself that the result will never be less than 5.

Answer

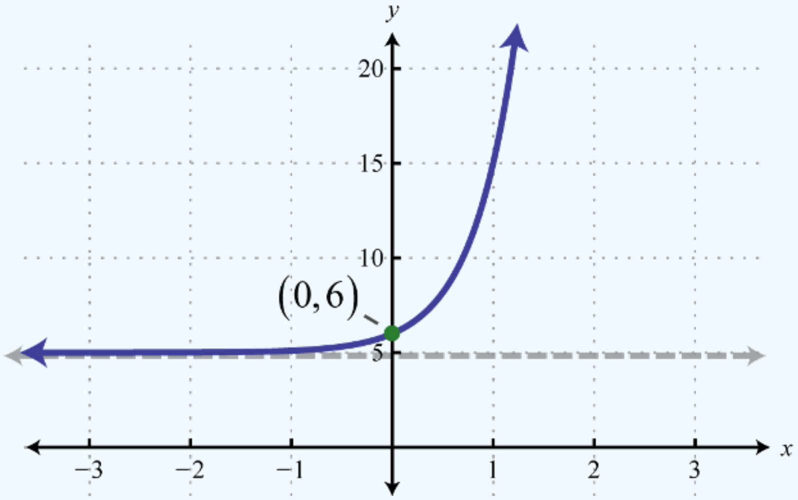

Figure 7.2.4

Domain: $(-\infty, \infty)$; Range: $(5, \infty)$

Next consider exponential functions with fractional bases $0 < b < 1$. For example, $f(x) = \left(\frac{1}{2}\right)^x$ is an exponential function with base $b = \frac{1}{2}$.

Table 7.2.2

x	y	$f(x) = \left(\frac{1}{2}\right)^x$	*Solutions*
-2	4	$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{-2} = \frac{1^{-2}}{2^{-2}} = \frac{2^2}{1^2} = 4$	$(-2, 4)$
-1	2	$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{-1} = \frac{1^{-1}}{2^{-1}} = \frac{2^1}{1^1} = 2$	$(-1, 2)$
0	1	$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{0} = 1$	$(0, 1)$
1	$\frac{1}{2}$	$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{1} = \frac{1}{2}$	$(1, \frac{1}{2})$
2	$\frac{1}{4}$	$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{2} = \frac{1}{4}$	$(2, \frac{1}{4})$

Plotting points we have,

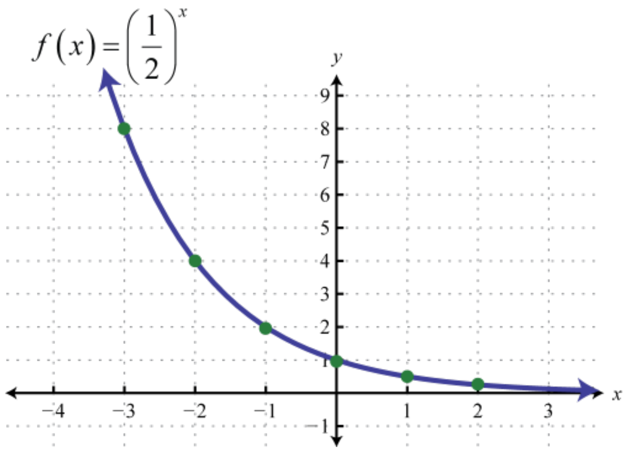

Figure 7.2.5

Reading the graph from left to right, it is interpreted as decreasing exponentially. The base affects the rate at which the exponential function decreases or decays. Below we have graphed $y=\left(\frac{1}{2}\right)^x$, $y=\left(\frac{1}{3}\right)^x$, and $y=\left(\frac{1}{10}\right)^x$ on the same set of axes.

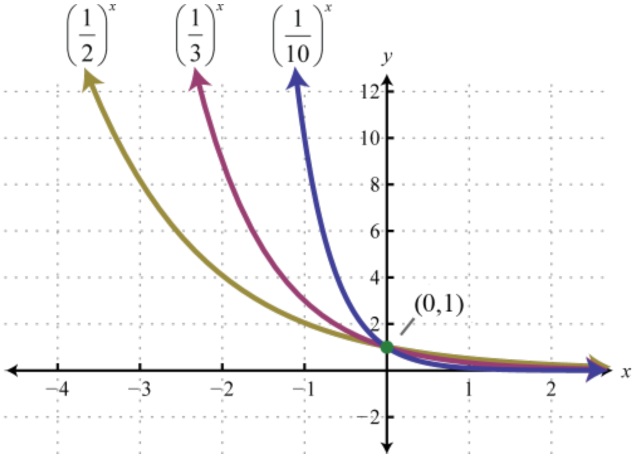

Figure 7.2.6

Recall that $x^{-1}=\frac{1}{x}$ and so we can express exponential functions with fractional bases using negative exponents. For example,

$$g(x)=\left(\frac{1}{2}\right)^x=\frac{1^x}{2^x}=\frac{1}{2^x}=2^{-x}$$

Furthermore, given that $f(x)=2^x$ we can see $g(x)=f(-x)=2^{-x}$ and can consider g to be a reflection of f about the y-axis.

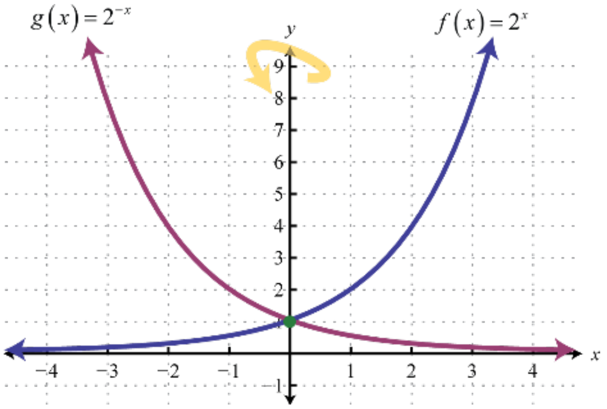

Figure 7.2.7

In summary, given $b>0$

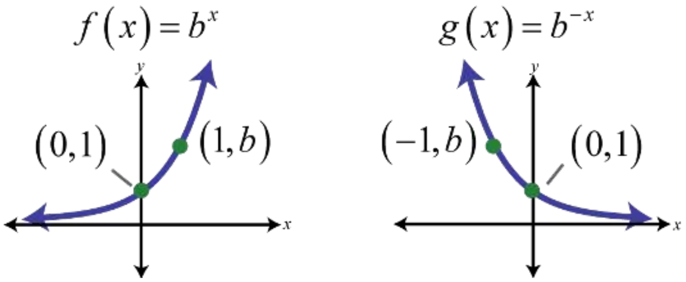

Figure 7.2.8

And for both cases,

$$\begin{aligned}\textit{Domain} :&(-\infty,\infty)\\ \textit{Range} :&(0,\infty)\\ y-\textit{intercept} :&(0,1)\\ \textit{Asymptote} :&y=0\end{aligned}$$

Furthermore, note that the graphs pass the horizontal line test and thus exponential functions are one-to-one. We use these basic graphs, along with the transformations, to sketch the graphs of exponential functions.

Example 7.2.2

Sketch the graph and determine the domain and range: $f(x)=5^{-x}-10$.

Solution

Begin with the basic graph $y=5^{-x}$ and shift it down 10 units.

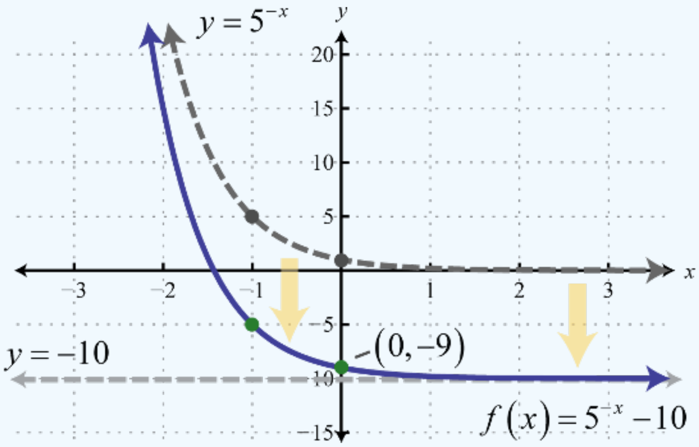

Figure 7.2.9

The y-intercept is $(0,-9)$ and the horizontal asymptote is $y=-10$.

Answer

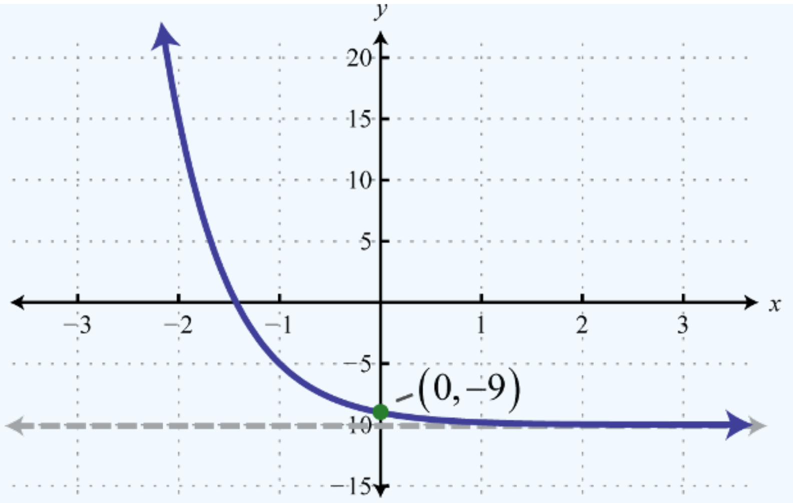

Figure 7.2.10

Domain: $(-\infty, \infty)$; Range: $(-10, \infty)$

Finding the x-intercept of the graph in the previous example is left for a later section in this chapter. For now, we are more concerned with the general shape of exponential functions.

Example 7.2.3

Sketch the graph and determine the domain and range of $g(x) = -2^{x-3}$.

Solution

Begin with the basic graph $y = 2^x$ and identify the transformations.

$$\begin{array}{ll} y = 2^x & \textit{Basic graph} \\ y = -2^x & \textit{Reflection about the } x-\textit{axis} \\ y = -2^{x-3} & \textit{Shift right 3 units} \end{array}$$

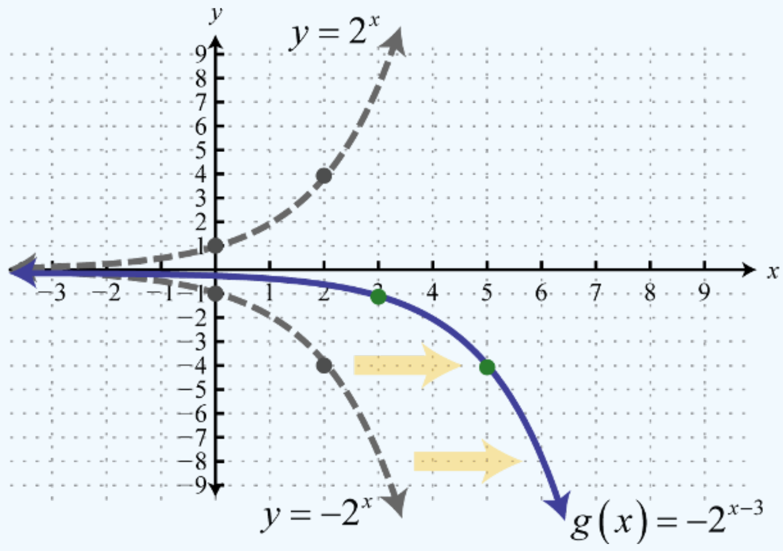

Figure 7.2.11

Note that the horizontal asymptote remains the same for all of the transformations. To finish we usually want to include the y-intercept. Remember that to find the y-intercept we set $x = 0$.

$$\begin{aligned} g(0) &= -2^{0-3} \\ &= -2^{-3} \\ &= -\frac{1}{2^3} \\ &= -\frac{1}{8} \end{aligned}$$

Therefore the y-intercept is $(0, -\frac{1}{8})$.

Answer

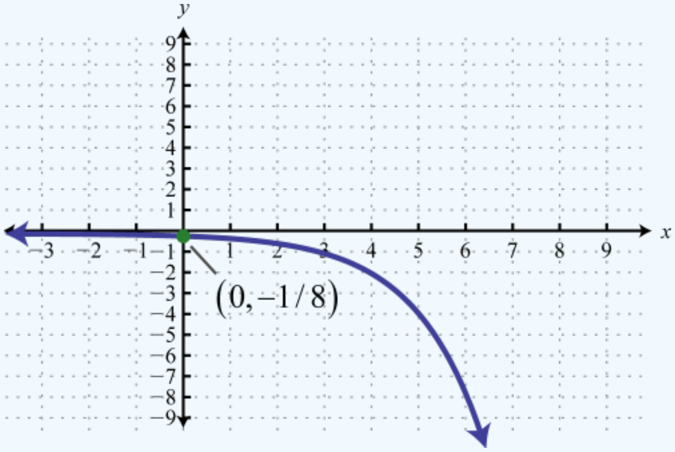

Figure 7.2.12

Domain: $(-\infty, \infty)$; Range: $(-\infty, 0)$

Exercise 7.2.1

Sketch the graph and determine the domain and range: $f(x) = 2^{x-1} + 3$

Answer

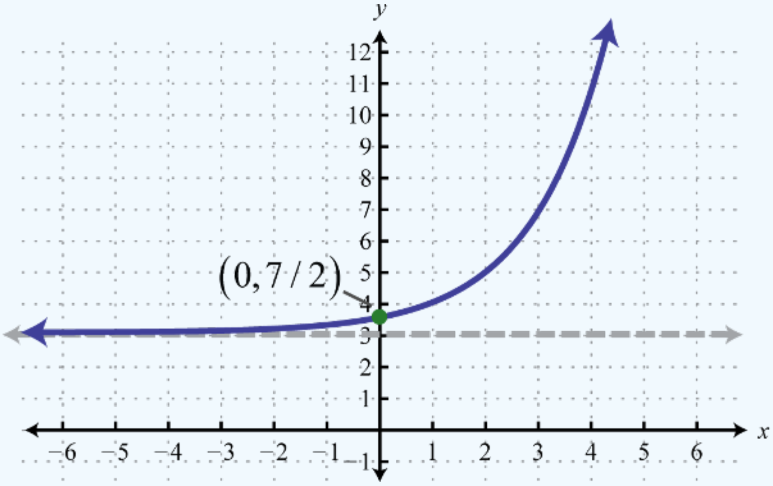

Figure 7.2.13

Domain: $(-\infty, \infty)$; Range: $(3, \infty)$

www.youtube.com/v/WDcObqDtE7M

Natural Base e

Some numbers occur often in common applications. One such familiar number is pi (π), which we know occurs when working with circles. This irrational number has a dedicated button on most calculators π and approximated to five decimal places, $\pi \approx 3.14159$. Another important number e occurs when working with exponential growth and decay models. It is an irrational number and approximated to five decimal places, $e \approx 2.71828$. This constant occurs naturally in many real-world applications and thus is called the **natural base**. Sometimes e is called Euler's constant in honor of Leonhard Euler (pronounced "Oiler").

***Figure** 7.2.14**:** Leonhard Euler (1707-1783)*

In fact, the **natural exponential funnction:**

$$f(x) = e^x$$

is so important that you will find a button e^x dedicated to it on any modern scientific calculator. In this section, we are interested in evaluating the natural exponential function for given real numbers and sketching its graph. To evaluate the natural exponential function, defined by $f(x) = e^x$ where $x = -2$ using a calculator, you may need to apply the shift button. On many scientific calculators the caret will display as follows,

$$f(-2) = e^{\wedge}(-2) \approx 0.13534$$

After learning how to use your particular calculator, you can now sketch the graph by plotting points. (Round off to the nearest hundredth.)

Table 7.2.3

x	y	$f(x) = e^x$	*Solutions*
-2	0.14	$f(-2) = e^{-2} = 0.14$	$(-2, 0.14)$
-1	0.37	$f(-1) = e^{-1} = 0.37$	$(-1, 0.37)$
0	1	$f(0) = e^0 = 1$	$(0, 1)$
1	2.72	$f(1) = e^1 = 2.72$	$(1, 2.72)$
2	7.39	$f(2) = e^2 = 7.39$	$(2, 7.39)$

Plot the points and sketch the graph.

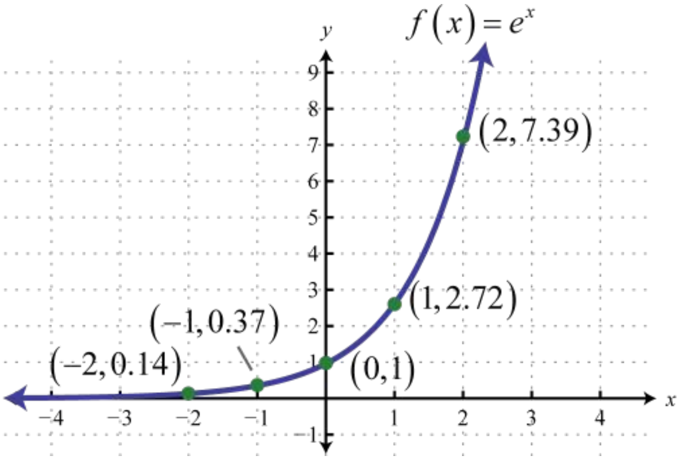

Figure 7.2.15

Note that the function is similar to the graph of $y=3^x$. The domain consists of all real numbers and the range consists of all positive real numbers. There is an asymptote at $y=0$ and a y-intercept at $(0,1)$. We can use the transformations to sketch the graph of more complicated exponential functions.

Example 7.2.4:

Sketch the graph and determine the domain and range: $g(x)=e^{x+2}-3$.

Solution

Identify the basic transformations.

$$\begin{array}{ll} y=e^x & \color{#1AA3E5}{Basic\,graph} \\ y=e^{x+2} & \color{#1AA3E5}{Shift\,left\,2\,units} \\ y=e^{x+2}-3 & \color{#1AA3E5}{Shift\,down\,3\,units} \end{array}$$

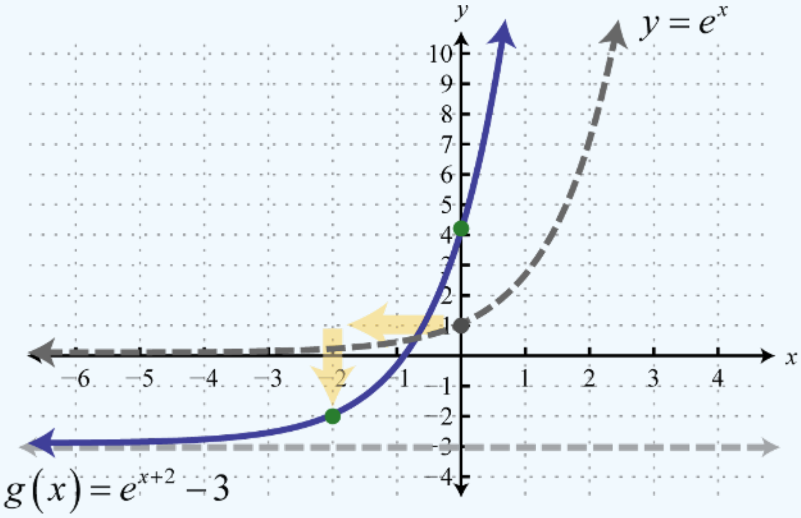

Figure 7.2.16

To determine the y-intercept set $x=0$.

$$\begin{aligned} g(\color{#1AA3E5}{0}) &= e^{\color{#1AA3E5}{0}+2}-3 \\ &= e^2-3 \\ &\approx 4.39 \end{aligned}$$

Therefore the y-intercept is $(0,e^2-3)$.

Answer

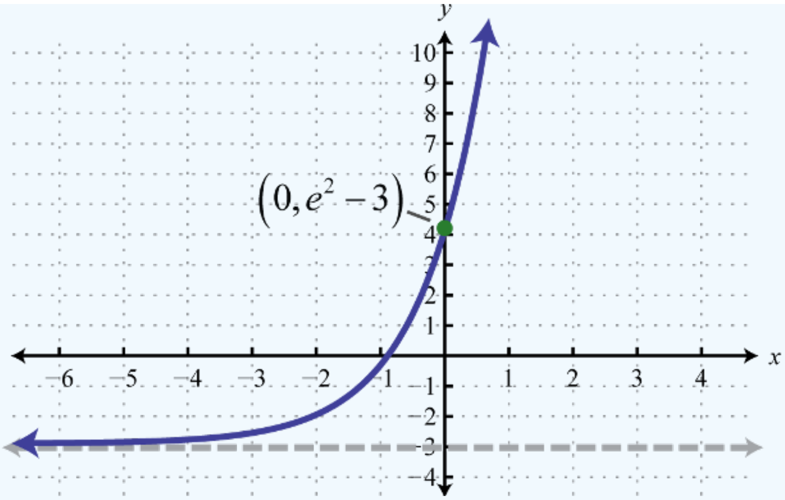

Figure 7.2.17

Domain: $(-\infty, \infty)$; Range: $(-3, \infty)$

Exercise 7.2.2

Sketch the graph and determine the domain and range: $f(x) = e^{-x} + 2$.

Answer

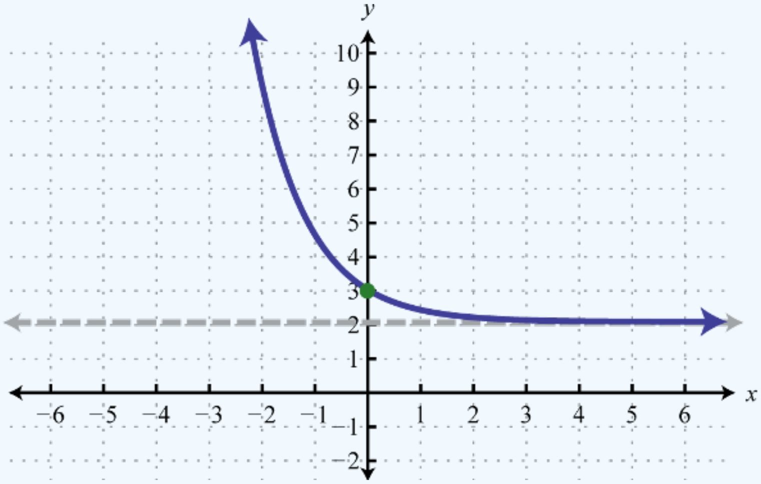

Figure 7.2.18

Domain: $(-\infty, \infty)$; Range: $(2, \infty)$

www.youtube.com/v/FkwBpg2Xpb8

Compound Interest Formulas

Exponential functions appear in formulas used to calculate interest earned in most regular savings accounts. Compound interest occurs when interest accumulated for one period is added to the principal investment before calculating interest for the next period. The amount accrued in this manner over time is modeled by the **compound interest formula**[6]:

$$A(t) = P\left(1 + \tfrac{r}{n}\right)^{nt}$$

Here the amount A depends on the time t in years the principal P is accumulating compound interest at an annual interest rate r. The value n represents the number of times the interest is compounded in a year.

Example 7.2.5:

An investment of $\$500$ is made in a 6-year CD that earns $4\frac{1}{2}\%$ annual interest that is compounded monthly. How much will the CD be worth at the end of the 6-year term?

Solution

Here the principal $P = \$500$, the interest rate $r = 4\frac{1}{2}\% = 0.045$, and because the interest is compounded monthly, $n = 12$. The investment is modeled by the following,

$$A(t) = 500\left(1 + \frac{0.045}{12}\right)^{12t}$$

To determine the amount in the account after 6 years evaluate $A(6)$ and round off to the nearest cent.

$$\begin{aligned} A(6) &= 500\left(1 + \frac{0.045}{12}\right)^{12(6)} \\ &= 500(1.00375)^{72} \\ &= 654.65 \end{aligned}$$

Answer

The CD will be worth $\$654.65$ at the end of the 6-year term.

Next we explore the effects of increasing n in the formula. For the sake of clarity we let P and r equal 1 and calculate accordingly.

Table $7.2.4$

Annual Compounding	$\left(1 + \frac{1}{n}\right)^n$
Yearly $(n = 1)$	$\left(1 + \frac{1}{1}\right)^1 = 2$
Semi-annually $(n = 2)$	$\left(1 + \frac{1}{2}\right)^2 = 2.25$
Quarterly $(n = 4)$	$\left(1 + \frac{1}{4}\right)^4 \approx 2.44140$
Monthly $(n = 12)$	$\left(1 + \frac{1}{12}\right)^{12} \approx 2.61304$
Weekly $(n = 52)$	$\left(1 + \frac{1}{52}\right)^{52} \approx 2.69260$
Daily $(n = 365)$	$\left(1 + \frac{1}{365}\right)^{365} \approx 2.71457$
Hourly $(n = 8760)$	$\left(1 + \frac{1}{8760}\right)^{8760} \approx 2.71813$

Continuing this pattern, as n increases to say compounding every minute or even every second, we can see that the result tends toward the natural base $e \approx 2.71828$. Compounding interest every instant leads to the **continuously compounding interest formula**[7],

$$A(t) = Pe^{rt}$$

Here P represents the initial principal amount invested, r represents the annual interest rate, and t represents the time in years the investment is allowed to accrue continuously compounded interest.

Example 7.2.6:

An investment of $\$500$ is made in a 6-year CD that earns $4\frac{1}{2}\%$ annual interest that is compounded continuously. How much will the CD be worth at the end of the 6-year term?

Solution

Here the principal $P = \$500$, and the interest rate $r = 4\frac{1}{2}\% = 0.045$. Since the interest is compounded continuously we will use the formula $A(t) = Pe^{rt}$. The investment is modeled by the following,

$$A(t) = 500e^{0.045t}$$

To determine the amount in the account after 6 years, evaluate $A(6)$ and round off to the nearest cent.

$$\begin{aligned} A(6) &= 500e^{0.045(6)} \\ &= 500e^{0.27} \\ &= 654.98 \end{aligned}$$

Answer

The CD will be worth \$$654.98$ at the end of the 6-year term.

Compare the previous two examples and note that compounding continuously may not be as beneficial as it sounds. While it is better to compound interest more often, the difference is not that profound. Certainly, the interest rate is a much greater factor in the end result.

Exercise 7.2.3

How much will a \$$1,200$ CD, earning 5.2% annual interest compounded continuously, be worth at the end of a 10-year term?

Answer

\$$2,018.43$

www.youtube.com/v/1alUOjTq5wc

Key Takeaways

- Exponential functions have definitions of the form $f(x)=b^{x}$ where $b>0$ and $b\neq 1$. The domain consists of all real numbers $(-\infty,\infty)$ and the range consists of positive numbers $(0,\infty)$. Also, all exponential functions of this form have a y-intercept of $(0,1)$ and are asymptotic to the x-axis.
- If the base of an exponential function is greater than $1(b>1)$, then its graph increases or grows as it is read from left to right.
- If the base of an exponential function is a proper fraction $(0<b<1)$, then its graph decreases or decays as it is read from left to right.
- The number 10 is called the common base and the number e is called the natural base.
- The natural exponential function defined by $f(x)=e^{x}$ has a graph that is very similar to the graph of $g(x)=3^{x}$.
- Exponential functions are one-to-one.

Exercise 7.2.4

Evaluate.

1. $f(x)=3^{x}$ where $f(-2), f(0)$, and $f(2)$.
2. $f(x)=10^{x}$ where $f(-1), f(0)$, and $f(1)$.
3. $g(x)=\left(\frac{1}{3}\right)^{x}$ where $g(-1), g(0)$, and $g(3)$.
4. $g(x)=\left(\frac{3}{4}\right)^{x}$ where $g(-2), g(-1)$, and $g(0)$.
5. $h(x)=9^{-x}$ where $h(-1), h(0)$, and $h\left(\frac{1}{2}\right)$.
6. $h(x)=4^{-x}$ where $h(-1), h\left(-\frac{1}{2}\right)$, and $h(0)$.
7. $f(x)=-2^{x}+1$ where $f(-1), f(0)$, and $f(3)$.
8. $f(x)=2-3^{x}$ where $f(-1), f(0)$, and $f(2)$.
9. $g(x)=10^{-x}+20$ where $g(-2), g(-1)$, and $g(0)$.
10. $g(x)=1-2^{-x}$ where $g(-1), g(0)$, and $g(1)$.

Answer

1. $f(-2)=\frac{1}{9}, f(0)=1, f(2)=9$

3. $g(-1)=3, g(0)=1, g(3)=\frac{1}{27}$

5. $h(-1)=9, h(0)=1, h\left(\frac{1}{2}\right)=\frac{1}{3}$

7. $f(-1)=\frac{1}{2}, f(0)=0, f(3)=-7$

9. $g(-2)=120, g(-1)=30, g(0)=21$

Exercise 7.2.5

Use a calculator to approximate the following to the nearest hundredth.

1. $f(x)=2^x+5$ where $f(2.5)$.
2. $f(x)=3^x-10$ where $f(3.2)$.
3. $g(x)=4^x$ where $g(\sqrt{2})$.
4. $g(x)=5^x-1$ where $g(\sqrt{3})$.
5. $h(x)=10^x$ where $h(\pi)$.
6. $h(x)=10^x+1$ where $h\left(\frac{\pi}{3}\right)$.
7. $f(x)=10^{-x}-2$ where $f(1.5)$.
8. $f(x)=5^{-x}+3$ where $f(1.3)$.
9. $f(x)=\left(\frac{2}{3}\right)^x+1$ where $f(-2.7)$.
10. $f(x)=\left(\frac{3}{5}\right)^{-x}-1$ where $f(1.4)$.

Answer

1. 10.66

3. 7.10

5. 1385.46

7. -1.97

9. 3.99

Exercise 7.2.6

Sketch the function and determine the domain and range. Draw the horizontal asymptote with a dashed line.

1. $f(x)=4^x$
2. $g(x)=3^x$
3. $f(x)=4^x+2$
4. $f(x)=3^x-6$
5. $f(x)=2^{x-2}$
6. $f(x)=4^{x+2}$
7. $f(x)=3^{x+1}-4$
8. $f(x)=10^{x-4}+2$
9. $h(x)=2^{x-3}-2$
10. $h(x)=3^{x+2}+4$
11. $f(x)=\left(\frac{1}{4}\right)^x$
12. $h(x)=\left(\frac{1}{3}\right)^x$
13. $f(x)=\left(\frac{1}{4}\right)^x-2$
14. $h(x)=\left(\frac{1}{3}\right)^x+2$
15. $g(x)=2^{-x}-3$
16. $g(x)=3^{-x}+1$
17. $f(x)=6-10^{-x}$
18. $g(x)=5-4^{-x}$
19. $f(x)=5-2^x$
20. $f(x)=3-3^x$

Answer

1. Domain: $(-\infty,\infty)$; Range: $(0,\infty)$

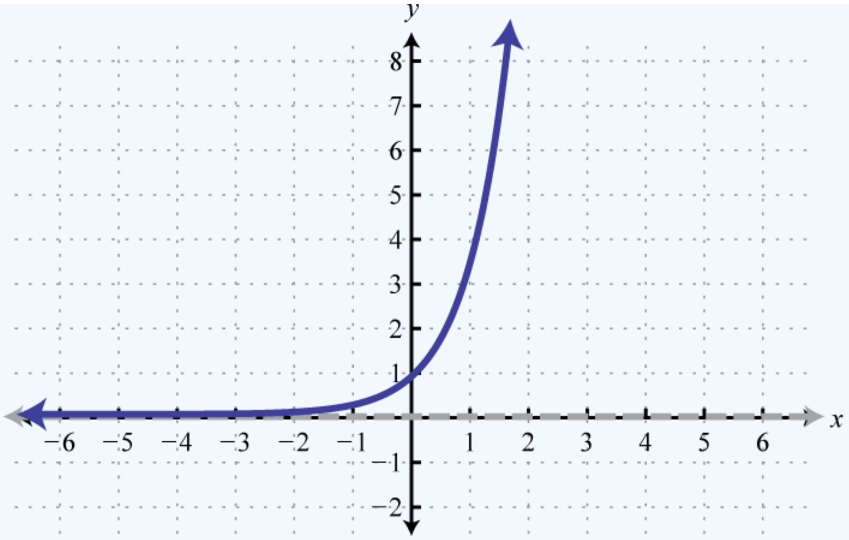

Figure 7.2.19

3. Domain: $(-\infty, \infty)$; Range: $(2, \infty)$

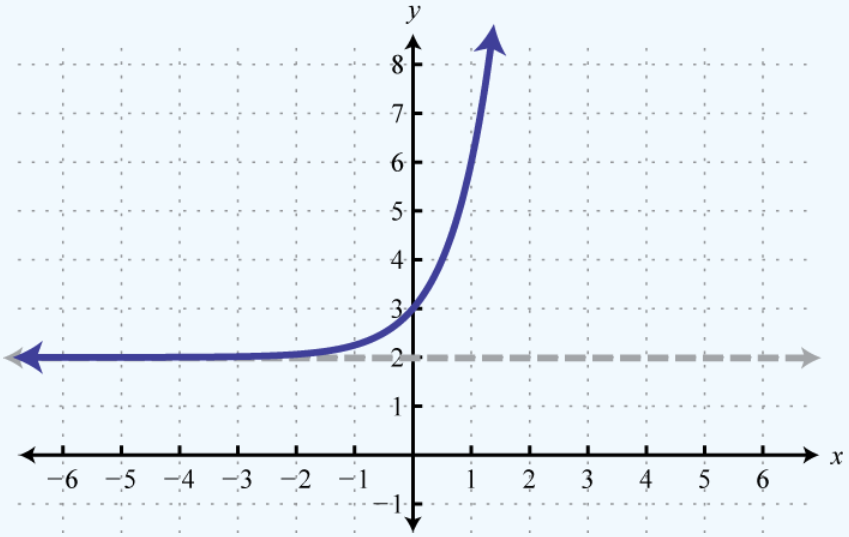

Figure 7.2.20

5. Domain: $(-\infty, \infty)$; Range: $(0, \infty)$

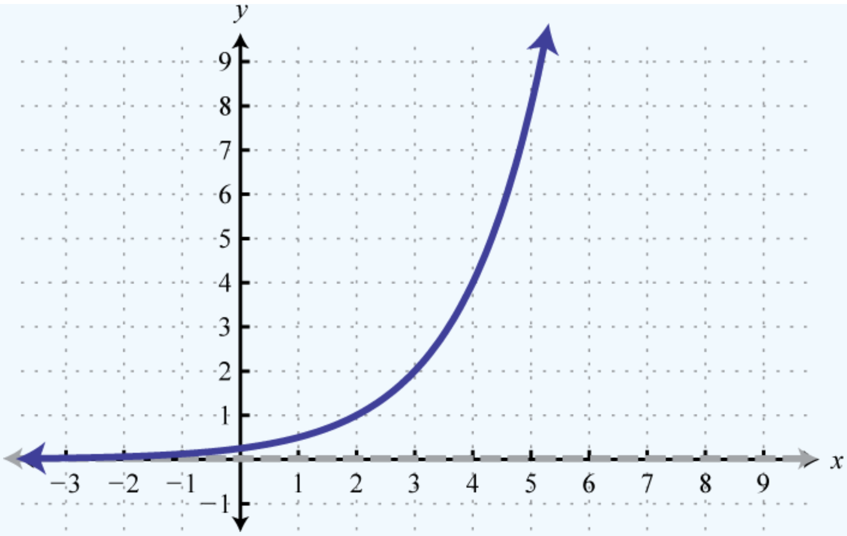

Figure 7.2.21

7. Domain: $(-\infty, \infty)$; Range: $(-4, \infty)$

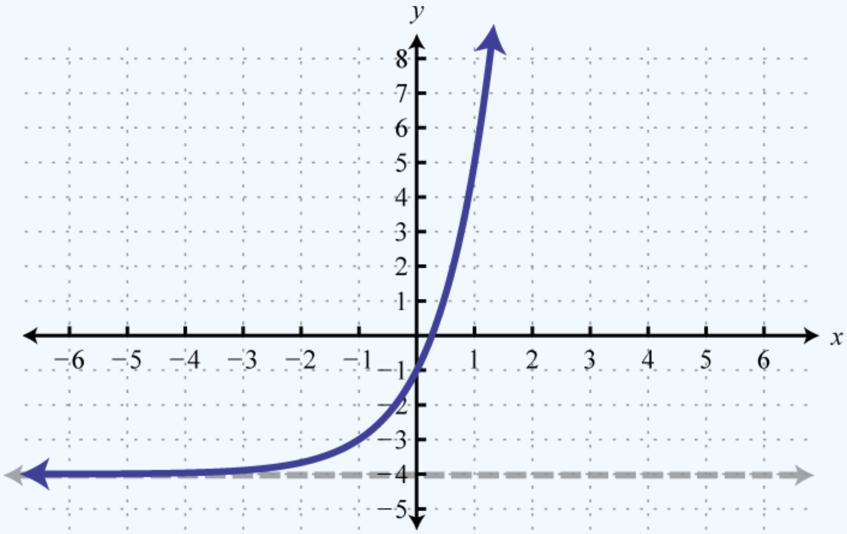

Figure 7.2.22

9. Domain: $(-\infty, \infty)$; Range: $(-2, \infty)$

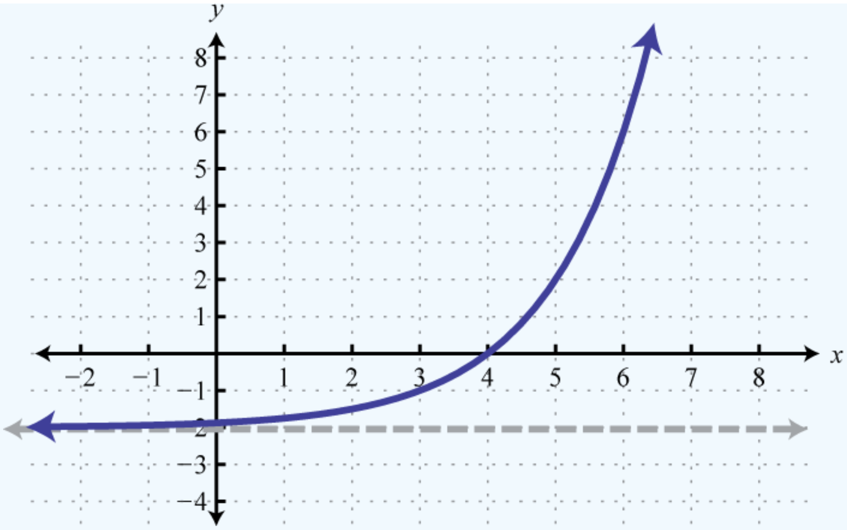

Figure 7.2.23

11. Domain: $(-\infty, \infty)$; Range: $(0, \infty)$

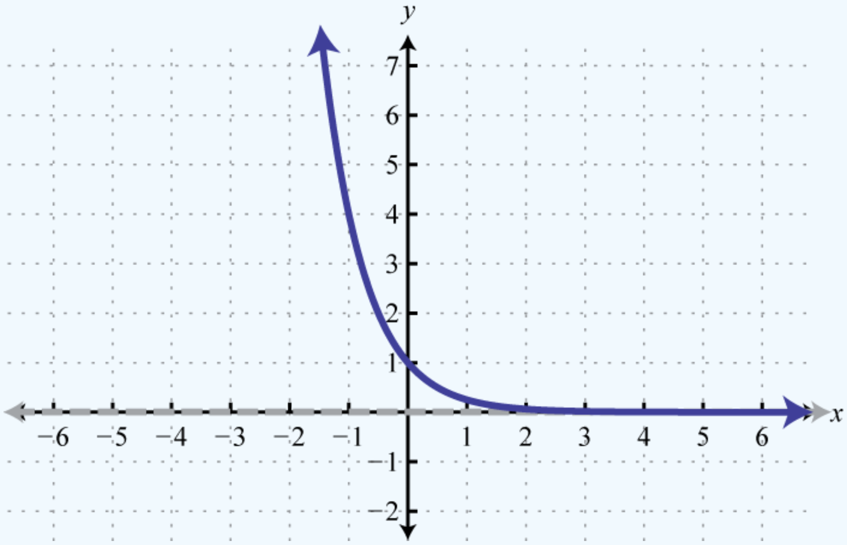

Figure 7.2.24

13. Domain: $(-\infty, \infty)$; Range: $(-2, \infty)$

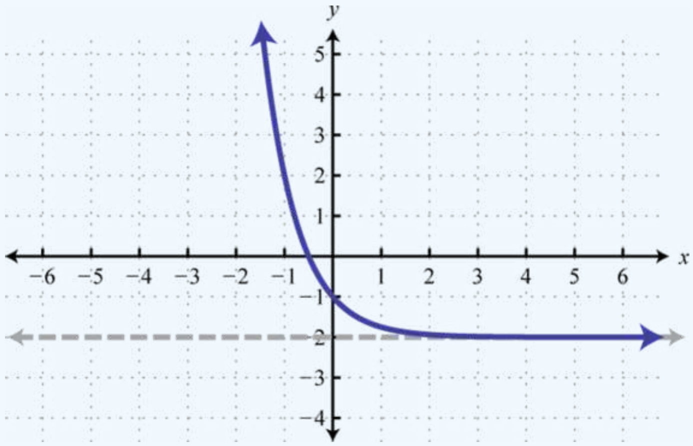

Figure 7.2.25

15. Domain: $(-\infty, \infty)$; Range: $(-3, \infty)$

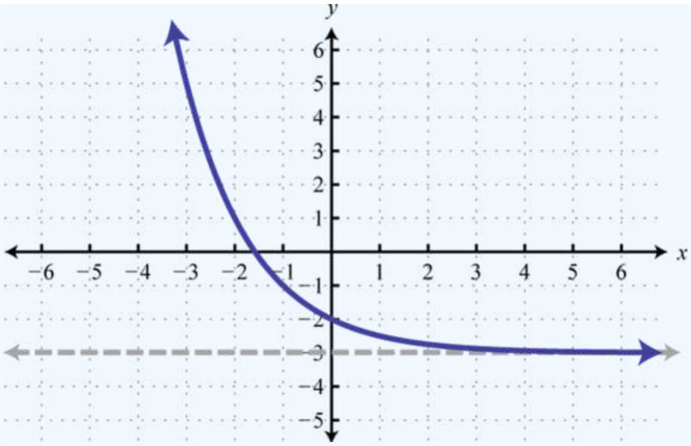

Figure 7.2.26

17. Domain: $(-\infty, \infty)$; Range: $(-\infty, 6)$

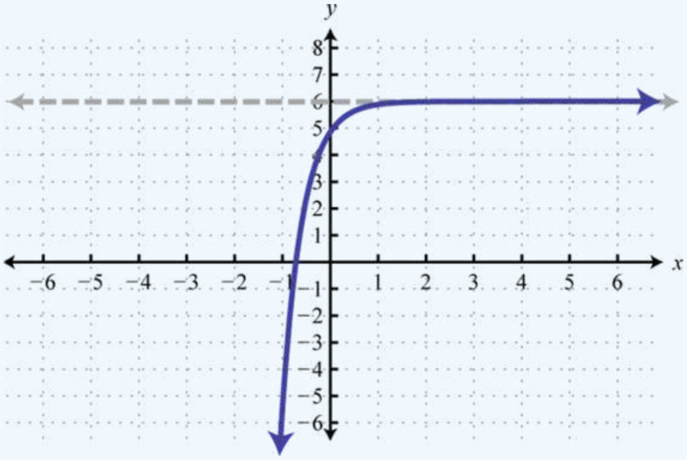

Figure 7.2.27

19. Domain: $(-\infty, \infty)$; Range: $(-\infty, 5)$

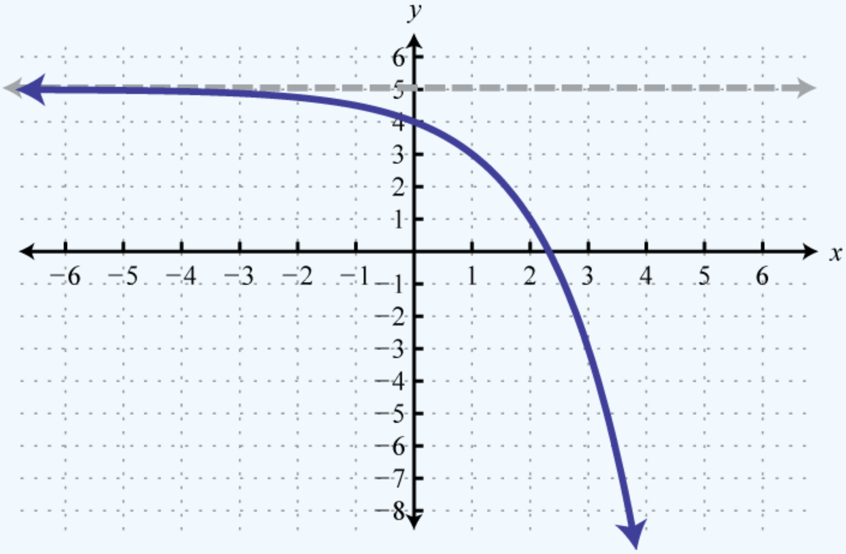

Figure 7.2.28

Exercise 7.2.7

Find $f(-1)$, $f(0)$, and $f(\frac{3}{2})$ for the given function. Use a calculator where appropriate to approximate to the nearest hundredth.

1. $f(x)=e^{x}+2$
2. $f(x)=e^{x}-4$
3. $f(x)=5-3e^{x}$
4. $f(x)=e^{-x}+3$
5. $f(x)=1+e^{-x}$
6. $f(x)=3-2e^{-x}$
7. $f(x)=e^{-2x}+2$
8. $f(x)=e^{-x^{2}}-1$

Answer

1. $f(-1)\approx 2.37, f(0)=3, f\left(\frac{3}{2}\right)\approx 6.48$

3. $f(-1)\approx 3.90, f(0)=2, f\left(\frac{3}{2}\right)\approx -8.45$

5. $f(-1)\approx 3.72, f(0)=2, f\left(\frac{3}{2}\right)\approx 1.22$

7. $f(-1)\approx 9.39, f(0)=3, f\left(\frac{3}{2}\right)\approx 2.05$

Exercise 7.2.8

Sketch the function and determine the domain and range. Draw the horizontal asymptote with a dashed line.

1. $f(x)=e^{x}-3$
2. $f(x)=e^{x}+2$
3. $f(x)=e^{x+1}$
4. $f(x)=e^{x-3}$
5. $f(x)=e^{x-2}+1$
6. $f(x)=e^{x+2}-1$
7. $g(x)=-e^{x}$
8. $g(x)=e^{-x}$
9. $h(x)=-e^{x+1}$
10. $h(x)=-e^{x}+3$

Answer

1. Domain: $(-\infty, \infty)$; Range: $(-3, \infty)$

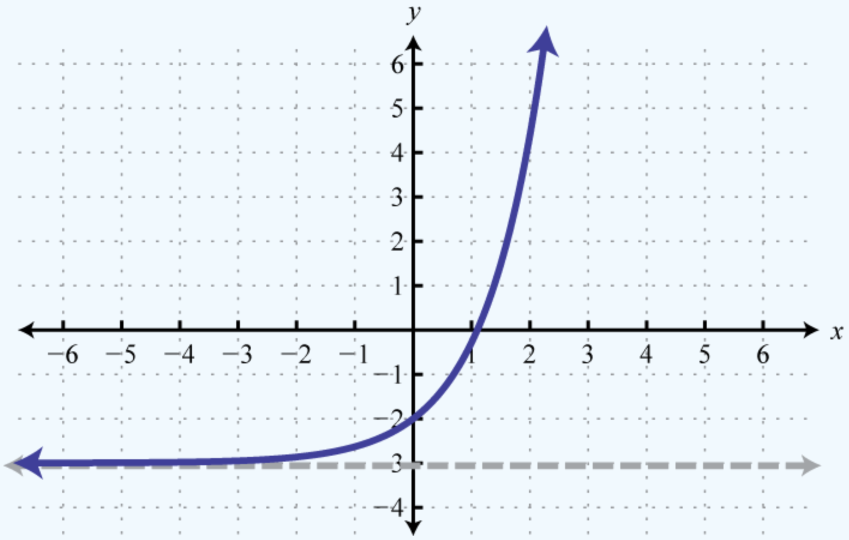

Figure 7.2.29

3. Domain: $(-\infty, \infty)$; Range: $(0, \infty)$

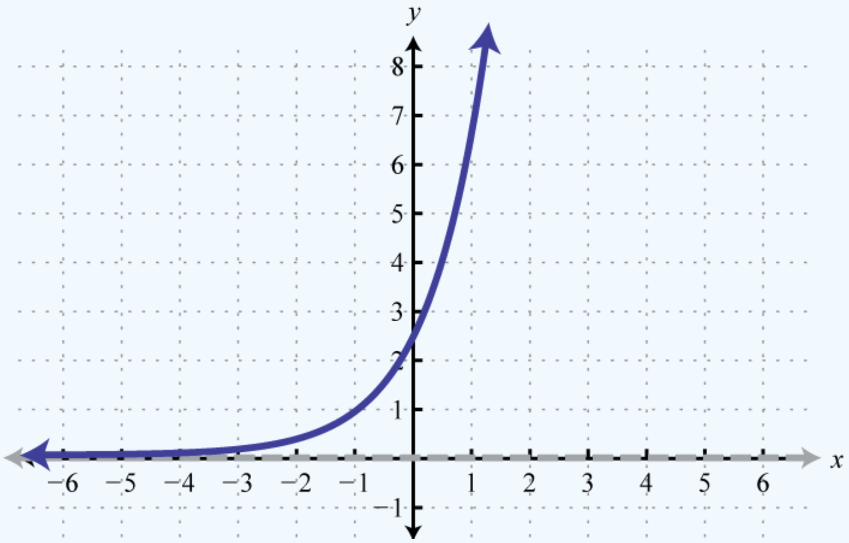

Figure 7.2.30

5. Domain: $(-\infty, \infty)$; Range: $(1, \infty)$

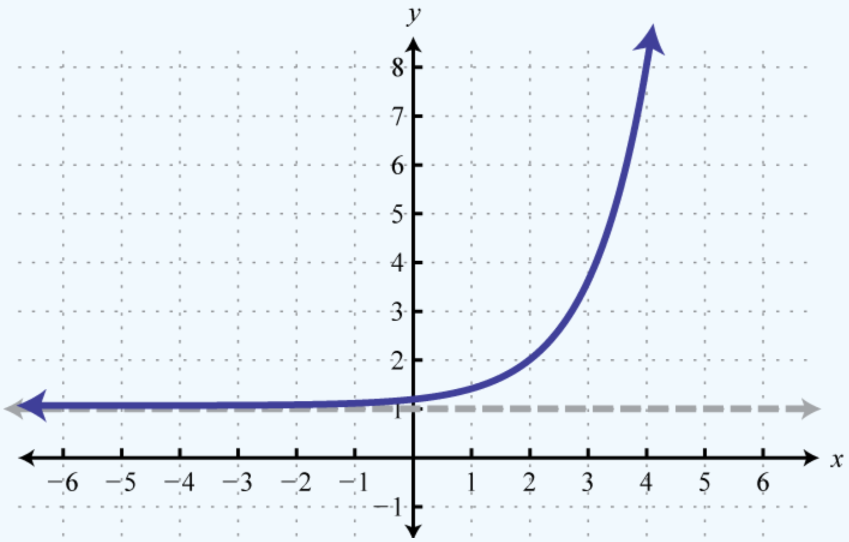

Figure 7.2.31

7. Domain: $(-\infty, \infty)$; Range: $(-\infty, 0)$

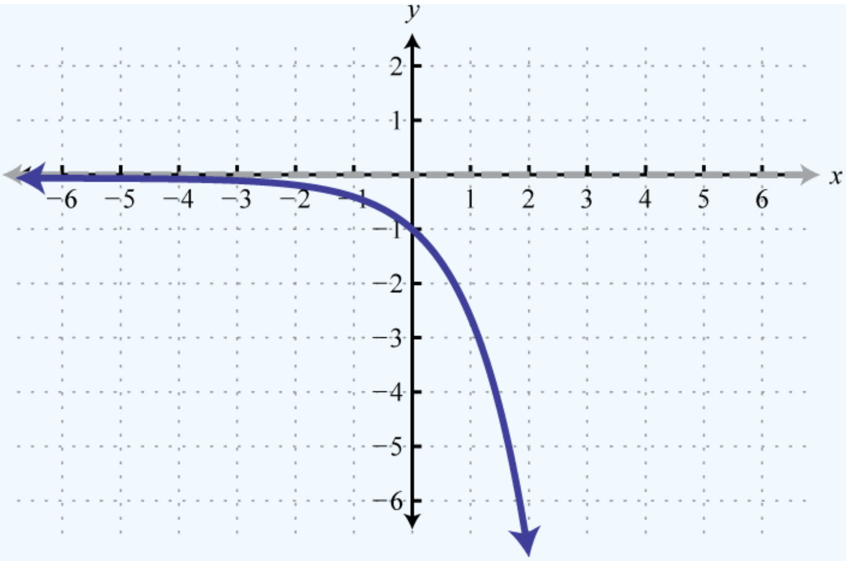

Figure 7.2.32

9. Domain: $(-\infty, \infty)$; Range: $(-\infty, 0)$

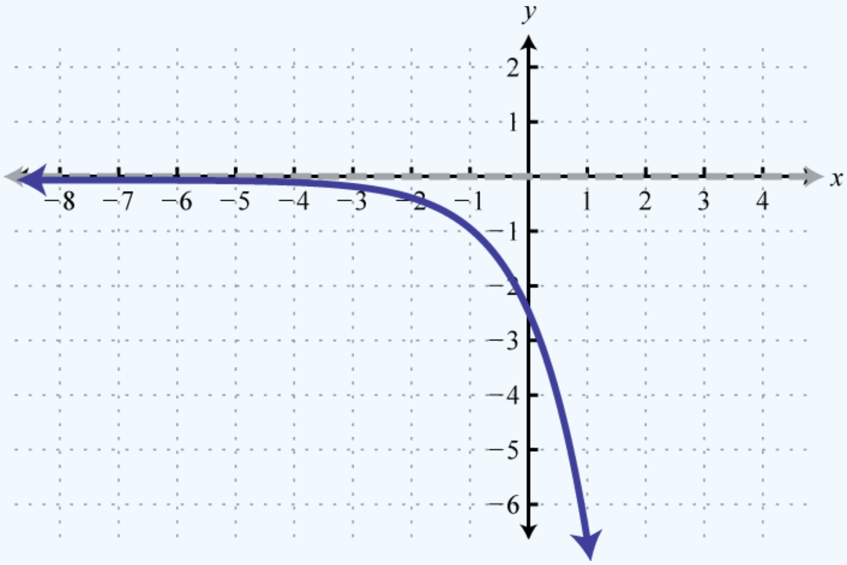

Figure 7.2.33

Exercise 7.2.9

1. Jim invested \$$750$ in a 3-year CD that earns 4.2% annual interest that is compounded monthly. How much will the CD be worth at the end of the 3-year term?
2. Jose invested \$$2,450$ in a 4-year CD that earns 3.6% annual interest that is compounded semi-annually. How much will the CD be worth at the end of the 4-year term?
3. Jane has her \$$5,350$ savings in an account earning $3\frac{5}{8}$% annual interest that is compounded quarterly. How much will be in the account at the end of 5 years?
4. Bill has \$$12,400$ in a regular savings account earning $4\frac{2}{3}$% annual interest that is compounded monthly. How much will be in the account at the end of 3 years?
5. If \$$85,200$ is invested in an account earning 5.8% annual interest compounded quarterly, then how much interest is accrued in the first 3 years?
6. If \$$124,000$ is invested in an account earning 4.6% annual interest compounded monthly, then how much interest is accrued in the first 2 years?

7. Bill invested $\$1,400$ in a 3-year CD that earns 4.2% annual interest that is compounded continuously. How much will the CD be worth at the end of the 3-year term?
8. Brooklyn invested $\$2,850$ in a 5-year CD that earns 5.3% annual interest that is compounded continuously. How much will the CD be worth at the end of the 5-year term?
9. Omar has his $\$4,200$ savings in an account earning $4\frac{3}{8}\%$ annual interest that is compounded continuously. How much will be in the account at the end of $2\frac{1}{2}$ years?
10. Nancy has her $\$8,325$ savings in an account earning $5\frac{7}{8}\%$ annual interest that is compounded continuously. How much will be in the account at the end of $5\frac{1}{2}$ years?
11. If $\$12,500$ is invested in an account earning 3.8% annual interest compounded continuously, then how much interest is accrued in the first 10 years?
12. If $\$220,000$ is invested in an account earning 4.5% annual interest compounded continuously, then how much interest is accrued in the first 2 years?
13. The population of a certain small town is growing according to the function $P(t)=12,500(1.02)^t$ where t represents time in years since the last census. Use the function to determine the population on the day of the census (when $t=0$) and estimate the population in 6 years from that time.
14. The population of a certain small town is decreasing according to the function $P(t)=22,300(0.982)^t$ where t represents time in years since the last census. Use the function to determine the population on the day of the census (when $t=0$) and estimate the population in 6 years from that time.
15. The decreasing value, in dollars, of a new car is modeled by the formula $V(t)=28,000(0.84)^t$ where t represents the number of years after the car was purchased. Use the formula to determine the value of the car when it was new ($t=0$) and the value after 4 years.
16. The number of unique visitors to the college website can be approximated by the formula $N(t)=410(1.32)^t$ where t represents the number of years after 1997 when the website was created. Approximate the number of unique visitors to the college website in the year 2020.
17. If left unchecked, a new strain of flu virus can spread from a single person to others very quickly. The number of people affected can be modeled using the formula $P(t)=e^{0.22t}$ where t represents the number of days the virus is allowed to spread unchecked. Estimate the number of people infected with the virus after 30 days and after 60 days.
18. If left unchecked, a population of 24 wild English rabbits can grow according to the formula $P(t)=24e^{0.19t}$ where the time t is measured in months. How many rabbits would be present $3\frac{1}{2}$ years later?
19. The population of a certain city in 1975 was $65,000$ people and was growing exponentially at an annual rate of 1.7%. At the time, the population growth was modeled by the formula $P(t)=65,000e^{0.017t}$ where t represented the number of years since 1975. In the year 2000, the census determined that the actual population was $104,250$ people. What population did the model predict for the year 2000 and what was the actual error?
20. Because of radioactive decay, the amount of a 10 milligram sample of Iodine-131 decreases according to the formula $A(t)=10e^{-0.087t}$ where t represents time measured in days. How much of the sample remains after 10 days?
21. The number of cells in a bacteria sample is approximated by the logistic growth model $N(t)=\frac{1.2\times10^5}{1+9e^{-0.32t}}$ where t represents time in hours. Determine the initial number of cells and then determine the number of cells 6 hours later.
22. The market share of a product, as a percentage, is approximated by the formula $P(t)=\frac{100}{2+e^{-0.44t}}$ where t represents the number of months after an aggressive advertising campaign is launched. By how much can we expect the market share to increase after the first three months of advertising?

Answer

1. $\$850.52$

3. $\$6,407.89$

5. $\$16,066.13$

7. $\$1,588.00$

9. $\$4,685.44$

11. $\$5,778.56$

13. Initial population: $12,500$; Population 6 years later: $14,077$

15. New: $\$28,000$; In 4 years: $\$13,940.40$

17. After 30 days: 735 people; After 60 days: $540,365$ people

19. Model: $99,423$ people; error: $4,827$ people

21. Initially there are $12,000$ cells and 6 hours later there are $51,736$ cells.

Exercise 7.2.10

1. Why is $b=1$ excluded as a base in the definition of exponential functions? Explain.
2. Explain why an exponential function of the form $y=b^x$ can never be negative.
3. Research and discuss the derivation of the compound interest formula.
4. Research and discuss the logistic growth model. Provide a link to more information on this topic.
5. Research and discuss the life and contributions of Leonhard Euler.

Answer

1. Answer may vary

3. Answer may vary

5. Answer may vary

Footnotes

[5] Any function with a definition of the form $f(x)=b^x$ where $b>0$ and $b \neq 1$.

[6] A formula that gives the amount accumulated by earning interest on principal and interest over time: $A(t)=P\left(1+\frac{r}{n}\right)^{nt}$.

[7] A formula that gives the , amount accumulated by earning continuously compounded interest: $A(t)=Pe^{rt}$.

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7.3: Logarithmic Functions and Their Graphs

Learning Objectives

- Define and evaluate logarithms.
- Identify the common and natural logarithm.
- Sketch the graph of logarithmic functions.

Definition of the Logarithm

We begin with the exponential function defined by $f(x)=2^x$ and note that it passes the horizontal line test.

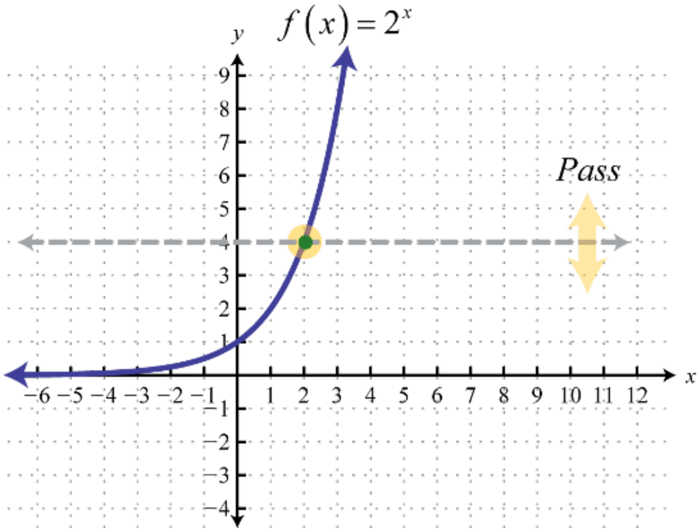

Figure $7.3.1$

Therefore it is one-to-one and has an inverse. Reflecting $y=2^x$ about the line $y=x$ we can sketch the graph of its inverse. Recall that if (x,y) is a point on the graph of a function, then (y,x) will be a point on the graph of its inverse.

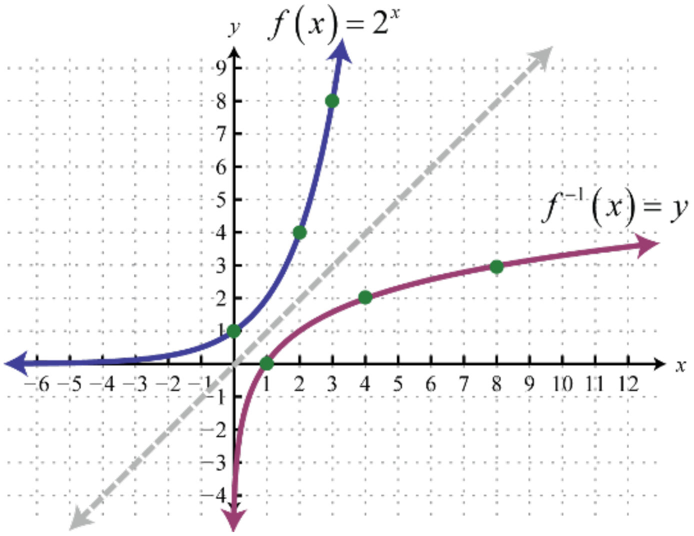

Figure $7.3.2$

To find the inverse algebraically, begin by interchanging x and y and then try to solve for y.

$$f(x)=2^x$$
$$y=2^x \quad \Rightarrow \quad x=2^y$$

We quickly realize that there is no method for solving for y. This function seems to "transcend" algebra. Therefore, we define the inverse to be the base-2 logarithm, denoted $\log_2 x$. The following are equivalent:

$$y=\log_2 x \Leftrightarrow x=2^y$$

This gives us another transcendental function defined by $f^{-1}(x)=\log_2 x$, which is the inverse of the exponential function defined by $f(x)=2^x$.

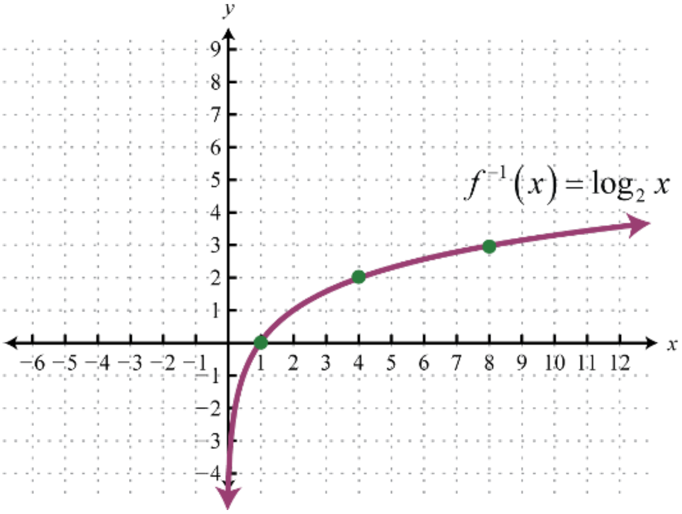

Figure $7.3.3$

The domain consists of all positive real numbers $(0,\infty)$ and the range consists of all real numbers $(-\infty,\infty)$. John Napier is widely credited for inventing the term logarithm.

Figure $7.3.4$: John Napier (1550-1617)

In general, given base $b>0$ where $b\neq 1$, the **logarithm base b**[8] is defined as follows:

$$y=\log_b x \quad \textit{if and only if} \quad x=b^y$$

Use this definition to convert logarithms to exponential form and back.

Table $7.3.1$

Logarithmic Form	Exponential Form
$\log_2 16=4$	$2^4=16$
$\log_5 25=2$	$5^2=25$
$\log_6 1=0$	$6^0=1$
$\log_3 \sqrt{3}=\frac{1}{2}$	$3^{1/2}=\sqrt{3}$
$\log_7\left(\frac{1}{49}\right)=-2$	$7^{-2}=\frac{1}{49}$

It is useful to note that the logarithm is actually the exponent y to which the base b is raised to obtain the argument x.

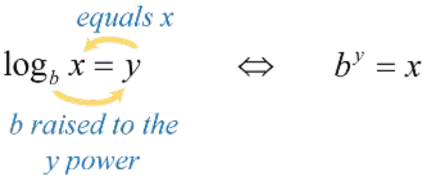

Figure 7.3.5

Example 7.3.1

Evaluate:

a. $\log_5 125$
b. $\log_2\left(\frac{1}{8}\right)$
c. $\log_4 2$
d. $\log_{11} 1$

Solution

a. $\log_5 125 = 3$ because $5^3 = 125$.
b. $\log_2\left(\frac{1}{8}\right) = -3$ because $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$.
c. $\log_4 2 = \frac{1}{2}$ because $4^{1/2} = \sqrt{4} = 2$.
d. $\log_{11} 1 = 0$ because $11^0 = 1$

Note that the result of a logarithm can be negative or even zero. However, the argument of a logarithm is not defined for negative numbers or zero:

$$\begin{aligned}\log_2(-4) = ?\Longrightarrow 2^{?} = -4 \\ \log_2(0) = ?\Longrightarrow 2^{?} = 0\end{aligned} \tag{7.3.1}$$

There is no power of two that results in -4 or 0. Negative numbers and zero are not in the domain of the logarithm. At this point it may be useful to go back and review all of the rules of exponents.

Example 7.3.2

Find x:

a. $\log_7 x = 2$
b. $\log_{16} x = \frac{1}{2}$
c. $\log_{1/2} x = -5$

Solution

Convert each to exponential form and then simplify using the rules of exponents.

a. $\log_7 x = 2$ is equivalent to $7^2 = x$ and thus $x = 49$.
b. $\log_{16} x = \frac{1}{2}$ is equivalent to $16^{1/2} = x$ or $\sqrt{16} = x$ and thus $x = 4$.
c. $\log_{1/2} x = -5$ is equivalent to $\left(\frac{1}{2}\right)^{-5} = x$ or $2^5 = x$ and thus $x = 32$,

Exercise 7.3.1

Evaluate: $\log_5\left(\frac{1}{\sqrt[3]{5}}\right)$.

Answer

$-\frac{1}{3}$

www.youtube.com/v/1Bxv8PY91bw

The Common and Natural Logarithm

A logarithm can have any positive real number, other than 1, as its base. If the base is 10, the logarithm is called the **common logarithm**[9].

$$f(x)=\log_{10}x=\log x \quad \textit{Common logarithm}$$

When a logarithm is written without a base it is assumed to be the common logarithm.

> This convention varies with respect to the subject in which it appears. For example, computer scientists often let $\log\ x$ represent the logarithm base 2.

Example 7.3.3:

Evaluate:

a. $\log 10^5$
b. $\log\sqrt{10}$
c. $\log 0.01$

Solution

a. $\log 10^5=5$ because $10^5=10^5$.
b. $\log\sqrt{10}=\frac{1}{2}$ because $10^{1/2}=\sqrt{10}$.
c. $\log 0.01=\log\left(\frac{1}{100}\right)=-2$ because $10^{-2}=\frac{1}{10^2}=\frac{1}{100}=0.01$

The result of a logarithm is not always apparent. For example, consider $log\, 75$.

$$\begin{array}{c}\log 10=1\\ \log 75=?\\ \log 100=2\end{array}$$

We can see that the result of $log\, 75$ is somewhere between 1 and 2. On most scientific calculators there is a common logarithm button LOG .Use it to find the $log\, 75$ as follows:

$$LOG\ 75=1.87506$$

Therefore, rounded off to the nearest thousandth, $log\ 75\approx 1.875$. As a check, we can use a calculator to verify that $10^\wedge 1.875\approx 75$.

If the base of a logarithm is e, the logarithm is called the **natural logarithm**[10].

$$f(x)=\log_e x=\ln x \quad \textit{Natural logarithm}$$

The natural logarithm is widely used and is often abbreviated $ln\ x$.

Example 7.3.4

Evaluate:

a. $\ln e$
b. $\ln\sqrt[3]{e^2}$
c. $\ln\left(\frac{1}{e^4}\right)$

Solution

a. $\ln e=1$ because $\ln e=\log_e e=1$ and $e^1=e$.
b. $\ln\left(\sqrt[3]{e^2}\right)=\frac{2}{3}$ because $e^{2/3}=\sqrt[3]{e^2}$.
c. $\ln\left(\frac{1}{e^4}\right)=-4$ because $e^{-4}=\frac{1}{e^4}$.

On a calculator you will find a button for the natural logarithm LN.

$$LN\ 75\ =\ 4.317488$$

Therefore, rounded off to the nearest thousandth, $ln\ (75)\approx 4.317$. As a check, we can use a calculator to verify that $e^\wedge 4.317\approx 75$.

Example 7.3.5

Find x. Round answers to the nearest thousandth.

a. $\log x = 3.2$
b. $\ln x = -4$
c. $\log x = -\frac{2}{3}$

Solution

Convert each to exponential form and then use a calculator to approximate the answer.

a. $\log x = 3.2$ is equivalent to $10^{3.2} = x$ and thus $x \approx 1584.893$.
b. $\ln x = -4$ is equivalent to $e^{-4} = x$ and thus $x \approx 0.018$.
c. $\log x = -\frac{2}{3}$ is equivalent to $10^{-2/3} = x$ and thus $x \approx 0.215$.

Exercise 7.3.2

Evaluate: $\ln\left(\frac{1}{\sqrt{e}}\right)$

Answer

$-frac12$

www.youtube.com/v/fLW7_Zv7QME

Graphing Logarithmic Functions

We can use the translations to graph logarithmic functions. When the base $b > 1$, the graph of $f(x) = log_b x$ has the following general shape:

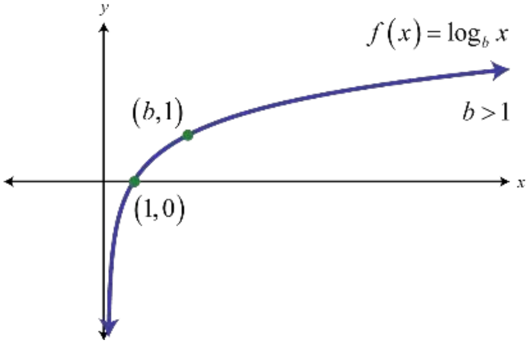

Figure 7.3.6

The domain consists of positive real numbers, $(0, \infty)$ and the range consists of all real numbers, $(-\infty, \infty)$. The y-axis, or $x = 0$, is a vertical asymptote and the x-intercept is $(1, 0)$. In addition, $f(b) = log_b b = 1$ and so $(b, 1)$ is a point on the graph no matter what the base is.

Example 7.3.6

Sketch the graph and determine the domain and range: $f(x) = \log_3(x+4) - 1$.

Solution

Begin by identifying the basic graph and the transformations.

$$\begin{array}{ll} y = \log_3 x & \textit{Basic graph.} \\ y = \log_3(x+4) & \textit{Shift left 4 units.} \\ y = \log_3(x+4) - 1 & \textit{Shift down 1 unit.} \end{array}$$

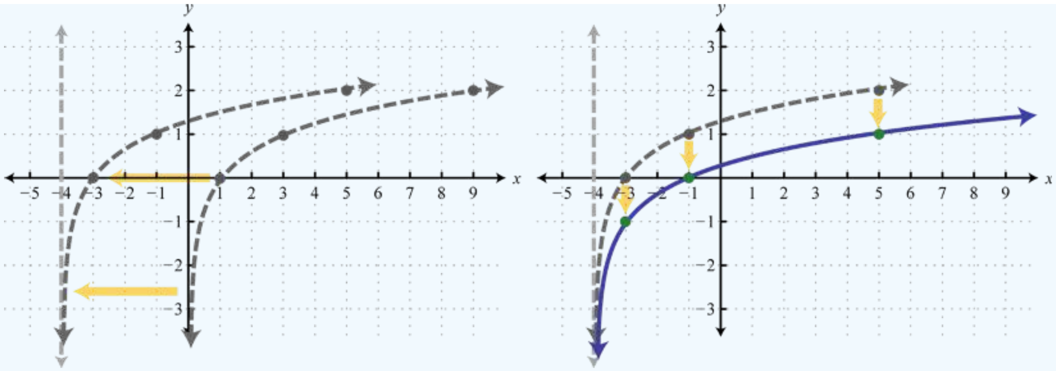

Figure 7.3.7

Notice that the asymptote was shifted 4 units to the left as well. This defines the lower bound of the domain. The final graph is presented without the intermediate steps.

Answer:

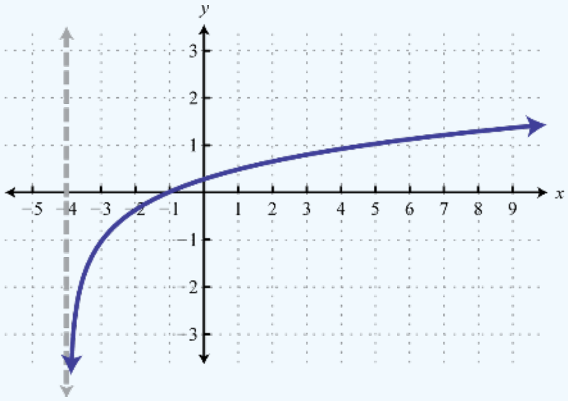

Figure 7.3.8

Domain: $(-4, \infty)$; Range: $(-\infty, \infty)$

Finding the intercepts of the graph in the previous example is left for a later section in this chapter. For now, we are more concerned with the general shape of logarithmic functions.

Example 7.3.7

Sketch the graph and determine the domain and range: $f(x) = -\log(x-2)$.

Solution

Begin by identifying the basic graph and the transformations.

$$\begin{array}{ll} y = \log x & \color{cyan}{\textit{Basic graph.}} \\ y = -\log x & \color{cyan}{\textit{Reflection about the } x-\textit{axis.}} \\ y = -\log(x-2) & \color{cyan}{\textit{Shift right } 2 \textit{ units.}} \end{array}$$

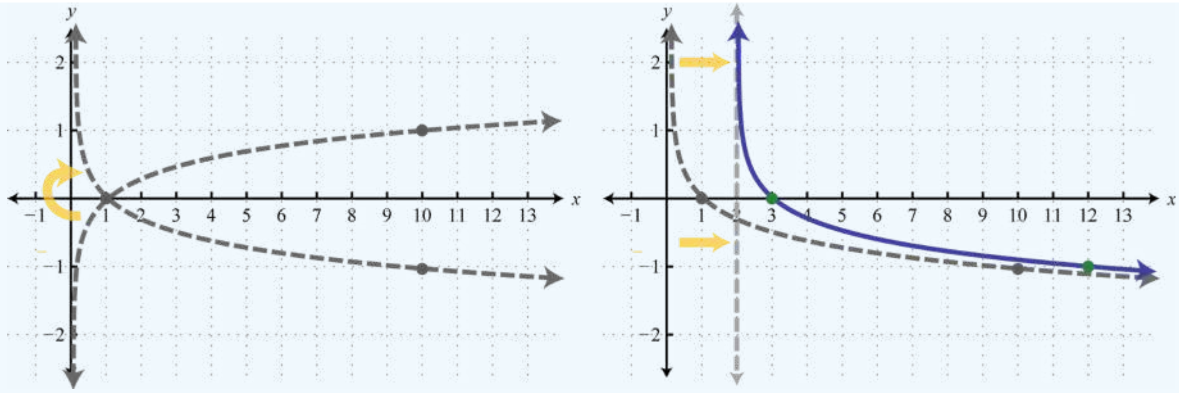

Figure 7.3.9

Here the vertical asymptote was shifted two units to the right. This defines the lower bound of the domain.

Answer

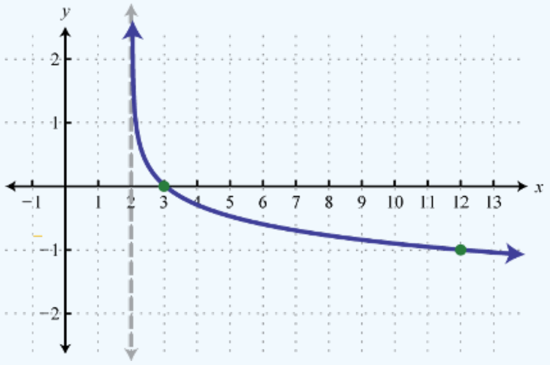

Figure 7.3.10

Domain: $(2, \infty)$; Range: $(-\infty, \infty)$

Exercise 7.3.3

Sketch the graph and determine the domain and range: $g(x) = \ln(-x) + 2$.

Answer

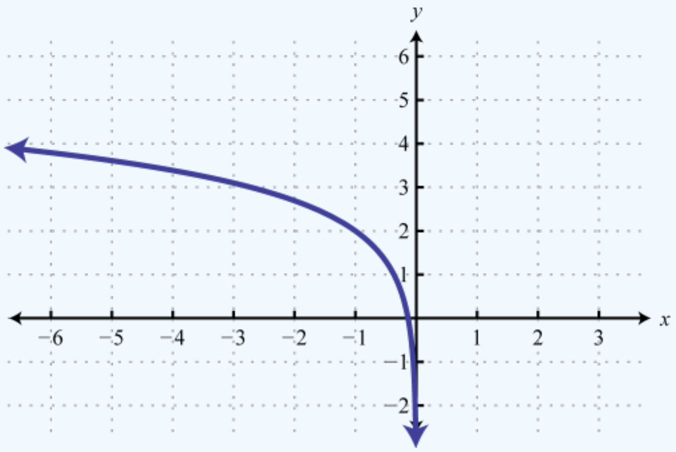

Figure 7.3.11

Domain: $(-\infty, 0)$; Range: $(-\infty, \infty)$

www.youtube.com/v/6obhBJfCDe4

Next, consider exponential functions with fractional bases, such as the function defined by $f(x)=(\frac{1}{2})^x$. The domain consists of all real numbers. Choose some values for x and then find the corresponding y-values.

Table 7.3.2

x	y		*Solutions*
-2	4	$f(-2)=\left(\frac{1}{2}\right)^{-2}=2^2=4$	$(-2,4)$
-1	2	$f(-1)=\left(\frac{1}{2}\right)^{-1}=2^1=2$	$(-1,2)$
0	1	$f(0)=\left(\frac{1}{2}\right)^{0}=1$	$(0,1)$
1	$\frac{1}{2}$	$f(1)=\left(\frac{1}{2}\right)^{1}=\frac{1}{2}$	$\left(1,\frac{1}{2}\right)$
2	$\frac{1}{4}$	$f(2)=\left(\frac{1}{2}\right)^{2}=\frac{1}{4}$	$\left(2,\frac{1}{4}\right)$

Use these points to sketch the graph and note that it passes the horizontal line test.

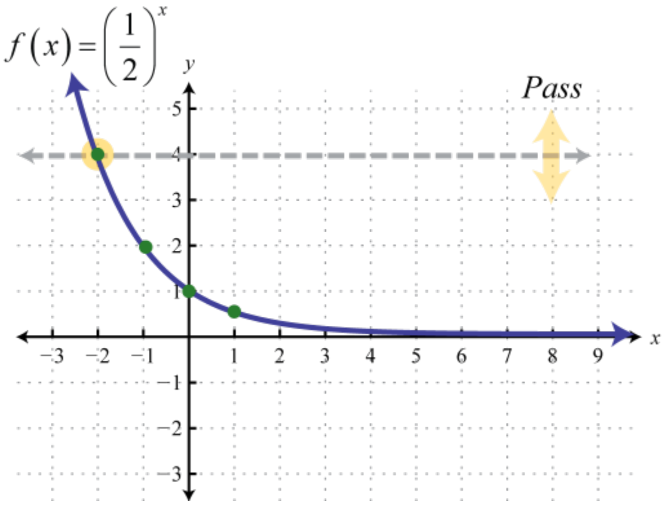

Figure 7.3.12

Therefore this function is one-to-one and has an inverse. Reflecting the graph about the line $y=x$ we have:

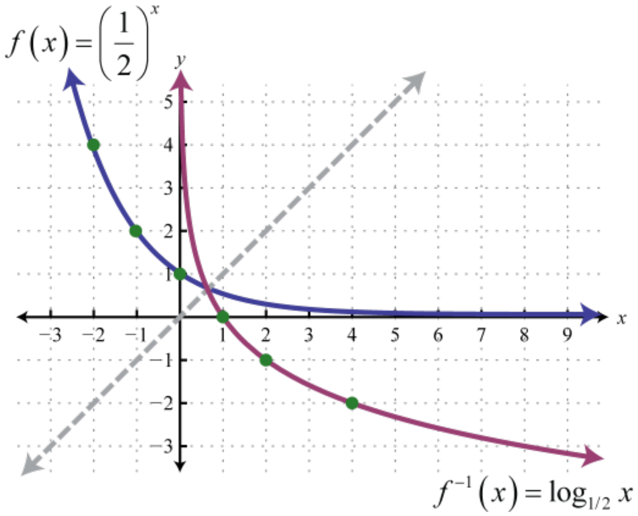

Figure 7.3.13

which gives us a picture of the graph of $f^{-1}(x)=\log_{1/2}x$. In general, when the base $b>1$, the graph of the function defined by $g(x)=\log_{1/b}x$ has the following shape.

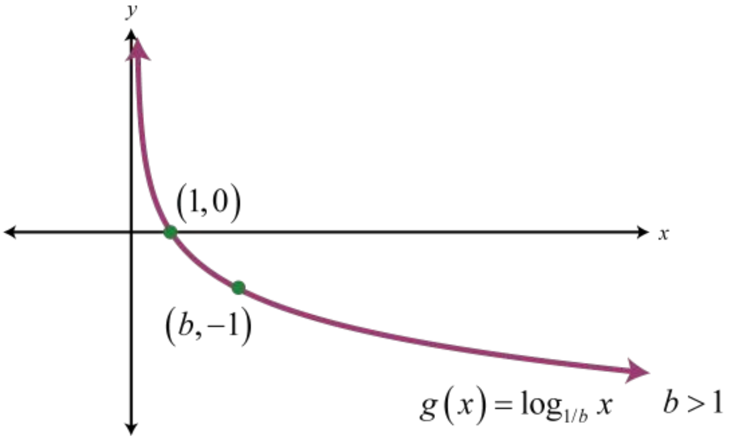

Figure 7.3.14

The domain consists of positive real numbers, $(0, \infty)$ and the range consists of all real numbers, $(-\infty, \infty)$. The y-axis, or $x = 0$, is a vertical asymptote and the x-intercept is $(1, 0)$. In addition, $f(b) = \log_{1/b} b = -1$ and so $(b, -1)$ is a point on the graph.

Example 7.3.8:

Sketch the graph and determine the domain and range: $f(x) = \log_{1/3}(x+3) + 2$.

Solution

Begin by identifying the basic graph and the transformations.

$$\begin{array}{ll} y = \log_{1/3} x & \color{#00a0e0}{\textit{Basic graph.}} \\ y = \log_{1/3}(x+3) & \color{#00a0e0}{\textit{Shift left 3 units.}} \\ y = \log_{1/3}(x+3) + 2 & \color{#00a0e0}{\textit{Shift up 2 units.}} \end{array} \tag{7.3.2}$$

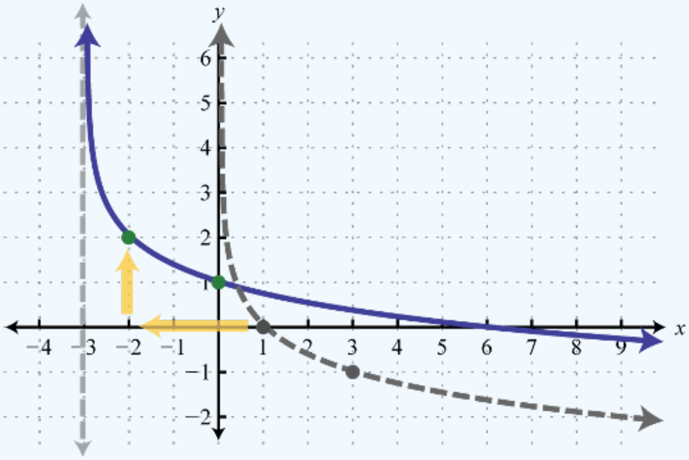

Figure 7.3.15

In this case the shift left 3 units moved the vertical asymptote to $x = -3$ which defines the lower bound of the domain.

Answer:

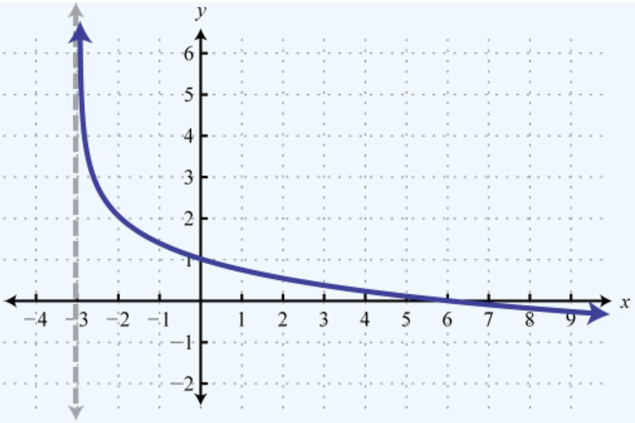

Figure 7.3.16

Domain: $(-3, \infty)$; Range: $(-\infty, \infty)$

In summary, if $b > 1$

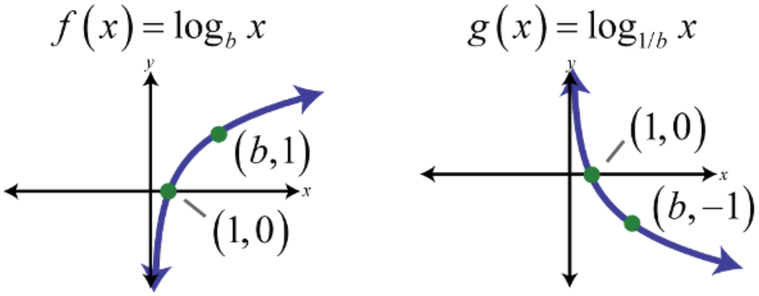

Figure 7.3.17

And for both cases,

$$\begin{aligned} \textit{Domain} &:(0, \infty) \\ \textit{Range} &:(-\infty, \infty) \\ x-\textit{intercept} &:(1,0) \\ \textit{Asymptote} &:x=0 \end{aligned} \tag{7.3.3}$$

Exercise 7.3.4

Sketch the graph and determine the domain and range: $f(x) = \log_{1/3}(x-1)$.

Answer

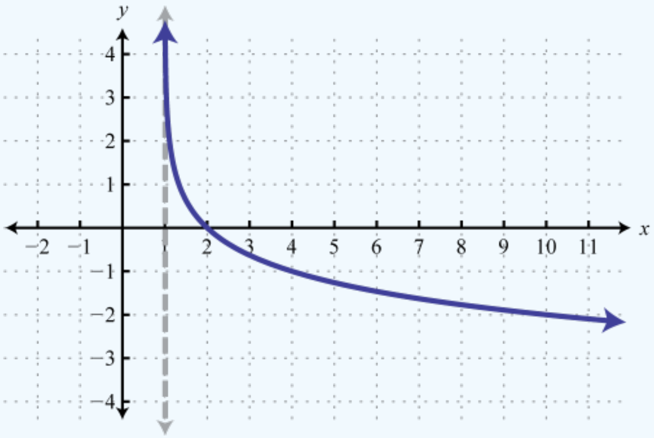

Figure 7.3.18

www.youtube.com/v/0oEVfInCAXw

Key Takeaways

- The base-b logarithmic function is defined to be the inverse of the base-b exponential function. In other words, $y=\log_b x$ if and only if $b^y=x$ where $b>0$ and $b\neq 1$.
- The logarithm is actually the exponent to which the base is raised to obtain its argument.
- The logarithm base 10 is called the common logarithm and is denoted $\log\ x$.
- The logarithm base e is called the natural logarithm and is denoted $\ln\ x$.
- Logarithmic functions with definitions of the form $f(x)=\log_b x$ have a domain consisting of positive real numbers $(0,\infty)$ and a range consisting of all real numbers $(-\infty,\infty)$.
- The y-axis, or $x=0$, is a vertical asymptote and the x-intercept is $(1,0)$.
- To graph logarithmic functions we can plot points or identify the basic function and use the transformations. Be sure to indicate that there is a vertical asymptote by using a dashed line. This asymptote defines the boundary of the domain.

Exercise 7.3.5

Evaluate.

1. $\log_3 9$
2. $\log_7 49$
3. $\log_4 4$
4. $\log_5 1$
5. $\log_5 625$
6. $\log_3 243$
7. $\log_2\left(\frac{1}{16}\right)$
8. $\log_3\left(\frac{1}{9}\right)$
9. $\log_5\left(\frac{1}{125}\right)$
10. $\log_2\left(\frac{1}{64}\right)$
11. $\log_4 4^{10}$
12. $\log_9 9^5$
13. $\log_5 \sqrt[3]{5}$
14. $\log_2 \sqrt{2}$
15. $\log_7\left(\frac{1}{\sqrt{7}}\right)$
16. $\log_9\left(\frac{1}{\sqrt[3]{9}}\right)$
17. $\log_{1/2} 4$
18. $\log_{1/3} 27$
19. $\log_{2/3}\left(\frac{2}{3}\right)$
20. $\log_{3/4}\left(\frac{9}{16}\right)$
21. $\log_{25} 5$
22. $\log_8 2$
23. $\log_4\left(\frac{1}{2}\right)$
24. $\log_{27}\left(\frac{1}{3}\right)$
25. $\log_{1/9} 1$
26. $\log_{3/5}\left(\frac{5}{3}\right)$

Answer

1. 2

3. 1

5. 4

7. -4

9. -3

11. 10

13. $\frac{1}{3}$

15. $-\frac{1}{2}$

17. -2

19. 1

21. $\frac{1}{2}$

23. $-\frac{1}{2}$

25. 0

Exercise $7.3.6$

Find x.

1. $\log_3 x = 4$
2. $\log_2 x = 5$
3. $\log_5 x = -3$
4. $\log_6 x = -2$
5. $\log_{12} x = 0$
6. $\log_7 x = -1$
7. $\log_{1/4} x = -2$
8. $\log_{2/5} x = 2$
9. $\log_{1/9} x = \frac{1}{2}$
10. $\log_{1/4} x = \frac{3}{2}$
11. $\log_{1/3} x = -1$
12. $\log_{1/5} x = 0$

Answer

1. 81

3. $\frac{1}{125}$

5. 1

7. 16

9. $\frac{1}{3}$

11. 3

Exercise $7.3.7$

Evaluate. Round off to the nearest hundredth where appropriate.

1. $\log 1000$
2. $\log 100$
3. $\log 0.1$
4. $\log 0.0001$
5. $\log 162$
6. $\log 23$
7. $\log 0.025$
8. $\log 0.235$
9. $\ln e^4$
10. $\ln 1$
11. $\ln\left(\frac{1}{e}\right)$
12. $\ln\left(\frac{1}{e^5}\right)$
13. $\ln(25)$

14. $\ln(100)$
15. $\ln(0.125)$
16. $\ln(0.001)$

Answer

1. 3
3. -1
5. 2.21
7. -1.60
9. 4
11. -1
13. 3.22
15. -2.08

Exercise 7.3.8

Find x. Round off to the nearest hundredth.

1. $\log x = 2.5$
2. $\log x = 1.8$
3. $\log x = -1.22$
4. $\log x = -0.8$
5. $\ln x = 3.1$
6. $\ln x = 1.01$
7. $\ln x = -0.69$
8. $\ln x = -1$

Answer

1. 316.23
3. 0.06
5. 22.20
7. 0.50

Exercise 7.3.9

Find a without using a calculator.

1. $\log_3\left(\frac{1}{27}\right) = a$
2. $\ln e = a$
3. $\log_2 a = 8$
4. $\log_2 \sqrt[5]{2} = a$
5. $\log 10^{12} = a$
6. $\ln a = 9$
7. $\log_{1/8}\left(\frac{1}{64}\right) = a$
8. $\log_6 a = -3$
9. $\ln a = \frac{1}{5}$
10. $\log_{4/9}\left(\frac{2}{3}\right) = a$

Answer

1. -3

3. 256

5. 12

7. 2

9. $\sqrt[5]{e}$

Exercise 7.3.10

In 1935 Charles Richter developed a scale used to measure earthquakes on a seismograph. The magnitude M of an earthquake is given by the formula,

$$M=\log\left(\frac{I}{I_0}\right)$$

Here I represents the intensity of the earthquake as measured on the seismograph 100 km from the epicenter and I_0 is the minimum intensity used for comparison. For example, if an earthquake intensity is measured to be 100 times that of the minimum, then $I=100I_0$ and

$$M=\log\left(\frac{100I_0}{I_0}\right)=\log(100)=2$$

The earthquake would be said to have a magnitude 2 on the Richter scale. Determine the magnitudes of the following intensities on the Richter scale. Round off to the nearest tenth.

1. I is 3 million times that of the minimum intensity.
2. I is 6 million times that of the minimum intensity.
3. I is the same as the minimum intensity.
4. I is 30 million times that of the minimum intensity.

Answer

1. 6.5

3. 0

Exercise 7.3.11

In chemistry, pH is a measure of acidity and is given by the formula,

$$\mathrm{pH}=-\log\left(H^{+}\right)$$

Here H^{+} represents the hydrogen ion concentration (measured in moles of hydrogen per liter of solution.) Determine the pH given the following hydrogen ion concentrations.

1. Pure water: $H^{+}=0.0000001$
2. Blueberry: $H^{+}=0.0003162$
3. Lemon Juice: $H^{+}=0.01$
4. Battery Acid: $H^{+}=0.1$

Answer

1. 7

3. 2

Exercise 7.3.12

Sketch the function and determine the domain and range. Draw the vertical asymptote with a dashed line.

1. $f(x)=\log_2(x+1)$
2. $f(x)=\log_3(x-2)$
3. $f(x)=\log_2 x-2$
4. $f(x)=\log_3 x+3$

5. $f(x)=\log_2(x-2)+4$
6. $f(x)=\log_3(x+1)-2$
7. $f(x)=-\log_2 x+1$
8. $f(x)=-\log_3(x+3)$
9. $f(x)=\log_2(-x)+1$
10. $f(x)=2-\log_3(-x)$
11. $f(x)=\log x+5$
12. $f(x)=\log x-1$
13. $f(x)=\log(x+4)-8$
14. $f(x)=\log(x-5)+10$
15. $f(x)=-\log(x+2)$
16. $f(x)=-\log(x-1)+2$
17. $f(x)=\ln(x-3)$
18. $f(x)=\ln x+3$
19. $f(x)=\ln(x-2)+4$
20. $f(x)=\ln(x+5)$
21. $f(x)=2-\ln x$
22. $f(x)=-\ln(x-1)$
23. $f(x)=\log_{1/2} x$
24. $f(x)=\log_{1/3} x+2$
25. $f(x)=\log_{1/2}(x-2)$
26. $f(x)=\log_{1/3}(x+1)-1$
27. $f(x)=2-\log_{1/4} x$
28. $f(x)=1+\log_{1/4}(-x)$
29. $f(x)=1-\log_{1/3}(x-2)$
30. $f(x)=1+\log_{1/2}(-x)$

Answer

1. Domain: $(-1,\infty)$; Range: $(-\infty,\infty)$

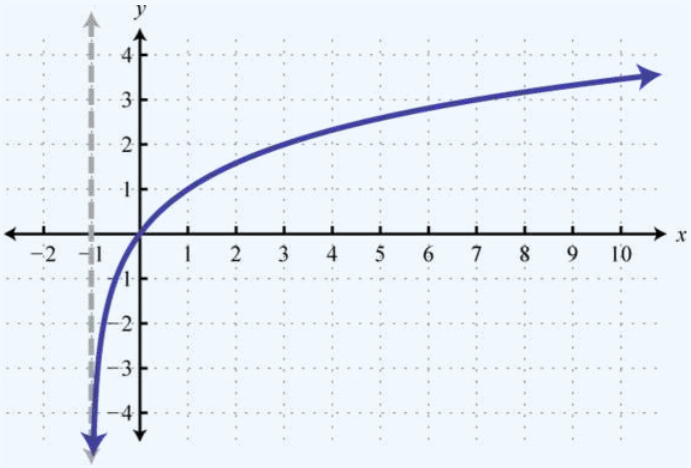

Figure $7.3.19$

3. Domain: $(0,\infty)$; Range: $(-\infty,\infty)$

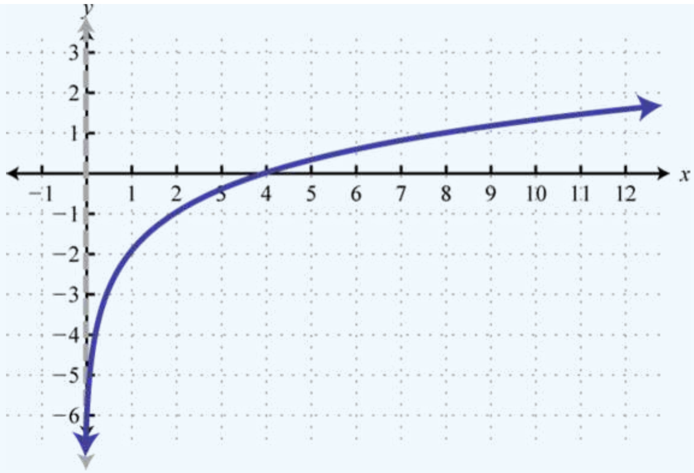

Figure 7.3.20

5. Domain: $(2, \infty)$; Range: $(-\infty, \infty)$

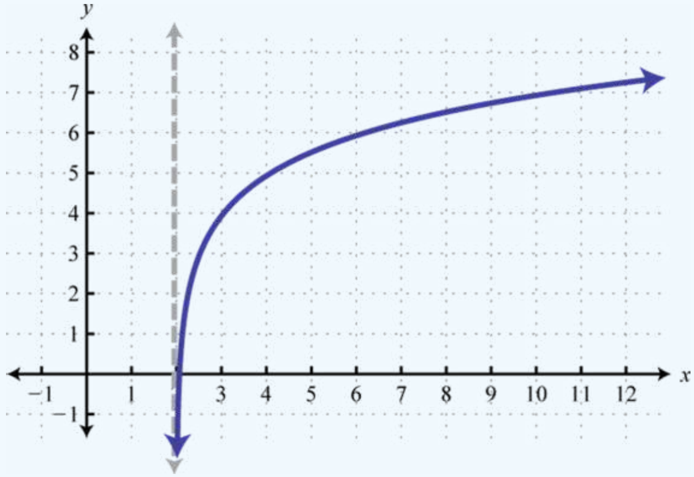

Figure 7.3.21

7. Domain: $(0, \infty)$; Range: $(-\infty, \infty)$

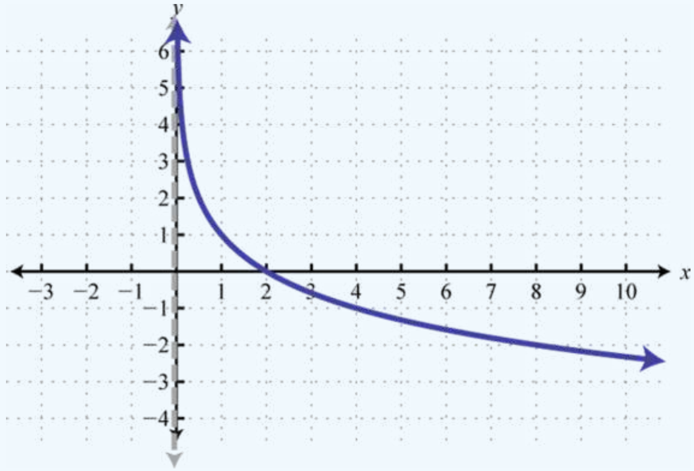

Figure 7.3.22

9. Domain: $(-\infty, 0)$; Range: $(-\infty, \infty)$

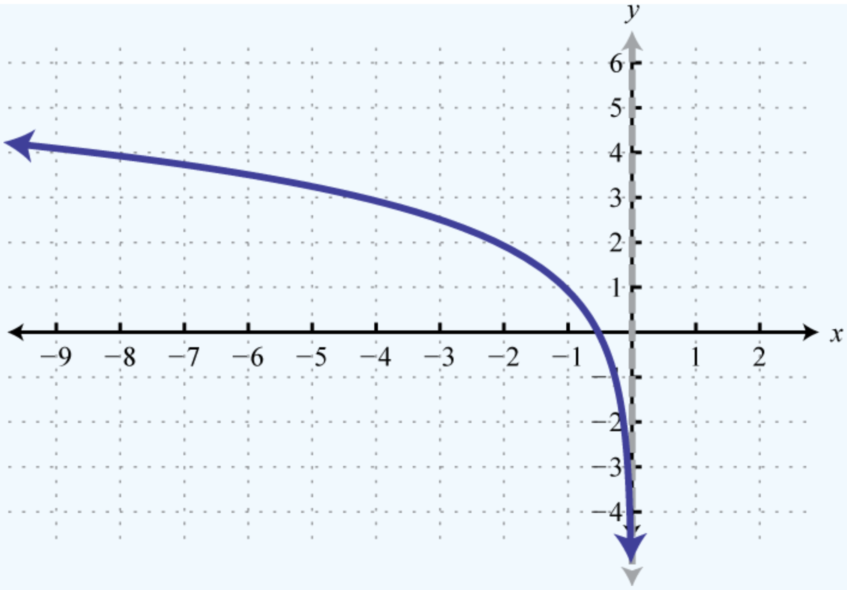

Figure 7.3.23

11. Domain: $(0, \infty)$; Range: $(-\infty, \infty)$

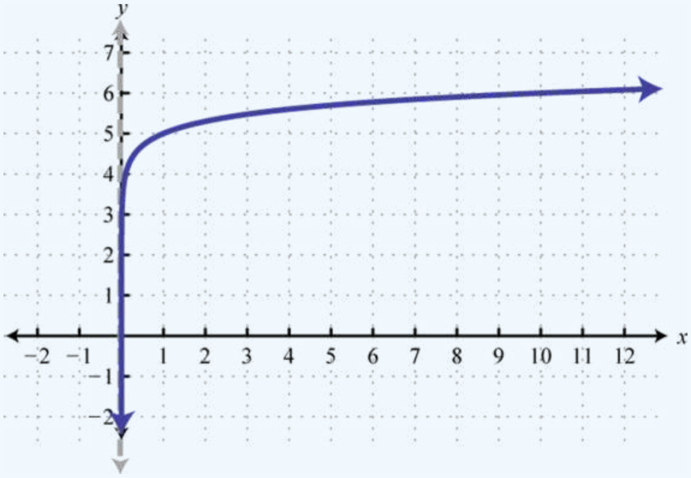

Figure 7.3.24

13. Domain: $(-4, \infty)$; Range: $(-\infty, \infty)$

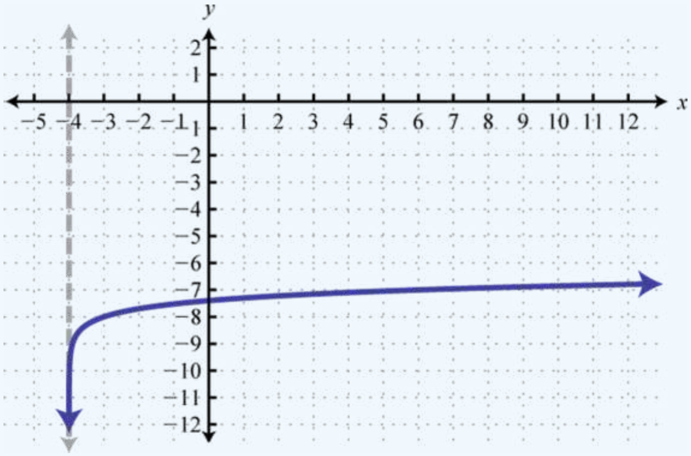

Figure 7.3.25

15. Domain: $(-2, \infty)$; Range: $(-\infty, \infty)$

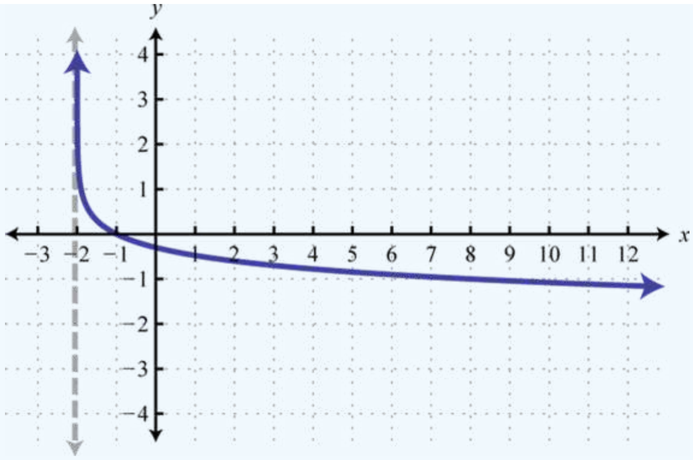

Figure 7.3.26

17. Domain: $(3, \infty)$; Range: $(-\infty, \infty)$

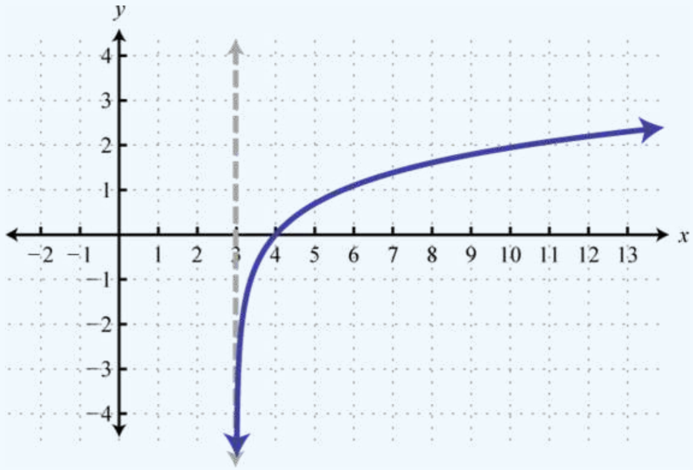

Figure 7.3.27

19. Domain: $(2, \infty)$; Range: $(-\infty, \infty)$

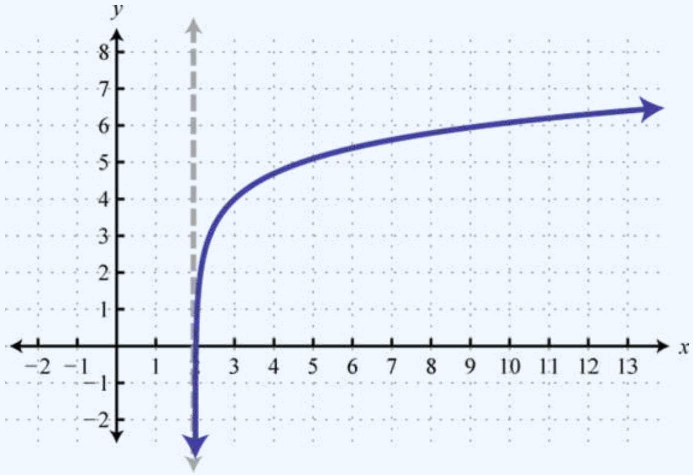

Figure 7.3.28

21. Domain: $(0, \infty)$; Range: $(-\infty, \infty)$

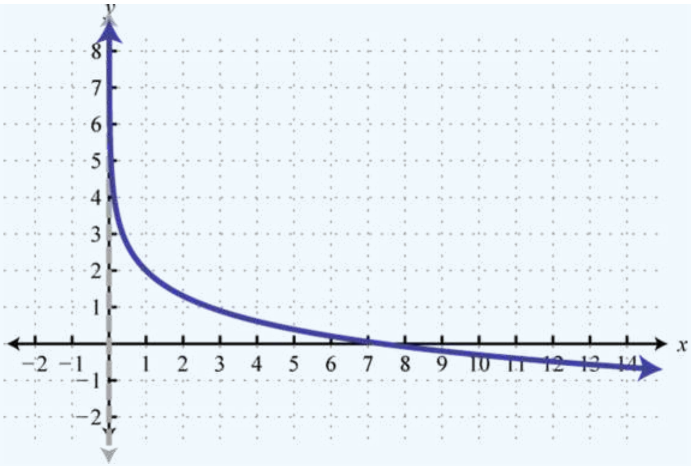

Figure 7.3.29

23. Domain: $(0, \infty)$; Range: $(-\infty, \infty)$

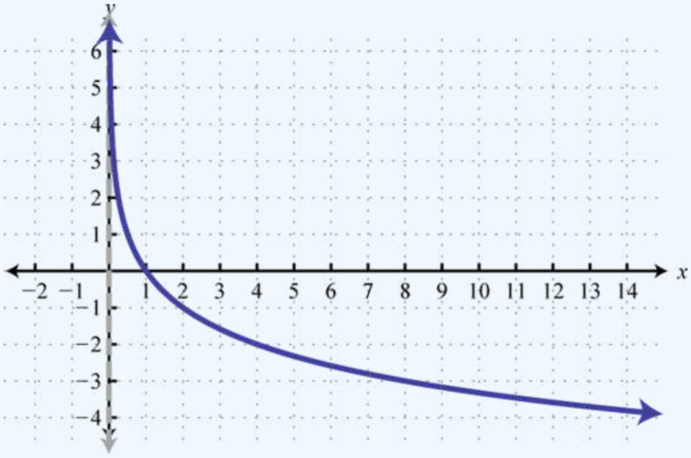

Figure 7.3.30

25. Domain: $(2, \infty)$; Range: $(-\infty, \infty)$

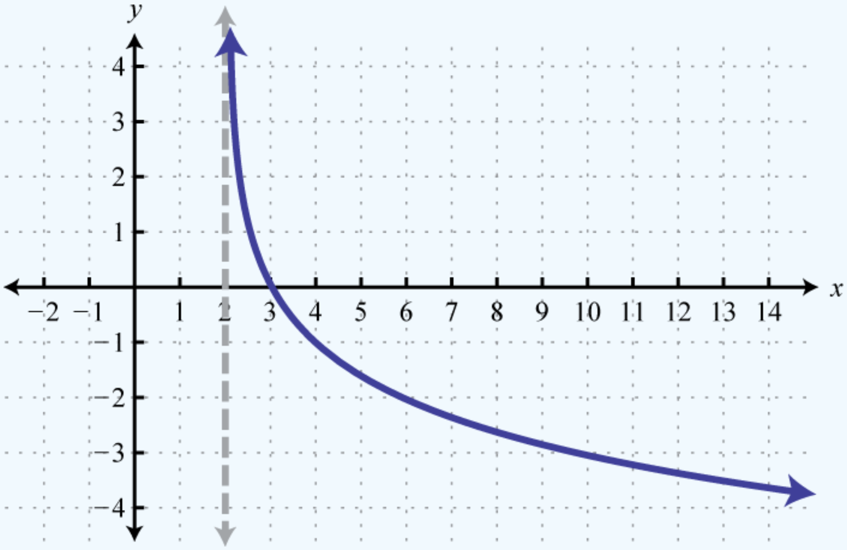

Figure 7.3.31

27. Domain: $(0, \infty)$; Range: $(-\infty, \infty)$

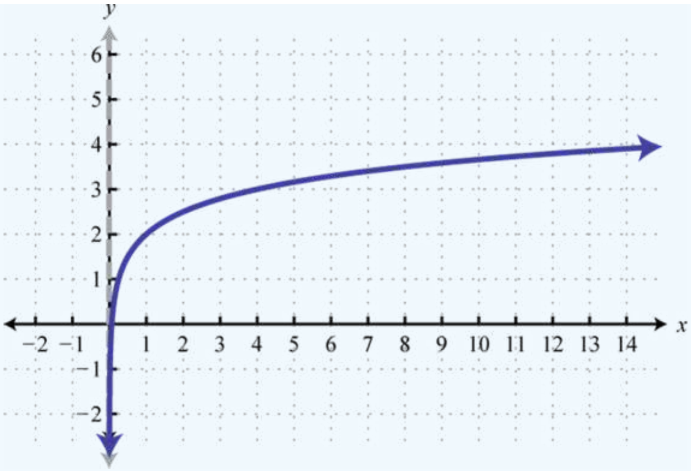

Figure 7.3.32

29. Domain: $(2, \infty)$; Range: $(-\infty, \infty)$

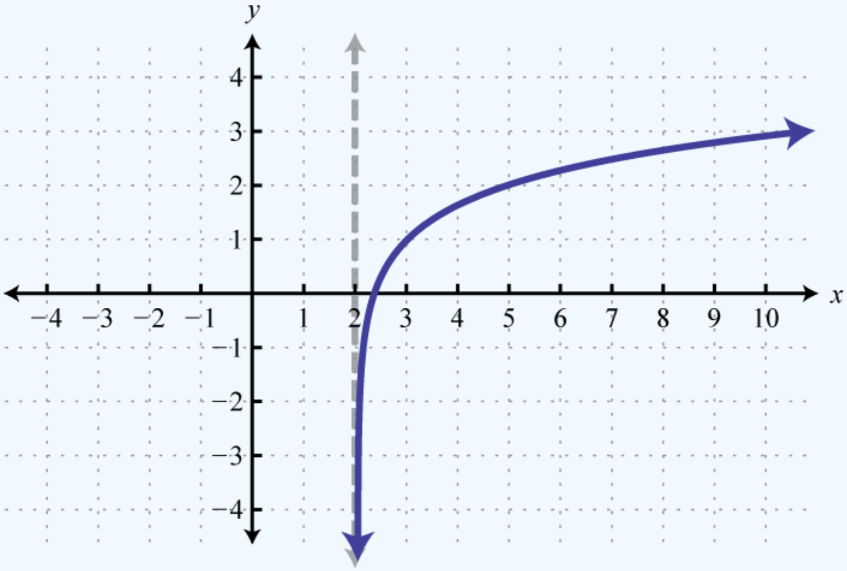

Figure 7.3.33

Exercise 7.3.13

1. Research and discuss the origins and history of the logarithm. How did students work with them before the common availability of calculators?
2. Research and discuss the history and use of the Richter scale. What does each unit on the Richter scale represent?
3. Research and discuss the life and contributions of John Napier.

Answer

1. Answer may vary

3. Answer may vary

Footnotes

[8] The exponent to which the base b is raised in order to obtain a specific value. In other words, $y = log_b x$ is equivalent to $b^y = x$.

[9] The logarithm base 10, denoted $log\ x$.

[10] The logarithm base e, denoted $ln\ x$.

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7.4: Properties of the Logarithm

Learning Objectives

- Apply the inverse properties of the logarithm.
- Expand logarithms using the product, quotient, and power rule for logarithms.
- Combine logarithms into a single logarithm with coefficient 1.

Logarithms and Their Inverse Properties

Recall the definition of the base-b logarithm: given $b>0$ where $b\neq 1$,

$$y=\log_b x \text{ if and only if } x=b^y$$

Use this definition to convert logarithms to exponential form. Doing this, we can derive a few properties:

$$\begin{aligned}\log_b 1=0 &\text{ because } b^0=1\\ \log_b b=1 &\text{ because } b^1=b\\ \log_b\left(\frac{1}{b}\right)=-1 &\text{ because } b^{-1}=\frac{1}{b}\end{aligned}$$

Example 7.4.1

Evaluate:

a. $\log 1$
b. $\ln e$
c. $\log_5\left(\frac{1}{5}\right)$

Solution

a. When the base is not written, it is assumed to be 10. This is the common logarithm,

$$\log 1=\log_{10} 1=0$$

b. The natural logarithm, by definition, has base e,

$$\ln e=\log_e e=1$$

c. Because $5^{-1}=\frac{1}{5}$ we have,

$$\log_5\left(\frac{1}{5}\right)=-1$$

Furthermore, consider fractional bases of the form $1/b$ where $b>1$.

$$\log_{1/b} b=-1 \quad \text{because } \left(\frac{1}{b}\right)^{-1}=\frac{1^{-1}}{b^{-1}}=\frac{b}{1}=b$$

Example 7.4.2

Evaluate:

a. $\log_{1/4} 4$
b. $\log_{2/3}\left(\frac{3}{2}\right)$

Solution

a. $\log_{1/4} 4=-1 \quad \text{because} \quad \left(\frac{1}{4}\right)^{-1}=4$
b. $\log_{2/3}\left(\frac{3}{2}\right)=-1$ because $\left(\frac{2}{3}\right)^{-1}=\frac{3}{2}$

Given an exponential function defined by $f(x)=b^x$, where $b>0$ and $b\neq 1$, its inverse is the base-b logarithm, $f^{-1}(x)=log_b x$. And because $f\left(f^{-1}(x)\right)=x$ and $f^{-1}(f(x))=x$, we have the following **inverse properties of the logarithm**[11]:

$$\begin{aligned}f^{-1}(f(x))&=\log_b b^x=x \text{ and}\\ f\left(f^{-1}(x)\right)&=b^{\log_b x}=x, x>0\end{aligned}$$

Since $f^{-1}(x) = \log_b x$ has a domain consisting of positive values $(0, \infty)$, the property $b^{\log_b x} = x$ is restricted to values where $x > 0$.

Example 7.4.3

Evaluate

a. $\log_5 625$
b. $5^{\log_5 3}$
c. $e^{\ln 5}$

Solution

Apply the inverse properties of the logarithm.

a. $\log_5 625 = \log_5 5^4 = 4$
b. $5^{\log_5 3} = 3$
c. $e^{\ln 5} = 5$

In summary, when $b > 0$ and $b \neq 1$, we have the following properties:

Table 7.4.1

$\log_b 1 = 0$	$\log_b b = 1$
$\log_{1/b} b = -1$	$\log_b \left(\frac{1}{b}\right) = -1$
$\log_b b^x = x$	$b^{\log_b x} = x, x > 0$

Exercise 7.4.1

Evaluate: $\log 0.00001$

Answer

-5

www.youtube.com/v/YFsZxKiMCTg

Product, Quotient, and Power Properties of Logarithms

In this section, three very important properties of the logarithm are developed. These properties will allow us to expand our ability to solve many more equations. We begin by assigning u and v to the following logarithms and then write them in exponential form:

$$\log_b x = u \Longrightarrow b^u = x$$
$$\log_b y = v \Longrightarrow b^v = y$$

Substitute $x = b^u$ and $y = b^v$ into the logarithm of a product $log_b(xy)$ and the logarithm of a quotient $\log_b\left(\frac{x}{y}\right)$.Then simplify using the rules of exponents and the inverse properties of the logarithm.

Table 7.4.2

Logarithm of a Product	Logarithm of a Quotient
$\begin{aligned} \log_b(xy) &= \log_b(b^u b^v) \\ &= \log_b b^{u+v} \\ &= u+v \\ &= \log_b x + \log_b y \end{aligned}$	$\begin{aligned} \log_b\left(\frac{x}{y}\right) &= \log_b\left(\frac{b^u}{b^v}\right) \\ &= \log_b b^{u-v} \\ &= u-v \\ &= \log_b x - \log_b y \end{aligned}$

This gives us two essential properties: the **product property of logarithms**[12],

$$\log_b(xy) = \log_b x + \log_b y$$

and the **quotient property of logarithms**[13],

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

In words, the logarithm of a product is equal to the sum of the logarithm of the factors. Similarly, the logarithm of a quotient is equal to the difference of the logarithm of the numerator and the logarithm of the denominator.

Example 7.4.4

Write as a sum $\log_2(8x)$.

Solution

Apply the product property of logarithms and then simplify.

$$\begin{aligned}\log_2(8x) &= \log_2 8 + \log_2 x \\ &= \log_2 2^3 + \log_2 x \\ &= 3 + \log_2 x\end{aligned}$$

Answer

$$3 + \log_2 x$$

Example 7.4.5

Write as a difference $\log\left(\frac{x}{10}\right)$.

Solution

Apply the quotient property of logarithms and then simplify.

$$\begin{aligned}\log\left(\frac{x}{10}\right) &= \log x - \log 10 \\ &= \log x - 1\end{aligned}$$

Answer

$$\log x - 1$$

Next we begin with $log_b x = u$ and rewrite it in exponential form. After raising both sides to the nth power, convert back to logarithmic form, and then back substitute.

$$\begin{aligned}\log_b x = u &\Longrightarrow b^u = x \\ (b^u)^n &= (x)^n \\ \log_b x^n = nu &\Longleftarrow b^{nu} = x^n \\ \log_b x^n &= n \log_b x\end{aligned}$$

This leads us to the **power property of logarithms**[14],

$$\log_b x^n = n \log_b x$$

In words, the logarithm of a quantity raised to a power is equal to that power times the logarithm of the quantity.

Example 7.4.6

Write as a product:

a. $\log_2 x^4$
b. $\log_5(\sqrt{x})$

Solution

a. Apply the power property of logarithms.

$$\log_2 x^4 = 4 \log_2 x \tag{7.4.1}$$

b. Recall that a square root can be expressed using rational exponents, $\sqrt{x} = x^{1/2}$. Make this replacement and then apply the power property of logarithms.

$$\begin{aligned}\log_5(\sqrt{x}) &= \log_5 x^{1/2} \\ &= \frac{1}{2}\log_5 x\end{aligned}$$

In summary,

Table 7.4.3

Product Property of Logarithms	$\log_b(xy) = \log_b x + \log_b y$
Quotient Property of Logarithms	$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$
Power Property of Logarithms	$\log_b x^n = n \log_b x$

We can use these properties to expand logarithms involving products, quotients, and powers using sums, differences and coefficients. A logarithmic expression is completely expanded when the properties of the logarithm can no further be applied.

Caution

It is important to point out the following:

$$\log(xy) \neq \log x \cdot \log y \tag{7.4.2}$$

and

$$\log\left(\frac{x}{y}\right) \neq \frac{\log x}{\log y} \tag{7.4.3}$$

Example 7.4.7

Expand completely: $\ln\left(2x^3\right)$.

Solution

Recall that the natural logarithm is a logarithm base e, $\ln x = \log_e x$. Therefore, all of the properties of the logarithm apply.

$$\begin{aligned} \ln\left(2x^3\right) &= \ln 2 + \ln x^3 && \textit{Product rule for logarithms} \\ &= \ln 2 + 3\ln x && \textit{Power rule for logarithms} \end{aligned}$$

Answer

$$\ln 2 + 3\ln x$$

Example 7.4.8:

Expand completely: $\log \sqrt[3]{10xy^2}$.

Solution

Begin by rewriting the cube root using the rational exponent $\frac{1}{3}$ and then apply the properties of the logarithm.

$$\begin{aligned} \log \sqrt[3]{10xy^2} &= \log\left(10xy^2\right)^{1/3} \\ &= \frac{1}{3}\log\left(10xy^2\right) \\ &= \frac{1}{3}\left(\log 10 + \log x + \log y^2\right) \\ &= \frac{1}{3}(1 + \log x + 2\log y) \\ &= \frac{1}{3} + \frac{1}{3}\log x + \frac{2}{3}\log y \end{aligned}$$

Answer

$$\frac{1}{3} + \frac{1}{3}\log x + \frac{2}{3}\log y$$

Example 7.4.9:

Expand completely: $\log_2\left(\frac{(x+1)^2}{5y}\right)$.

Solution

When applying the product property to the denominator, take care to distribute the negative obtained from applying the quotient property.

$$\begin{aligned}\log_2\left(\frac{(x+1)^2}{5y}\right) &= \log_2(x+1)^2 - \log_2(5y) \\ &= \log_2(x+1)^2 - (\log_2 5 + \log_2 y) \quad \textit{Distribute.} \\ &= \log_2(x+1)^2 - \log_2 5 - \log_2 y \\ &= 2\log_2(x+1) - \log_2 5 - \log_2 y\end{aligned}$$

Answer

$$2\log_2(x+1) - \log_2 5 - \log_2 y$$

Caution

There is no rule that allows us to expand the logarithm of a sum or difference. In other words,

$$\log(x \pm y) \neq \log x \pm \log y$$

Exercise 7.4.2

Expand completely: $\ln\left(\frac{5y^4}{\sqrt{x}}\right)$

Answer

$\ln 5 + 4\ln y - \frac{1}{2}\ln x$

www.youtube.com/v/h_Zu-SkATl0

Example 7.4.10:

Given that $\log_2 x = a$, $\log_2 y = b$, and that $\log_2 z = c$, write the following in terms of a, b, and c:

a. $\log_2(8x^2y)$
b. $\log_2\left(\frac{2x^4}{\sqrt{z}}\right)$

Solution

a. Begin by expanding using sums and coefficients and then replace a and b with the appropriate logarithm.

$$\begin{aligned}\log_2(8x^2y) &= \log_2 8 + \log_2 x^2 + \log_2 y \\ &= \log_2 8 + 2\log_2 x + \log_2 y \\ &= 3 + 2a + b\end{aligned} \tag{7.4.4}$$

b. Expand and then replace a, b, and c where appropriate.

$$\begin{aligned}\log_2\left(\frac{2x^4}{\sqrt{z}}\right) &= \log_2(2x^4) - \log_2 z^{1/2} \\ &= \log_2 2 + \log_2 x^4 - \log_2 z^{1/2} \\ &= \log_2 2 + 4\log_2 x - \frac{1}{2}\log_2 z \\ &= 1 + 4a - \frac{1}{2}b\end{aligned} \tag{7.4.5}$$

Next we will condense logarithmic expressions. As we will see, it is important to be able to combine an expression involving logarithms into a single logarithm with coefficient 1. This will be one of the first steps when solving logarithmic equations.

Example 7.4.11:

Write as a single logarithm with coefficient 1: $3\log_3 x - \log_3 y + 2\log_3 5$.

Solution

Begin by rewriting all of the logarithmic terms with coefficient 1. Use the power rule to do this. Then use the product and quotient rules to simplify further.

$$\begin{aligned} 3\log_3 x - \log_3 y + 2\log_3 5 &= \{\log_3 x^3 - \log_3 y\} + \log_3 5^2 && \textit{quotient property} \\ &= \left\{\log_3\left(\frac{x^3}{y}\right) + \log_3 25\right\} && \textit{product property} \\ &= \log_3\left(\frac{x^3}{y}\cdot 25\right) \\ &= \log_3\left(\frac{25x^3}{y}\right) \end{aligned}$$

Answer

$$\log_3\left(\frac{25x^3}{y}\right)$$

Example 7.4.12:

Write as a single logarithm with coefficient 1: $\frac{1}{2}\ln x - 3\ln y - \ln z$.

Solution

Begin by writing the coefficients of the logarithms as powers of their argument, after which we will apply the quotient rule twice working from left to right.

$$\begin{aligned} \frac{1}{2}\ln x - 3\ln y - \ln z &= \ln x^{1/2} - \ln y^3 - \ln z \\ &= \ln\left(\frac{x^{1/2}}{y^3}\right) - \ln z \\ &= \ln\left(\frac{x^{1/2}}{y^3} \div z\right) \\ &= \ln\left(\frac{x^{1/2}}{y^3}\cdot\frac{1}{z}\right) \\ &= \ln\left(\frac{x^{1/2}}{y^3 z}\right) \quad \text{or} \quad = \ln\left(\frac{\sqrt{x}}{y^3 z}\right) \end{aligned}$$

Answer

$$\ln\left(\frac{\sqrt{x}}{y^3 z}\right)$$

Exercise 7.4.3

Write as a single logarithm with coefficient 1: $3\log(x+y) - 6\log z + 2\log 5$

Answer

$\log\left(\frac{25(x+y)^3}{z^6}\right)$

www.youtube.com/v/o1AVbmS9AU8

Key Takeaways

- Given any base $b > 0$ and $b \neq 1$, we can say that $log_b 1 = 0$, $log_b b = 1$, $log_{1/b} b = -1$ and that $log_b(\frac{1}{b}) = -1$.
- The inverse properties of the logarithm are $log_b b^x = x$ and $b^{log_b x} = x$ where $x > 0$.
- The product property of the logarithm allows us to write a product as a sum: $\log_b(xy) = \log_b x + \log_b y$.
- The quotient property of the logarithm allows us to write a quotient as a difference: $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$.
- The power property of the logarithm allows us to write exponents as coefficients: $\log_b x^n = n\log_b x$.
- Since the natural logarithm is a base-e logarithm, $\ln x = \log_e x$, all of the properties of the logarithm apply to it.
- We can use the properties of the logarithm to expand logarithmic expressions using sums, differences, and coefficients. A logarithmic expression is completely expanded when the properties of the logarithm can no further be applied.

- We can use the properties of the logarithm to combine expressions involving logarithms into a single logarithm with coefficient 1. This is an essential skill to be learned in this chapter.

Exercise 7.4.4

Evaluate:

1. $\log_7 1$
2. $\log_{1/2} 2$
3. $\log 10^{14}$
4. $\log 10^{-23}$
5. $\log_3 3^{10}$
6. $\log_6 6$
7. $\ln e^7$
8. $\ln\left(\frac{1}{e}\right)$
9. $\log_{1/2}\left(\frac{1}{2}\right)$
10. $\log_{1/5} 5$
11. $\log_{3/4}\left(\frac{4}{3}\right)$
12. $\log_{2/3} 1$
13. $2^{\log_2 100}$
14. $3^{\log_3 1}$
15. $10^{\log 18}$
16. $e^{\ln 23}$
17. $e^{\ln x^2}$
18. $e^{\ln e^x}$

Answer

1. 0

3. 14

5. 10

7. 7

9. 1

11. -1

13. 100

15. 18

17. x^2

Exercise 7.4.5

Find a:

1. $\ln a = 1$
2. $\log a = -1$
3. $\log_9 a = -1$
4. $\log_{12} a = 1$
5. $\log_2 a = 5$
6. $\log a = 13$
7. $2^a = 7$
8. $e^a = 23$
9. $\log_a 4^5 = 5$
10. $\log_a 10 = 1$

Answer

1. e
3. $\frac{1}{9}$
5. $2^5 = 32$
7. $\log_2 7$
9. 4

Exercise 7.4.6

Expand completely.

1. $\log_4(xy)$
2. $\log(6x)$
3. $\log_3\left(9x^2\right)$
4. $\log_2\left(32x^7\right)$
5. $\ln\left(3y^2\right)$
6. $\log\left(100x^2\right)$
7. $\log_2\left(\frac{x}{y^2}\right)$
8. $\log_5\left(\frac{25}{x}\right)$
9. $\log\left(10x^2y^3\right)$
10. $\log_2\left(2x^4y^5\right)$
11. $\log_3\left(\frac{x^3}{yz^2}\right)$
12. $\log\left(\frac{x}{y^3z^2}\right)$
13. $\log_5\left(\frac{1}{x^2yz}\right)$
14. $\log_4\left(\frac{1}{16x^2z^3}\right)$
15. $\log_6\left[36(x+y)^4\right]$
16. $\ln\left[e^4(x-y)^3\right]$
17. $\log_7(2\sqrt{xy})$
18. $\ln(2x\sqrt{y})$
19. $\log_3\left(\frac{x^2\sqrt[3]{y}}{z}\right)$
20. $\log\left(\frac{2(x+y)^3}{z^2}\right)$
21. $\log\left(\frac{100x^3}{(y+10)^3}\right)$
22. $\log_7\left(\frac{x}{\sqrt[5]{(y+z)^3}}\right)$
23. $\log_5\left(\frac{x^3}{\sqrt[3]{yz^2}}\right)$
24. $\log\left(\frac{x^2}{\sqrt[5]{y^3z^2}}\right)$

Answer

1. $\log_4 x + \log_4 y$
3. $2 + 2\log_3 x$
5. $\ln 3 + 2\ln y$
7. $\log_2 x - 2\log_2 y$
9. $1 + 2\log x + 3\log y$
11. $3\log_3 x - \log_3 y - 2\log_3 z$
13. $-2\log_5 x - \log_5 y - \log_5 z$
15. $2 + 4\log_6(x+y)$
17. $\log_7 2 + \frac{1}{2}\log_7 x + \frac{1}{2}\log_7 y$

19. $2\log_3 x + \frac{1}{3}\log_3 y - \log_3 z$

21. $2 + 3\log x - 3\log(y+10)$

23. $3\log_5 x - \frac{1}{3}\log_5 y - \frac{2}{3}\log_5 z$

Exercise 7.4.7

Given $\log_3 x = a, \log_3 y = b$, and $\log_3 z = c$, write the following logarithms in terms of a, b, and and c.

1. $\log_3\left(27x^2y^3z\right)$
2. $\log_3\left(xy^3\sqrt{z}\right)$
3. $\log_3\left(\frac{9x^2y}{z^3}\right)$
4. $\log_3\left(\frac{\sqrt[3]{x}}{yz^2}\right)$

Answer

1. $3 + 2a + 3b + c$

3. $2 + 2a + b - 3c$

Exercise 7.4.8

Given $\log_b 2 = 0.43, \log_b 3 = 0.68$, and $\log_b 7 = 1.21$, calculate the following. (Hint: Expand using sums, differences, and quotients of the factors $2, 3$, and 7.)

1. $\log_b 42$
2. $\log_b(36)$
3. $\log_b\left(\frac{28}{9}\right)$
4. $\log_b\sqrt{21}$

Answer

1. 2.32

3. 0.71

Exercise 7.4.9

Expand using the properties of the logarithm and then approximate using a calculator to the nearest tenth.

1. $\log\left(3.10 \times 10^{25}\right)$
2. $\log\left(1.40 \times 10^{-33}\right)$
3. $\ln\left(6.2e^{-15}\right)$
4. $\ln\left(1.4e^{22}\right)$

Answer

1. $\log(3.1) + 25 \approx 25.5$

3. $\ln(6.2) - 15 \approx -13.2$

Exercise 7.4.10

Write as a single logarithm with coefficient 1.

1. $\log x + \log y$
2. $\log_3 x - \log_3 y$
3. $\log_2 5 + 2\log_2 x + \log_2 y$
4. $\log_3 4 + 3\log_3 x + \frac{1}{2}\log_3 y$
5. $3\log_2 x - 2\log_2 y + \frac{1}{2}\log_2 z$

6. $4\log x-\log y-\log 2$
7. $\log 5+3\log(x+y)$
8. $4\log_5(x+5)+\log_5 y$
9. $\ln x-6\ln y+\ln z$
10. $\log_3 x-2\log_3 y+5\log_3 z$
11. $7\log x-\log y-2\log z$
12. $2\ln x-3\ln y-\ln z$
13. $\frac{2}{3}\log_3 x-\frac{1}{2}\left(\log_3 y+\log_3 z\right)$
14. $\frac{1}{5}\left(\log_7 x+2\log_7 y\right)-2\log_7(z+1)$
15. $1+\log_2 x-\frac{1}{2}\log_2 y$
16. $2-3\log_3 x+\frac{1}{3}\log_3 y$
17. $\frac{1}{3}\log_2 x+\frac{2}{3}\log_2 y$
18. $-2\log_5 x+\frac{3}{5}\log_5 y$
19. $-\ln 2+2\ln(x+y)-\ln z$
20. $-3\ln(x-y)-\ln z+\ln 5$
21. $\frac{1}{3}(\ln x+2\ln y)-(3\ln 2+\ln z)$
22. $4\log 2+\frac{2}{3}\log x-4\log(y+z)$
23. $\log_2 3-2\log_2 x+\frac{1}{2}\log_2 y-4\log_2 z$
24. $2\log_5 4-\log_5 x-3\log_5 y+\frac{2}{3}\log_5 z$

Answer

1. $\log(xy)$

3. $\log_2\left(5x^2y\right)$

5. $\log_2\left(\frac{x^3\sqrt{z}}{y^2}\right)$

7. $\log\left[5(x+y)^3\right]$

9. $\ln\left(\frac{xz}{y^6}\right)$

11. $\log\left(\frac{x^7}{yz^2}\right)$

13. $\log_3\left(\frac{\sqrt[3]{x^2}}{\sqrt{yz}}\right)$

15. $\log_2\left(\frac{2x}{\sqrt{y}}\right)$

17. $\log_2\left(\sqrt[3]{xy^2}\right)$

19. $\ln\left(\frac{(x+y)^2}{2z}\right)$

21. $\ln\left(\frac{\sqrt[3]{xy^2}}{8z}\right)$

23. $\log_2\left(\frac{3\sqrt{y}}{x^2z^4}\right)$

Exercise 7.4.11

Express as a single logarithm and simplify.

1. $\log(x+1)+\log(x-1)$
2. $\log_2(x+2)+\log_2(x+1)$
3. $\ln\left(x^2+2x+1\right)-\ln(x+1)$
4. $\ln\left(x^2-9\right)-\ln(x+3)$
5. $\log_5\left(x^3-8\right)-\log_5(x-2)$
6. $\log_3\left(x^3+1\right)-\log_3(x+1)$
7. $\log x+\log(x+5)-\log\left(x^2-25\right)$
8. $\log(2x+1)+\log(x-3)-\log\left(2x^2-5x-3\right)$

Answer

1. $\log\left(x^{2}-1\right)$
3. $\ln(x+1)$
5. $\log_{5}\left(x^{2}+2x+4\right)$
7. $\log\left(\frac{x}{x-5}\right)$

Footnotes

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[11] Given $b>0$ we have $\log_{b} b^{x}=x$ and $b^{\log_{b} x}=x$ when $x>0$.

[12] $\log_{b}(xy)=\log_{b} x+\log_{b} y$; the logarithm of a product is equal to the sum of the logarithm of the factors.

[13] $\log_{b}\left(\frac{x}{y}\right)=\log_{b} x-\log_{b} y$; the logarithm of a quotient is equal to the difference of the logarithm of the numerator and the logarithm of the denominator.

[14] $\log_{b} x^{n}=n \log_{b} x$; the logarithm of a quantity raised to a power is equal to that power times the logarithm of the quantity.]

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7.5: Solving Exponential and Logarithmic Equations

Learning Objectives

- Solve exponential equations.
- Use the change of base formula to approximate logarithms.
- Solve logarithmic equations.

Solving Exponential Equations

An **exponential equation**[15] is an equation that includes a variable as one of its exponents. In this section we describe two methods for solving exponential equations. First, recall that exponential functions defined by $f(x) = b^x$ where $b > 0$ and $b \neq 1$, are one-to-one; each value in the range corresponds to exactly one element in the domain. Therefore, $f(x) = f(y)$ implies $x = y$. The converse is true because f is a function. This leads to the very important **one-to-one property of exponential functions**[16]:

$$b^{\mathrm{x}} = b^{\mathrm{y}} \quad \text{if and only if} \quad x = y$$

Use this property to solve special exponential equations where each side can be written in terms of the same base.

Example $7.5.1$:

Solve $3^{2x-1} = 27$.

Solution

Begin by writing 27 as a power of 3.

$$\begin{aligned} 3^{2x-1} &= 27 \\ 3^{2x-1} &= 3^3 \end{aligned}$$

Next apply the one-to-one property of exponential functions. In other words, set the exponents equal to each other and then simplify.

$$\begin{aligned} 2x-1 &= 3 \\ 2x &= 4 \\ x &= 2 \end{aligned}$$

Answer:

2

Example $7.5.2$:

Solve: $16^{1-3x} = 2$.

Solution

Begin by writing 16 as a power of 2 and then apply the power rule for exponents.

$$\begin{aligned} 16^{1-3x} &= 2 \\ \left(2^4\right)^{1-3x} &= 2 \\ 2^{4(1-3x)} &= 2^1 \end{aligned}$$

Now that the bases are the same we can set the exponents equal to each other and simplify.

$$\begin{aligned} 4(1-3x) &= 1 \\ 4-12x &= 1 \\ -12x &= -3 \\ x &= \frac{-3}{-12} = \frac{1}{4} \end{aligned}$$

Answer:

$\frac{1}{4}$

Exercise 7.5.1

Solve: $25^{2x+3} = 125$.

Answer

$-\frac{3}{4}$

www.youtube.com/v/yU_tJg6Unnc

In many cases we will not be able to equate the bases. For this reason we develop a second method for solving exponential equations. Consider the following equations:

$$\begin{aligned} 3^2 &= 9 \\ 3^{?} &= 12 \\ 3^3 &= 27 \end{aligned}$$

We can see that the solution to $3^x = 12$ should be somewhere between 2 and 3. A graphical interpretation follows.

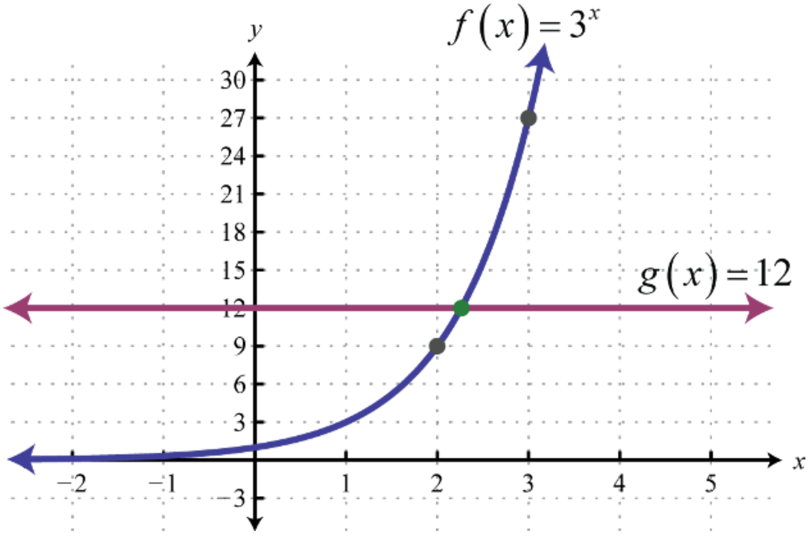

Figure $7.5.1$

To solve this we make use of fact that logarithms are one-to-one functions. Given $x, y > 0$ the **one-to-one property of logarithms**[17] follows:

$$\log_b x = \log_b y \quad \text{if and only if} \quad x = y$$

This property, as well as the properties of the logarithm, allows us to solve exponential equations. For example, to solve $3^x = 12$ apply the common logarithm to both sides and then use the properties of the logarithm to isolate the variable.

$$\begin{aligned} 3^x &= 12 \\ \log 3^x &= \log 12 \quad \textit{One-to-one property of logarithms} \\ x \log 3 &= \log 12 \quad \textit{Power rule for logarithms} \\ x &= \frac{\log 12}{\log 3} \end{aligned}$$

Approximating to four decimal places on a calculator.

$$x = \log(12)/\log(3) \approx 2.2619$$

An answer between 2 and 3 is what we expected. Certainly we can check by raising 3 to this power to verify that we obtain a good approximation of 12.

$$3 \wedge 2.2618 \approx 12 \checkmark$$

> Note that we are *not* multiplying both sides by “log”; we are applying the one-to-one property of logarithmic functions — which is often expressed as “*taking the log of both sides*.” The general steps for solving exponential equations are outlined in the following example.

Example 7.5.3:

Solve: $5^{2x-1}+2=9$.

Solution

Step 1: Isolate the exponential expression.

$$\begin{aligned} 5^{2x-1}+2&=9 \\ 5^{2x-1}&=7 \end{aligned}$$

Step 2: Take the logarithm of both sides. In this case, we will take the common logarithm of both sides so that we can approximate our result on a calculator.

$$\log 5^{2x-1}=\log 7$$

Step 3: Apply the power rule for logarithms and then solve.

$$\begin{aligned} \log 5^{2x-1}&=\log 7 \\ (2x-1)\log 5&=\log 7 \qquad \color{blue}{\textit{Distribute.}} \\ 2x\log 5-\log 5&=\log 7 \\ 2x\log 5&=\log 5+\log 7 \\ x&=\frac{\log 5+\log 7}{2\log 5} \end{aligned}$$

This is an irrational number which can be approximated using a calculator. Take care to group the numerator and the product in the denominator when entering this into your calculator. To do this, make use of the parenthesis buttons (and) :

$$x=(\log 5+\log(7))/(2*\log(5))\approx 1.1045$$

Answer:

$\frac{\log 5+\log 7}{2\log 5}\approx 1.1045$

Example 7.5.4:

Solve: $e^{5x+3}=1$.

Solution

The exponential function is already isolated and the base is e. Therefore, we choose to apply the natural logarithm to both sides.

$$\begin{aligned} e^{5x+3}&=1 \\ \ln e^{5x+3}&=\ln 1 \end{aligned}$$

Apply the power rule for logarithms and then simplify.

$$\begin{aligned} \ln e^{5x+3}&=\ln 1 \\ (5x+3)\ln e&=\ln 1 \quad \color{blue}{\textit{Recall } \ln e=1 \textit{ and } \ln 1=0.} \\ (5x+3)\cdot 1&=0 \\ 5x+3&=0 \\ x&=-\frac{3}{5} \end{aligned}$$

Answer:

$-\frac{3}{5}$

On most calculators there are only two logarithm buttons, the common logarithm LOG and the natural logarithm LN. If we want to approximate $\log_3 10$ we have to somehow change this base to 10 or e. The idea begins by rewriting the logarithmic function $y=\log_a x$, in exponential form.

$$\log_a x=y \Longrightarrow x=a^y$$

Here $x>0$ and so we can apply the one-to-one property of logarithms. Apply the logarithm base b to both sides of the function in exponential form.

$$x = a^y$$
$$\log_b x = \log_b a^y$$

And then solve for y.

$$\log_b x = y \log_b a$$
$$\frac{\log_b x}{\log_b a} = y$$

Replace y into the original function and we have the very important **change of base formula**[18]:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

We can use this to approximate $\log_3 10$ as follows.

$$\log_3 10 = \frac{\log 10}{\log 3} \approx 2.0959 \text{ or } \log_3 10 = \frac{\ln 10}{\ln 3} \approx 2.0959$$

Notice that the result is independent of the choice of base. In words, we can approximate the logarithm of any given base on a calculator by dividing the logarithm of the argument by the logarithm of that given base.

Example 7.5.5:

Approximate $\log_7 120$ the nearest hundredth.

Solution

Apply the change of base formula and use a calculator.

$$\log_7 120 = \frac{\log 120}{\log 7}$$

On a calculator,

$$\log(120)/\log(7) \approx 2.46$$

Answer:

2.46

Exercise 7.5.2

Solve: $2^{3x+1} - 4 = 1$. Give the exact and approximate answer rounded to four decimal places.

Answer

$\frac{\log 5 - \log 2}{3 \log 2} \approx 0.4406$

www.youtube.com/v/kQf1-cEQUg8

Solving Logarithmic Equations

A **logarithmic equation**[19] is an equation that involves a logarithm with a variable argument. Some logarithmic equations can be solved using the one-to-one property of logarithms. This is true when a single logarithm with the same base can be obtained on both sides of the equal sign.

Example 7.5.6:

Solve: $\log_2(2x-5) - \log_2(x-2) = 0$.

Solution

We can obtain two equal logarithms base 2 by adding $\log_2(x-2)$ to both sides of the equation.

$$\begin{aligned} \log_2(2x-5) - \log_2(x-2) &= 0 \\ \log_2(2x-5) &= \log_2(x-2) \end{aligned}$$

Here the bases are the same and so we can apply the one-to-one property and set the arguments equal to each other.

$$\begin{aligned} \log_2(2x-5) &= \log_2(x-2) \\ 2x-5 &= x-2 \\ x &= 3 \end{aligned}$$

Checking $x=3$ in the original equation:

$$\begin{aligned}\log_2(2(3)-5) &= \log_2((3)-2)\\ \log_2 1 &= \log_2 1\\ 0 &= 0 \ \checkmark\end{aligned}$$

Answer:

3

When solving logarithmic equations the check is very important because extraneous solutions can be obtained. The properties of the logarithm only apply for values in the domain of the given logarithm. And when working with variable arguments, such as $\log(x-2)$, the value of x is not known until the end of this process. The logarithmic expression $\log(x-2)$ is only defined for values $x > 2$.

Example 7.5.7:

Solve: $\log(3x-4)=\log(x-2)$.

Solution

Apply the one-to-one property of logarithms (set the arguments equal to each other) and then solve for x.

$$\begin{aligned}\log(3x-4) &= \log(x-2)\\ 3x-4 &= x-2\\ 2x &= 2\\ x &= 1\end{aligned}$$

When performing the check we encounter a logarithm of a negative number:

$$\begin{aligned}\log(x-2) &= \log(1-2)\\ &= \log(-1) \quad \mathit{Undefined}\end{aligned}$$

Try this on a calculator, what does it say? Here $x=1$ is not in the domain of $\log(x-2)$. Therefore our only possible solution is extraneous and we conclude that there are no solutions to this equation.

Answer

No solution, $\emptyset$.

Caution: Solving logarithmic equations sometimes leads to extraneous solutions — we must check our answers.

Exercise 7.5.3

Solve: $\ln\left(x^2-15\right)-\ln(2x)=0$.

Answer

5

www.youtube.com/v/-kBkk-q2pNs

In many cases we will not be able to obtain two equal logarithms. To solve such equations we make use of the definition of the logarithm. If $b>0$, where $b\neq 1$, then $\log_b x = y$ implies that $b^y = x$. Consider the following common logarithmic equations (base 10),

$$\begin{aligned}\log x = 0 &\Longrightarrow = 1 \qquad \mathit{Because}\ 10^{10} = 1.\\ \log x = 0.5 &\Longrightarrow = \ ?\\ \log x = 1 &\Longrightarrow = 10 \qquad \mathit{Because}\ 10^{1} = 10.\end{aligned}$$

We can see that the solution to $\log x = 0.5$ will be somewhere between 1 and 10. A graphical interpretation follows.

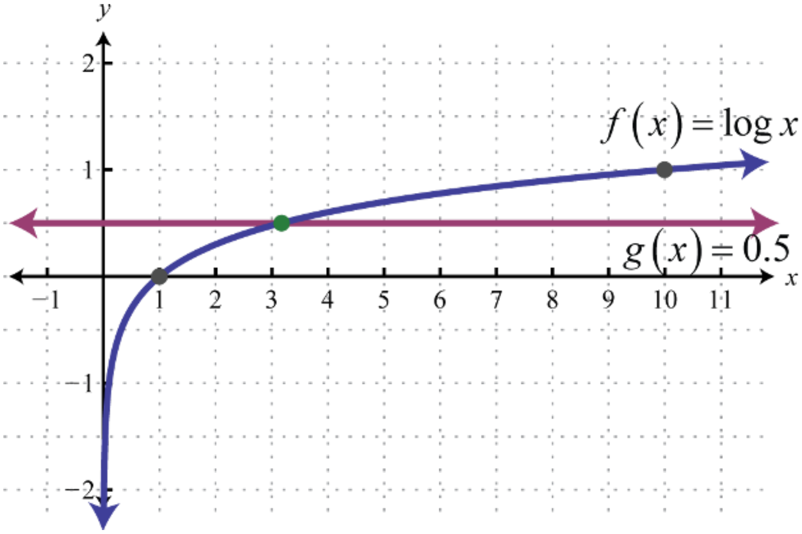

Figure 7.5.2

To find x we can apply the definition as follows.

$$\log_{10} x = 0.5 \Longrightarrow 10^{0.5} = x$$

This can be approximated using a calculator,

$$x = 10^{0.5} = 10\text{^}0.5 \approx 3.1623$$

An answer between 1 and 10 is what we expected. Check this on a calculator.

$$\log 3.1623 \approx 5 \quad \checkmark$$

Example 7.5.8:

Solve: $\log_3(2x-5) = 2$.

Solution

Apply the definition of the logarithm.

$$\log_3(2x-5) = 2 \Longrightarrow 2x - 5 = 3^2$$

Solve the resulting equation.

$$\begin{aligned} 2x - 5 &= 9 \\ 2x &= 14 \\ x &= 7 \end{aligned}$$

Check.

$$\begin{aligned} \log_3(2(7)-5) &\overset{?}{=} 2 \\ \log_3(9) &= 2 \quad \checkmark \end{aligned}$$

Answer:

7

In order to apply the definition, we will need to rewrite logarithmic expressions as a single logarithm with coefficient 1. The general steps for solving logarithmic equations are outlined in the following example.

Example 7.5.9:

Solve: $\log_2(x-2) + \log_2(x-3) = 1$.

Solution

Step 1: Write all logarithmic expressions as a single logarithm with coefficient 1. In this case, apply the product rule for logarithms.

$$\begin{aligned}\log_2(x-2)+\log_2(x-3)&=1\\ \log_2[(x-2)(x-3)]&=1\end{aligned}$$

Step 2: Use the definition and rewrite the logarithm in exponential form,

$$\log_2[(x-2)(x-3)]=1\Longrightarrow(x-2)(x-3)=2^1$$

Step 3: Solve the resulting equation. Here we can solve by factoring.

$$\begin{aligned}(x-2)(x-3)&=2\\ x^2-5x+6&=2\\ x^2-5x+4&=0\\ (x-4)(x-1)&=0\\ x-4&=0 \quad\text{or}\quad & x-1=0\\ x&=4 & x=1\end{aligned}$$

Step 4: Check. This step is required.

Table $7.5.1$

Check $x=4$	Check $x=1$
$\log_2(x-2)+\log_2(x-3)=1$ $\log_2(4-2)+\log_2(4-3)=1$ $\log_2(2)+\log_2(1)=1$ $1+0=1$ ✓	\(\begin{aligned}\log _{2}(x-2)+\log _{2}(x-3)&=1 \ \log _{2}(\color{Cerulean}{1}\color{black}{-}2)+\log _{2}(\color{Cerulean}{1}\color{black}{-}3)&=1 \ \log _{2}(-1)+\log _{2}(-2)&=1\N/A&\neq1\:\:\color{red}{✗}\end{aligned}\)

In this example, $x=1$ is not in the domain of the given logarithmic expression and is extraneous. The only solution is $x=4$.

Answer:

4

Example $7.5.10$:

Solve: $\log(x+15)-1=\log(x+6)$

Solution

Begin by writing all logarithmic expressions on one side and constants on the other.

$$\begin{aligned}\log(x+15)-1&=\log(x+6)\\ \log(x+15)-\log(x+6)&=1\end{aligned}$$

Apply the quotient rule for logarithms as a means to obtain a single logarithm with coefficient 1.

$$\begin{aligned}\log(x+15)-\log(x+6)&=1\\ \log\left(\frac{x+15}{x+6}\right)&=1\end{aligned}$$

This is a common logarithm; therefore use 10 as the base when applying the definition.

$$\begin{aligned}\frac{x+15}{x+6}&=10^1\\ x+15&=10(x+6)\\ x+15&=10x+60\\ -9x&=45\\ x&=-5\end{aligned}$$

Check.

$$\begin{aligned}\log(x+15)-1&=\log(x+6)\\ \log(-5+15)-1&=\log(-5+6)\\ \log 10-1&=\log 1\\ 1-1&=0\\ 0&=0\ ✓\end{aligned}$$

Answer:

-5

Exercise 7.5.4

Solve: $\log_2(x) + \log_2(x-1) = 1$.

Answer

2

www.youtube.com/v/Lzwh1jkj-EY

Example 7.5.11:

Find the inverse: $f(x) = \log_2(3x-4)$.

Solution

Begin by replacing the function notation $f(x)$ with y.

$$\begin{aligned} f(x) &= \log_2(3x-4) \\ y &= \log_2(3x-4) \end{aligned}$$

Interchange x and y and then solve for y.

$$\begin{aligned} x = \log_2(3y-4) \Longrightarrow 3y - 4 &= 2^x \\ 3y &= 2^x + 4 \\ y &= \frac{2^x+4}{3} \end{aligned}$$

The resulting function is the inverse of f. Present the answer using function notation.

Answer:

$f^{-1}(x) = \frac{2^x+4}{3}$

Key Takeaways

- If each side of an exponential equation can be expressed using the same base, then equate the exponents and solve.
- To solve a general exponential equation, first isolate the exponential expression and then apply the appropriate logarithm to both sides. This allows us to use the properties of logarithms to solve for the variable.
- The change of base formula allows us to use a calculator to calculate logarithms. The logarithm of a number is equal to the common logarithm of the number divided by the common logarithm of the given base.
- If a single logarithm with the same base can be isolated on each side of an equation, then equate the arguments and solve.
- To solve a general logarithmic equation, first isolate the logarithm with coefficient 1 and then apply the definition. Solve the resulting equation.
- The steps for solving logarithmic equations sometimes produce extraneous solutions. Therefore, the check is required.

Exercise 7.5.5

Solve using the one-to-one property of exponential functions.

1. $3^x = 81$
2. $2^{-x} = 16$
3. $5^{x-1} = 25$
4. $3^{x+4} = 27$
5. $2^{5x-2} = 16$
6. $2^{3x+7} = 8$
7. $81^{2x+1} = 3$
8. $64^{3x-2} = 2$
9. $9^{2-3x} - 27 = 0$
10. $8^{1-5x} - 32 = 0$
11. $16^{x^2} - 2 = 0$

12. $4^{x^2-1} - 64 = 0$
13. $9^{x(x+1)} = 81$
14. $4^{x(2x+5)} = 64$
15. $100^{x^2} - 10^{7x-3} = 0$
16. $e^{3(3x^2-1)} - e = 0$

Answer

1. 4

3. 3

5. $\frac{6}{5}$

7. $-\frac{3}{8}$

9. $\frac{1}{6}$

11. $\pm\frac{1}{2}$

13. $-2, 1$

15. $\frac{1}{2}, 3$

Exercise 7.5.6

Solve. Give the exact answer and the approximate answer rounded to the nearest thousandth.

1. $3^x = 5$
2. $7^x = 2$
3. $4^x = 9$
4. $2^x = 10$
5. $5^{x-3} = 13$
6. $3^{x+5} = 17$
7. $7^{2x+5} = 2$
8. $3^{5x-9} = 11$
9. $5^{4x+3} + 6 = 4$
10. $10^{7x-1} - 2 = 1$
11. $e^{2x-3} - 5 = 0$
12. $e^{5x+1} - 10 = 0$
13. $6^{3x+1} - 3 = 7$
14. $8 - 10^{9x+2} = 9$
15. $15 - e^{3x} = 2$
16. $7 + e^{4x+1} = 10$
17. $7 - 9e^{-x} = 4$
18. $3 - 6e^{-x} = 0$
19. $5^{x^2} = 2$
20. $3^{2x^2-x} = 1$
21. $100e^{27x} = 50$
22. $6e^{12x} = 2$
23. $\frac{3}{1+e^{-x}} = 1$
24. $\frac{2}{1+3e^{-x}} = 1$

Answer

1. $\frac{\log 5}{\log 3} \approx 1.465$

3. $\frac{\log 3}{\log 2} \approx 1.585$

5. $\frac{3\log 5+\log 13}{\log 5} \approx 4.594$

7. $\frac{\log 2-5\log 7}{2\log 7}\approx -2.322$

9. $\varnothing$

11. $\frac{3+\ln 5}{2}\approx 2.305$

13. $\frac{1-\log 6}{3\log 6}\approx 0.095$

15. $\frac{\ln 13}{3}\approx 0.855$

17. $\ln 3\approx 1.099$

19. $\pm\sqrt{\frac{\log 2}{\log 5}}\approx \pm 0.656$

21. $-\frac{\ln 2}{27}\approx -0.026$

23. $-\ln 2\approx -0.693$

Exercise 7.5.7

Find the x- and y-intercepts of the given function.

1. $f(x)=3^{x+1}-4$
2. $f(x)=2^{3x-1}-1$
3. $f(x)=10^{x+1}+2$
4. $f(x)=10^{4x}-5$
5. $f(x)=e^{x-2}+1$
6. $f(x)=e^{x+4}-4$

Answer

1. x-intercept: $\left(\frac{2\log 2-\log 3}{\log 3},0\right)$; y-intercept: $(0,-1)$

3. x-intercept: None; y-intercept: $(0,12)$

5. x-intercept: None; y-intercept: $\left(0,\frac{1+e^2}{e^2}\right)$

Exercise 7.5.8

Use a u-substitution to solve the following.

1. $3^{2x}-3^x-6=0$ Hint: Let $u=3^x$
2. $2^{2x}+2^x-20=0$
3. $10^{2x}+10^x-12=0$
4. $10^{2x}-10^x-30=0$
5. $e^{2x}-3e^x+2=0$
6. $e^{2x}-8e^x+15=0$

Answer

1. 1

3. $\log 3$

5. $0,\ln 2$

Exercise 7.5.9

Use the change of base formula to approximate the following to the nearest hundredth.

1. $\log_2 5$
2. $\log_3 7$
3. $\log_5\left(\frac{2}{3}\right)$
4. $\log_7\left(\frac{1}{5}\right)$

5. $\log_{1/2} 10$
6. $\log_{2/3} 30$
7. $\log_2 \sqrt{5}$
8. $\log_2 \sqrt[3]{6}$
9. If left unchecked, a new strain of flu virus can spread from a single person to others very quickly. The number of people affected can be modeled using the formula $P(t) = e^{0.22t}$, where t represents the number of days the virus is allowed to spread unchecked. Estimate the number of days it will take $1,000$ people to become infected.
10. The population of a certain small town is growing according to the function $P(t) = 12,500(1.02)^t$, where t represents time in years since the last census. Use the function to determine number of years it will take the population to grow to $25,000$ people.

Answer

1. 2.32
3. -0.25
5. -3.32
7. 1.16
9. Approximately 31 days

Exercise 7.5.10

Solve using the one-to-one property of logarithms.

1. $\log_5(2x+4) = \log_5(3x-6)$
2. $\log_4(7x) = \log_4(5x+14)$
3. $\log_2(x-2) - \log_2(6x-5) = 0$
4. $\ln(2x-1) = \ln(3x)$
5. $\log(x+5) - \log(2x+7) = 0$
6. $\ln\left(x^2+4x\right) = 2\ln(x+1)$
7. $\log_3 2 + 2\log_3 x = \log_3(7x-3)$
8. $2\log x - \log 36 = 0$
9. $\ln(x+3) + \ln(x+1) = \ln 8$
10. $\log_5(x-2) + \log_5(x-5) = \log_5 10$

Answer

1. 10
3. $\frac{3}{5}$
5. -2
7. $\frac{1}{2}, 3$
9. 1

Exercise 7.5.11

Solve.

1. $\log_2(3x-7) = 5$
2. $\log_3(2x+1) = 2$
3. $\log(2x+20) = 1$
4. $\log_4(3x+5) = \frac{1}{2}$
5. $\log_3 x^2 = 2$
6. $\log\left(x^2+3x+10\right) = 1$
7. $\ln\left(x^2-1\right) = 0$

8. $\log_5(x^2+20)-2=0$
9. $\log_2(x-5)+\log_2(x-9)=5$
10. $\log_2(x+5)+\log_2(x+1)=5$
11. $\log_4 x+\log_4(x-6)=2$
12. $\log_6 x+\log_6(2x-1)=2$
13. $\log_3(2x+5)-\log_3(x-1)=2$
14. $\log_2(x+1)-\log_2(x-2)=4$
15. $\ln x-\ln(x-1)=1$
16. $\ln(2x+1)-\ln x=2$
17. $2\log_3 x=2+\log_3(2x-9)$
18. $2\log_2 x=3+\log_2(x-2)$
19. $\log_2(x-2)=2-\log_2 x$
20. $\log_2(x+3)+\log_2(x+1)-1=0$
21. $\log x-\log(x+1)=1$
22. $\log_2(x+2)+\log_2(1-x)=1+\log_2(x+1)$

Answer

1. 13

3. -5

5. ± 3

7. $\pm\sqrt{2}$

9. 13

11. 8

13. 2

15. $\frac{e}{e-1}$

17. 9

19. $1\pm\sqrt{5}$

21. $\varnothing$

Exercise 7.5.12

Find the x- and y-intercepts of the given function.

1. $f(x)=\log(x+3)-1$
2. $f(x)=\log(x-2)+1$
3. $f(x)=\log_2(3x)-4$
4. $f(x)=\log_3(x+4)-3$
5. $f(x)=\ln(2x+5)-6$
6. $f(x)=\ln(x+1)+2$

Answer

1. x-intercept: $(7,0)$; y-intercept: $(0,\log 3-1)$

3. x-intercept: $(\frac{16}{3},0)$; y-intercept: None

5. x-intercept: $\left(\frac{e^6-5}{2},0\right)$; y-intercept: $(0,\ln 5-6)$

Exercise 7.5.13

Find the inverse of the following functions.

1. $f(x)=\log_2(x+5)$
2. $f(x)=4+\log_3 x$
3. $f(x)=\log(x+2)-3$
4. $f(x)=\ln(x-4)+1$
5. $f(x)=\ln(9x-2)+5$
6. $f(x)=\log_6(2x+7)-1$
7. $g(x)=e^{3x}$
8. $g(x)=10^{-2x}$
9. $g(x)=2^{x+3}$
10. $g(x)=3^{2x}+5$
11. $g(x)=10^{x+4}-3$
12. $g(x)=e^{2x-1}+1$

Answer

1. $f^{-1}(x)=2^x-5$

3. $f^{-1}(x)=10^{x+3}-2$

5. $f^{-1}(x)=\frac{e^{x-5}+2}{9}$

7. $g^{-1}(x)=\frac{\ln x}{3}$

9. $g^{-1}(x)=\log_2 x-3$

11. $g^{-1}(x)=\log(x+3)-4$

Exercise 7.5.14

Solve.

1. $\log(9x+5)=1+\log(x-5)$
2. $2+\log_2\left(x^2+1\right)=\log_2 13$
3. $e^{5x-2}-e^{3x}=0$
4. $3^{x^2}-11=70$
5. $2^{3x}-5=0$
6. $\log_7(x+1)+\log_7(x-1)=1$
7. $\ln(4x-1)-1=\ln x$
8. $\log(20x+1)=\log x+2$
9. $\frac{3}{1+e^{2x}}=2$
10. $2e^{-3x}=4$
11. $2e^{3x}=e^{4x+1}$
12. $2\log x+\log x-1=0$
13. $3\log x=\log(x-2)+2\log x$
14. $2\ln 3+\ln x^2=\ln\left(x^2+1\right)$
15. In chemistry, pH is a measure of acidity and is given by the formula $\mathrm{pH}=-\log\left(H^+\right)$, where H^+ is the hydrogen ion concentration (measured in moles of hydrogen per liter of solution.) Determine the hydrogen ion concentration if the pH of a solution is 4.
16. The volume of sound, L in decibels (dB), is given by the formula $L=10\log\left(I/10^{-12}\right)$ where I represents the intensity of the sound in watts per square meter. Determine the intensity of an alarm that emits 120 dB of sound.

Answer

1. 55

3. 1

5. $\frac{\log_2 5}{3}$

7. $\frac{1}{4-e}$

9. $\frac{\ln(1/2)}{2}$

11. $\ln 2-1$

13. $\emptyset$

15. 10^{-4} moles per liter

Exercise 7.5.15

1. Research and discuss the history and use of the slide rule.
2. Research and discuss real-world applications involving logarithms.

Answer

1. Answer may vary

Footnotes

[15] An equation which includes a variable as an exponent.

[16] Given $b>0$ and $b \neq 1$ we have $b^x = b^y$ if and only if $x=y$.

[17] Given $b>0$ and $b \neq 1$ where $x, y>0$ we have $log_b x = log_b y$ if and only if $x=y$.

[18] $\log_a x = \frac{\log_b x}{\log_b a}$; we can write any base-$a$ logarithm in terms of base-b logarithms using this formula.

[19] An equation that involves a logarithm with a variable argument.

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7.6: Applications

Learning Objectives

- Use the compound and continuous interest formulas.
- Calculate doubling time.
- Use the exponential growth/decay model.
- Calculate the rate of decay given half-life.

Compound and Continuous Interest Formulas

Recall that compound interest occurs when interest accumulated for one period is added to the principal investment before calculating interest for the next period. The amount A accrued in this manner over time t is modeled by the compound interest formula:

$$A(t)=P\left(1+\frac{r}{n}\right)^{nt}$$

Here the initial principal P is accumulating compound interest at an annual rate r where the value n represents the number of times the interest is compounded in a year.

Example 7.6.1:

Susan invested $\$500$ in an account earning $4\frac{1}{2}\%$ annual interest that is compounded monthly.

a. How much will be in the account after 3 years?

b. How long will it take for the amount to grow to $\$750$?

Solution

In this example, the principal $P=\$500$, the interest rate $r=4\frac{1}{2}\%=0.045$, and because the interest is compounded monthly, $n=12$. The investment can be modeled by the following function:

$$\begin{aligned}A(t)&=500\left(1+\frac{0.045}{12}\right)^{12t}\\A(t)&=500(1.00375)^{12t}\end{aligned}$$

a. Use this model to calculate the amount in the account after $t=3$ years.

$$\begin{aligned}A(3)&=500(1.00375)^{12(3)}\\&=500(1.00375)^{36}\\&\approx 572.12\end{aligned}$$

Rounded off to the nearest cent, after 3 years, the amount accumulated will be $\$572.12$.

b. To calculate the time it takes to accumulate $\$750$, set $A(t)=750$ and solve for t.

$$\begin{aligned}A(t)&=500(1.00375)^{12t}\\750&=500(1.00375)^{12t}\end{aligned}$$

This results in an exponential equation that can be solved by first isolating the exponential expression.

$$\begin{aligned}750&=500(1.00375)^{12t}\\\frac{750}{500}&=(1.00375)^{12t}\\1.5&=(1.00375)^{12t}\end{aligned}$$

At this point take the common logarithm of both sides, apply the power rule for logarithms, and then solve for t.

$$\begin{aligned}\log(1.5)&=\log(1.00375)^{12t}\\\log(1.5)&=12t\log(1.00375)\\\frac{\log(1.5)}{12\log(1.00375)}&=\frac{\cancel{12}\,t\ \cancel{\log(1.00375)}}{\cancel{12\ \log(1.00375)}}\\\frac{\log(1.5)}{12\ \log(1.00375)}&=t\end{aligned}$$

Using a calculator we can approximate the time it takes.

$$t = \log(1.5)/(12 * \log(1.00375)) \approx 9 \text{ years}$$

Answer:

a. $\$572.12$

b. Approximately 9 years

The period of time it takes a quantity to double is called the **doubling time**[20]. We next outline a technique for calculating the time it takes to double an initial investment earning compound interest.

Example 7.6.2:

Mario invested $\$1000$ in an account earning 6.3% annual interest, that is compounded semi-annually. How long will it take the investment to double?

Solution

Here the principal $P = \$1,000$, the interest rate $r = 6.3\% = 0.063$, and because the interest is compounded semi-annually $n = 2$. This investment can be modeled as follows:

$$\begin{aligned} A(t) &= 1,000\left(1+\tfrac{0.063}{2}\right)^{2t} \\ A(t) &= 1,000(1.0315)^{2t} \end{aligned}$$

Since we are looking for the time it takes to double $\$1,000$, substitute $\$2,000$ for the resulting amount $A(t)$ and then solve for t.

$$\begin{aligned} 2,000 &= 1,000(1.0315)^{2t} \\ \frac{2,000}{1,000} &= (1.0315)^{2t} \\ 2 &= (1.0315)^{2t} \end{aligned}$$

At this point we take the common logarithm of both sides.

$$\begin{aligned} 2 &= (1.0315)^{2t} \\ \log 2 &= \log(1.0315)^{2t} \\ \log 2 &= 2t\log(1.0315) \\ \frac{\log 2}{2\log(1.0315)} &= t \end{aligned}$$

Using a calculator we can approximate the time it takes:

$$t = \log(2)/(2 * \log(1.0315)) \approx 11.17 \text{ years}$$

Answer:

Approximately 11.17 years to double at 6.3%.

If the investment in the previous example was one million dollars, how long would it take to double? To answer this we would use $P = \$1,000,000$ and $A(t) = \$2,000,000$:

$$\begin{aligned} A(t) &= 1,000(1.0315)^{2t} \\ 2,000,000 &= 1,000,000(1.0315)^{2t} \end{aligned}$$

Dividing both sides by $1,000,000$ we obtain the same exponential function as before.

$$2 = (1.0315)^{2t}$$

Hence, the result will be the same, about 11.17 years. In fact, doubling time is independent of the initial investment P.

Interest is typically compounded semi-annually $(n = 2)$, quarterly $(n = 4)$, monthly $(n = 12)$, or daily $(n = 365)$. However if interest is compounded every instant we obtain a formula for continuously compounding interest:

$$A(t) = Pe^{rt}$$

Here P represents the initial principal amount invested, r represents the annual interest rate, and t represents the time in years the investment is allowed to accrue continuously compounded interest.

Example 7.6.3:

Mary invested $\$200$ in an account earning $5\frac{3}{4}\%$ annual interest that is compounded continuously. How long will it take the investment to grow to $\$350$?

Solution

Here the principal $P = \$200$ and the interest rate $r = 5\frac{3}{4}\% = 5.75\% = 0.0575$. Since the interest is compounded continuously, use the formula $A(t) = Pe^{rt}$. Hence, the investment can be modeled by the following,

$$A(t) = 200e^{0.0575t}$$

To calculate the time it takes to accumulate to $\$350$, set $A(t) = 350$ and solve for t.

$$\begin{aligned} A(t) &= 200e^{0.0575t} \\ 350 &= 200e^{0.0575t} \end{aligned}$$

Begin by isolating the exponential expression.

$$\begin{aligned} \frac{350}{200} &= e^{0.0575t} \\ \frac{7}{4} &= e^{0.0575t} \\ 1.75 &= e^{0.0575t} \end{aligned}$$

Because this exponential has base e, we choose to take the natural logarithm of both sides and then solve for t.

$$\begin{aligned} \ln(1.75) &= \ln e^{0.0575t} && \textit{Apply the power rule for logarithms.} \\ \ln(1.75) &= 0.0575t \ln e && \textit{Recall that } \ln e = 1. \\ \ln(1.75) &= 0.0575t \cdot 1 \\ \frac{\ln(1.75)}{0.0575} &= t \end{aligned}$$

Using a calculator we can approximate the time it takes:

$$t = \ln(1.75)/0.0575 \approx 9.73 \quad years$$

Answer:

It will aprroximately 9.73 years.

When solving applications involving compound interest, look for the keyword "continuous," or the keywords that indicate the number of annual compoundings. It is these keywords that determine which formula to choose.

Exercise 7.6.1

Mario invested $\$1,000$ in an account earning 6.3% annual interest that is compounded continuously. How long will it take the investment to double?

Answer

Approximately 11 years.

www.youtube.com/v/Z_ilJHIQvWQ

Modeling Exponential Growth and Decay

In the sciences, when a quantity is said to grow or decay exponentially, it is specifically meant to be modeled using the **exponential growth/decay formula**[21]:

$$P(t) = P_0e^{kt}$$

Here P_0, read "P naught," or "P zero," represents the initial amount, k represents the growth rate, and t represents the time the initial amount grows or decays exponentially. If k is negative, then the function models exponential decay. Notice that the function looks very similar to that of continuously compounding interest formula. We can use this formula to model population growth when conditions are optimal.

Example 7.6.4:

It is estimated that the population of a certain small town is $93,000$ people with an annual growth rate of 2.6%. If the population continues to increase exponentially at this rate:

a. Estimate the population in 7 years' time.
b. Estimate the time it will take for the population to reach 120,000 people.

Solution

We begin by constructing a mathematical model based on the given information. Here the initial population $P_0 = 93,000$ people and the growth rate $r = 2.6\% = 0.026$. The following model gives population in terms of time measured in years:

$$P(t) = 93,000e^{0.026t}$$

a. Use this function to estimate the population in $t = 7$ years.

$$\begin{aligned} P(t) &= 93,000e^{0006(7)} \\ &= 93,000e^{0.182} \\ &\approx 111,564 \quad people \end{aligned}$$

b. Use the model to determine the time it takes to reach $P(t) = 120,000$ people.

$$\begin{aligned} P(t) &= 93,000e^{0.026t} \\ 120,000 &= 93,000e^{0.026t} \\ \frac{120,000}{93,000} &= e^{0.026t} \\ \frac{40}{31} &= e^{0.026t} \end{aligned}$$

Take the natural logarithm of both sides and then solve for t.

$$\begin{aligned} \ln\left(\tfrac{40}{31}\right) &= \ln e^{0.026t} \\ \ln\left(\tfrac{40}{31}\right) &= 0.026t \ln e \\ \ln\left(\tfrac{40}{31}\right) &= 0.026t \cdot 1 \\ \frac{\ln\left(\frac{40}{31}\right)}{0.026} &= t \end{aligned}$$

Using a calculator,

$$t = \ln(40/31)/0.026 \approx 9.8 \quad years$$

Answer:

a. $111,564$ people
b. 9.8 years

Often the growth rate k is not given. In this case, we look for some other information so that we can determine it and then construct a mathematical model. The general steps are outlined in the following example.

Example 7.6.5:

Under optimal conditions *Escherichia coli (E. coli)* bacteria will grow exponentially with a doubling time of 20 minutes. If $1,000$ *E. coli* cells are placed in a Petri dish and maintained under optimal conditions, how many *E. coli* cells will be present in 2 hours?

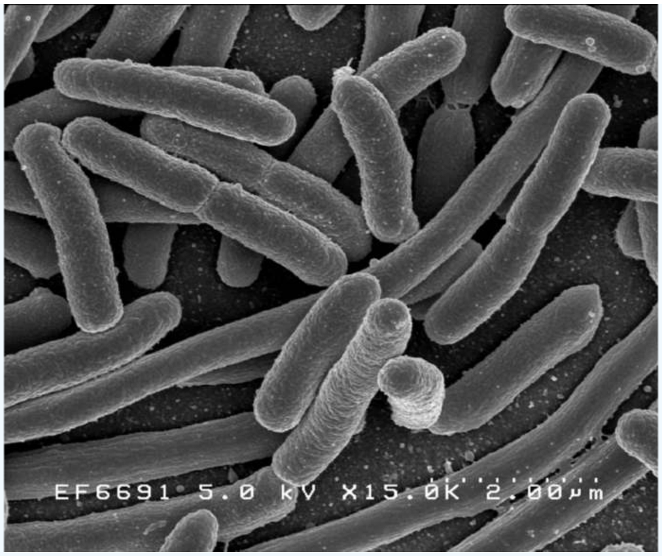

Figure *$7.6.1$: Escherichia coli (E. coli)*

Solution

The goal is to use the given information to construct a mathematical model based on the formula $P(t)=P_0e^{kt}$.

Step 1: Find the growth rate k. Use the fact that the initial amount, $P_0=1,000$ cells, doubles in 20 minutes. That is, $P(t)$ $=2,000$ cells when $t=20$ minutes.

$$\begin{aligned}P(t)&=P_0e^{kt}\\ 2,000&=1,000e^{k20}\end{aligned}$$

Solve for the only variable k.

$$\begin{aligned}2,000&=1,000e^{k20}\\ \frac{2,000}{1,000}&=e^{k20}\\ 2&=e^{k20}\\ \ln(2)&=\ln e^{k20}\\ \ln(2)&=k20\ln e\\ \ln(2)&=k20\cdot 1\\ \frac{\ln(2)}{20}&=k\end{aligned}$$

Step 2: Write a mathematical model based on the given information. Here $k\approx 0.0347$, which is about 3.5% growth rate per minute. However, we will use the exact value for k in our model. This will allow us to avoid round-off error in the final result. Use $P_0=1,000$ and $k=\ln(2)/20$:

$$P(t)=1,000e^{(\ln(2)/20)t}$$

This equation models the number of *E. coli* cells in terms of time in minutes.

Step 3: Use the function to answer the questions. In this case, we are asked to find the number of cells present in 2 hours. Because time is measured in minutes, use $t=120$ minutes to calculate the number of *E. coli* cells.

$$\begin{aligned}P(120)&=1,000e^{(\ln(2)/20)(120)}\\ &=1,000e^{\ln(2)\cdot 6}\\ &=1,000e^{\ln 2^6}\\ &=1,000\cdot 2^6\\ &=64,000\text{ cells}\end{aligned}$$

Answer:

In two hours $64,000$ cells will be present.

When the growth rate is negative the function models exponential decay. We can describe decreasing quantities using a **half-life**[22], or the time it takes to decay to one-half of a given quantity.

Example $7.6.6$:

Due to radioactive decay, caesium-137 has a half-life of 30 years. How long will it take a 50-milligram sample to decay to 10 milligrams?

Solution

Use the half-life information to determine the rate of decay k. In $t = 30$ years the initial amount $P_0 = 50$ milligrams will decay to half $P(30) = 25$ milligrams.

$$\begin{aligned} P(t) &= P_0 e^{kt} \\ 25 &= 50e^{k30} \end{aligned}$$

Solve for the only variable, k.

$$\begin{aligned} 25 &= 50e^{130} \\ \frac{25}{50} &= e^{30k} \\ \ln\left(\frac{1}{2}\right) &= \ln e^{30k} \\ \ln\left(\frac{1}{2}\right) &= 30k \ln e \\ \frac{\ln 1 - \ln 2}{30} &= k \qquad \textit{Recall that } \ln 1 = 0. \\ -\frac{\ln 2}{30} &= k \end{aligned}$$

Note that $k = -\ln\frac{2}{30} \approx -0.0231$ is negative. However, we will use the exact value to construct a model that gives the amount of cesium-137 with respect to time in years.

$$P(t) = 50e^{(-\ln 2/30)t}$$

Use this model to find t when $P(t) = 10$ milligrams.

$$\begin{aligned} 10 &= 50e^{(-\ln 2/30)t} \\ \frac{10}{50} &= e^{(-\ln 2/30)t} \\ \ln\left(\frac{1}{5}\right) &= \ln e^{(-\ln 2/30)t} \\ \ln 1 - \ln 5 &= \left(-\frac{\ln 2}{30}\right) t \ln e \quad \textit{Recall that } \ln e = 1. \\ -30\frac{(\ln 1 - \ln 5)}{\ln 2} &= t \\ -\frac{30(0 - \ln 5)}{\ln 2} &= t \\ \frac{30 \ln 5}{\ln 2} &= t \end{aligned}$$

Answer:

Using a calculator, it will take $t \approx 69.66$ years to decay to 10 milligrams.

Radiocarbon dating is a method used to estimate the age of artifacts based on the relative amount of carbon-14 present in it. When an organism dies, it stops absorbing this naturally occurring radioactive isotope, and the carbon-14 begins to decay at a known rate. Therefore, the amount of carbon-14 present in an artifact can be used to estimate the age of the artifact.

Example $7.6.7$:

An ancient bone tool is found to contain 25% of the carbon-14 normally found in bone. Given that carbon-14 has a half-life of $5,730$ years, estimate the age of the tool.

Solution

Begin by using the half-life information to find k. Here the initial amount P_0 of carbon-14 is not given, however, we know that in $t = 5,730$ years, this amount decays to half, $\frac{1}{2}P_0$.

$$\begin{aligned} P(t) &= P_0e^{kt} \\ \tfrac{1}{2}P_0 &= P_0e^{k5,730} \end{aligned}$$

Dividing both sides by P_0 leaves us with an exponential equation in terms of k. This shows that half-life is independent of the initial amount.

$$\tfrac{1}{2} = e^{k5,730}$$

Solve for k.

$$\begin{aligned} \ln\left(\frac{1}{2}\right) &= \ln e^{k5,730} \\ \ln 1 - \ln 2 &= 5,730k\ln e \\ \frac{0-\ln 2}{5,730} &= k \\ -\frac{\ln 2}{5,730} &= k \end{aligned}$$

Therefore we have the model,

$$P(t) = P_0e^{(-\ln 2/5,730)t}$$

Next we wish to the find time it takes the carbon-14 to decay to 25% of the initial amount, or $P(t) = 0.25P_0$

$$.0.25P_0 = P_0e^{(-\ln 2/5,730)t}$$

Divide both sides by P_0 and solve for t.

$$\begin{aligned} 0.25 &= e^{(-\ln 2/5,730)t} \\ \ln(0.25) &= \ln e^{(-\ln 2/5,730)t} \\ \ln(0.25) &= \left(-\frac{\ln 2}{5,730}\right)_{t\ln e} \\ -\frac{5,730\ln(0.25)}{\ln 2} &= t \\ 11,460 &\approx t \end{aligned}$$

Answer:

The tool is approximately $11,460$ years old.

Exercise 7.6.2

The half-life of strontium-90 is about 28 years. How long will it take a 36 milligram sample of strontium-90 to decay to 30 milligrams?

Answer

7.4 years

www.youtube.com/v/Oto0ihiyvBc

Key Takeaways

- When interest is compounded a given number of times per year use the formula $A(t) = P\left(1+\frac{r}{n}\right)^{nt}$.
- When interest is to be compounded continuously use the formula $A(t) = Pe^{rt}$.
- Doubling time is the period of time it takes a given amount to double. Doubling time is independent of the principal.
- When amounts are said to be increasing or decaying exponentially, use the formula $P(t) = P_0e^{kt}$.
- Half-life is the period of time it takes a given amount to decrease to one-half. Half-life is independent of the initial amount.
- To model data using the exponential growth/decay formula, use the given information to determine the growth/decay rate k. Once k is determined, a formula can be written to model the problem. Use the formula to answer the questions.

Exercise 7.6.3

1. Jill invested $\$1,450$ in an account earning $4\frac{5}{8}\%$ annual interest that is compounded monthly.
    1. How much will be in the account after 6 years?
    2. How long will it take the account to grow to $\$2,200$?
2. James invested $\$825$ in an account earning $5\frac{2}{5}\%$ annual interest that is compounded monthly.
    1. How much will be in the account after 4 years?
    2. How long will it take the account to grow to $\$1,500$?
3. Raul invested $\$8,500$ in an online money market fund earning 4.8% annual interest that is compounded continuously.
    1. How much will be in the account after 2 years?
    2. How long will it take the account to grow to $\$10,000$?
4. Ian deposited $\$500$ in an account earning 3.9% annual interest that is compounded continuously.
    1. How much will be in the account after 3 years?
    2. How long will it take the account to grow to $\$1,500$?
5. Bill wants to grow his $\$75,000$ inheritance to $\$100,000$ before spending any of it. How long will this take if the bank is offering 5.2% annual interest compounded quarterly?
6. Mary needs $\$25,000$ for a down payment on a new home. If she invests her savings of $\$21,350$ in an account earning 4.6% annual interest that is compounded semi-annually, how long will it take to grow to the amount that she needs?
7. Joe invested his $\$8,700$ savings in an account earning $6\frac{3}{4}\%$ annual interest that is compounded continuously. How long will it take to earn $300 in interest?
8. Miriam invested $\$12,800$ in an account earning $5\frac{1}{4}\%$ annual interest that is compounded monthly. How long will it take to earn $\$1,200$ in interest?
9. Given that the bank is offering 4.2% annual interest compounded monthly, what principal is needed to earn $\$25,000$ in interest for one year?
10. Given that the bank is offering 3.5% annual interest compounded continuously, what principal is needed to earn $ $12,000$ in interest for one year?
11. Jose invested his $\$3,500$ bonus in an account earning $5\frac{1}{2}\%$ annual interest that is compounded quarterly. How long will it take to double his investment?
12. Maria invested her $\$4,200$ savings in an account earning $6\frac{3}{4}\%$ annual interest that is compounded semi-annually. How long will it take to double her savings?
13. If money is invested in an account earning 3.85% annual interest that is compounded continuously, how long will it take the amount to double?
14. If money is invested in an account earning 6.82% annual interest that is compounded continuously, how long will it take the amount to double?
15. Find the annual interest rate at which an account earning continuously compounding interest has a doubling time of 9 years.
16. Find the annual interest rate at which an account earning interest that is compounded monthly has a doubling time of 10 years.
17. Alice invested her savings of $\$7,000$ in an account earning 4.5% annual interest that is compounded monthly. How long will it take the account to triple in value?
18. Mary invested her $\$42,000$ bonus in an account earning 7.2% annual interest that is compounded continuously. How long will it take the account to triple in value?
19. Calculate the doubling time of an investment made at 7% annual interest that is compounded:
    1. monthly
    2. continuously
20. Calculate the doubling time of an investment that is earning continuously compounding interest at an annual interest rate of:
    1. 4%
    2. 6%
21. Billy’s grandfather invested in a savings bond that earned 5.5% annual interest that was compounded annually. Currently, 30 years later, the savings bond is valued at $\$10,000$. Determine what the initial investment was.

22. In 1935 Frank opened an account earning 3.8% annual interest that was compounded quarterly. He rediscovered this account while cleaning out his garage in 2005. If the account is now worth $\$11,294.30$, how much was his initial deposit in 1935?

Answer

1. (1) $\$1,912.73$ (2) 9 years
3. (1) $\$9,356.45$ (2) 3.4 years
5. 5.6 years
7. $\frac{1}{2}$ year
9. $\$583,867$
11. 12.7 years
13. 18 years
15. 7.7%
17. 24.5 years
19. (1) 9.93 years (2) 9.90 years
21. $\$2,006.44$

Exercise 7.6.4

1. The population of a small town of $24,000$ people is expected grow exponentially at a rate of 1.6% per year. Construct an exponential growth model and use it to:
    1. Estimate the population in 3 years' time.
    2. Estimate the time it will take for the population to reach $30,000$ people.
2. During the exponential growth phase, certain bacteria can grow at a rate of 4.1% per hour. If $10,000$ cells are initially present in a sample, construct an exponential growth model and use it to:
    1. Estimate the population in 5 hours.
    2. Estimate the time it will take for the population to reach $25,000$ cells.
3. In 2000, the world population was estimated to be 6.115 billion people and in 2010 the estimate was 6.909 billion people. If the world population continues to grow exponentially, estimate the total world population in 2020.
4. In 2000, the population of the United States was estimated to be 282 million people and in 2010 the estimate was 309 million people. If the population of the United States grows exponentially, estimate the population in 2020.
5. An automobile was purchased new for $\$42,500$ and 2 years later it was valued at $\$33,400$. Estimate the value of the automobile in 5 years if it continues to decrease exponentially.
6. A new PC was purchased for $\$1,200$ and in 1.5 years it was worth $\$520$. Assume the value is decreasing exponentially and estimate the value of the PC four years after it is purchased.
7. The population of the downtown area of a certain city decreased from $12,500$ people to $10,200$ people in two years. If the population continues to decrease exponentially at this rate, what would we expect the population to be in two more years?
8. A new MP3 player was purchased for $\$320$ and in 1 year it was selling used online for $\$210$. If the value continues to decrease exponentially at this rate, determine the value of the MP3 player 3 years after it was purchased.
9. The half-life of radium-226 is about $1,600$ years. How long will a 5-milligram sample of radium-226 take to decay to 1 milligram?
10. The half-life of plutonium-239 is about $24,000$ years. How long will a 5-milligram sample of plutonium-239 take to decay to 1 milligram?
11. The half-life of radioactive iodine-131 is about 8 days. How long will it take a 28-gram initial sample of iodine-131 to decay to 12 grams?
12. The half-life of caesium-137 is about 30 years. How long will it take a 15-milligram sample of caesium-137 to decay to 5 milligrams?

13. The Rhind Mathematical Papyrus is considered to be the best example of Egyptian mathematics found to date. This ancient papyrus was found to contain 64% of the carbon-14 normally found in papyrus. Given that carbon-14 has a half-life of $5,730$ years, estimate the age of the papyrus.
14. A wooden bowl artifact carved from oak was found to contain 55% of the carbon-14 normally found in oak. Given that carbon-14 has a half-life of $5,730$ years, estimate the age of the bowl.
15. The half-life of radioactive iodine-131 is about 8 days. How long will it take a sample of iodine-131 to decay to 10% of the original amount?
16. The half-life of caesium-137 is about 30 years. How long will it take a sample of caesium-137 to decay to 25% of the original amount?
17. The half-life of caesium-137 is about 30 years. What percent of an initial sample will remain in 100 years?
18. The half-life of radioactive iodine-131 is about 8 days. What percent of an initial sample will remain in 30 days?
19. If a bone is 100 years old, what percent of its original amount of carbon-14 do we expect to find in it?
20. The half-life of plutonium-239 is about $24,000$ years. What percent of an initial sample will remain in $1,000$ years?
21. Find the amount of time it will take for 10% of an initial sample of plutonium-239 to decay. (Hint: If 10% decays, then 90% will remain.)
22. Find the amount of time it will take for 10% of an initial sample of carbon-14 to decay.

Answer

1. (1) About $25,180$ people (2) About 14 years

3. About 7.806 billion people

5. About $\$23,269.27$

7. $8,323$ people

9. $3,715$ years

11. 9.8 days

13. About $3,689$ years old

15. 26.6 days

17. 9.9%

19. 98.8%

21. $3,648$ years

Exercise 7.6.5

Solve for the given variable:

1. Solve for $t: A=Pe^{rt}$
2. Solve for $t: A=P(1+r)^{t}$
3. Solve for $I: M=\log\left(\frac{I}{I_0}\right)$
4. Solve for $H^{+}: pH=-\log\left(H^{+}\right)$
5. Solve for $t: P=\frac{1}{1+e^{-t}}$
6. Solve for $I: L=10\log\left(I/10^{-12}\right)$
7. The number of cells in a certain bacteria sample is approximated by the logistic growth model $N(t)=\frac{1.2\times10^{5}}{1+9e^{-0.32t}}$, where t represents time in hours. Determine the time it takes the sample to grow to $24,000$ cells.
8. The market share of a product, as a percentage, is approximated by the formula $P(t)=\frac{100}{3+e^{-0.44t}}$ where t represents the number of months after an aggressive advertising campaign is launched.
   1. What was the initial market share?
   2. How long would we expect to see a 3.5% increase in market share?
9. In chemistry, pH is a measure of acidity and is given by the formula $\mathrm{pH}=-\log\left(H^{+}\right)$, where H^{+} is the hydrogen ion concentration (measured in moles of hydrogen per liter of solution.) What is the hydrogen ion concentration of seawater with a pH of 8?

10. Determine the hydrogen ion concentration of milk with a pH of 6.6.
11. The volume of sound, L in decibels (dB), is given by the formula $L = 10\log\left(I/10^{-12}\right)$ where I represents the intensity of the sound in watts per square meter. Determine the sound intensity of a hair dryer that emits 70 dB of sound.
12. The volume of a chainsaw measures 110 dB. Determine the intensity of this sound.

Answer

1. $t = \frac{\ln(A)-\ln(P)}{r}$

3. $I = I_0 \cdot 10^M$

5. $t = \ln\left(\frac{P}{1-P}\right)$

7. Approximately 2.5 hours

9. 10^{-8} moles per liter

11. 10^{-5} watts per square meter

Exercise $7.6.6$

1. Which factor affects the doubling time the most, the annual compounding n or the interest rate r? Explain.
2. Research and discuss radiocarbon dating. Post something interesting you have learned as well as a link to more information.
3. Is exponential growth sustainable over an indefinite amount of time? Explain.
4. Research and discuss the half-life of radioactive materials.

Answer

1. Answer may vary

3. Answer may vary

Footnotes

[20] The period of time it takes a quantity to double.

[21] A formula that models exponential growth or decay: $P(t) = P_0 e^{kt}$.

[22] The period of time it takes a quantity to decay to one-half of the initial amount.

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7.E: Exponential and Logarithmic Functions (Exercises)

Exercise $7.E.1$

Given f and g find $(f\circ g)(x)$ and $(g\circ f)(x)$.

1. $f(x)=6x-5, g(x)=2x+1$
2. $f(x)=5-6x, g(x)=\frac{3}{2}x$
3. $f(x)=2x^2+x-2, g(x)=5x$
4. $f(x)=x^2-x-6, g(x)=x-3$
5. $f(x)=\sqrt{x+2}, g(x)=8x-2$
6. $f(x)=\frac{x-1}{3x-1}, g(x)=\frac{1}{x}$
7. $f(x)=x^2+3x-1, g(x)=\frac{1}{x-2}$
8. $f(x)=\sqrt[3]{3(x+2)}, g(x)=9x^3-2$

Answer

1. $(f\circ g)(x)=12x+1; (g\circ f)(x)=12x-9$

3. $(f\circ g)(x)=50x^2+5x-2;\ (g\circ f)(x)=10x^2+5x-10$

5. $(f\circ g)(x)=2\sqrt{2x};\ (g\circ f)(x)=8\sqrt{x+2}-2$

7. $(f\circ g)(x)=-\frac{x^2-7x+9}{(x-2)^2};\ (g\circ f)(x)=\frac{1}{x^2+3x-3}$

Exercise $7.E.1$

Are the given functions one-to-one? Explain.

1.

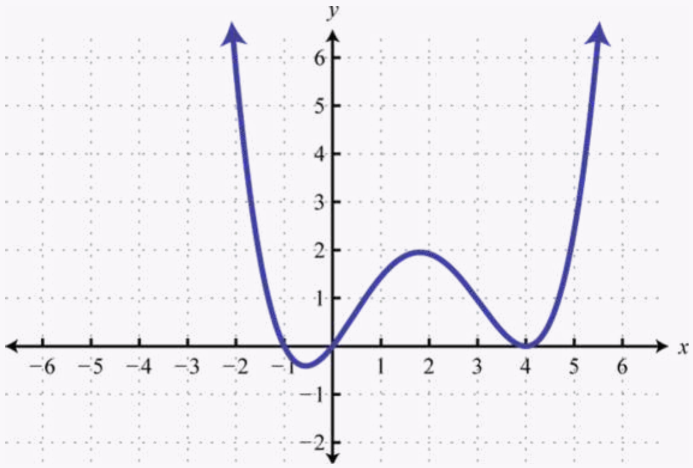

Figure 7.E.1

2.

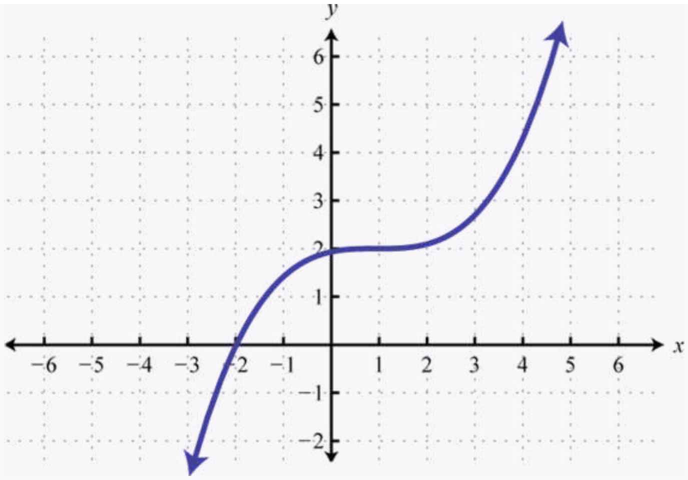

Figure 7.E.2

3.

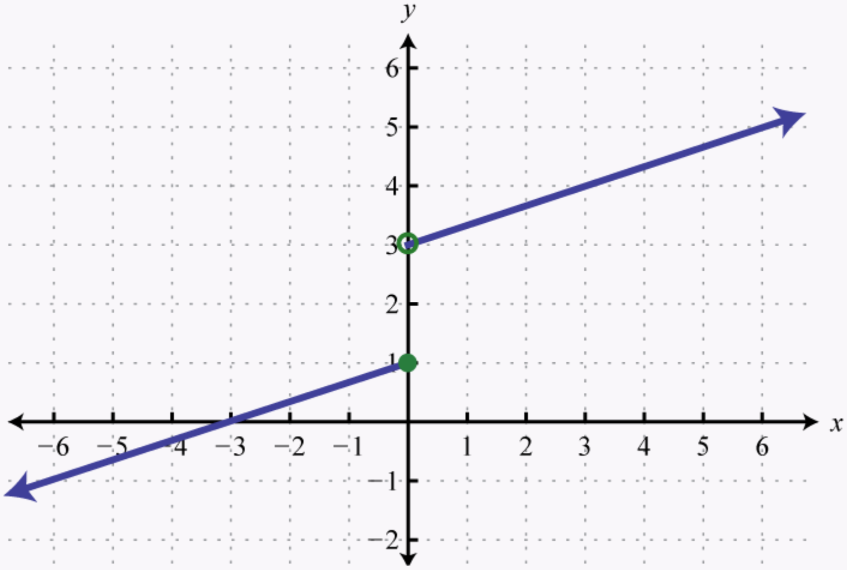

Figure 7.E.3

4.

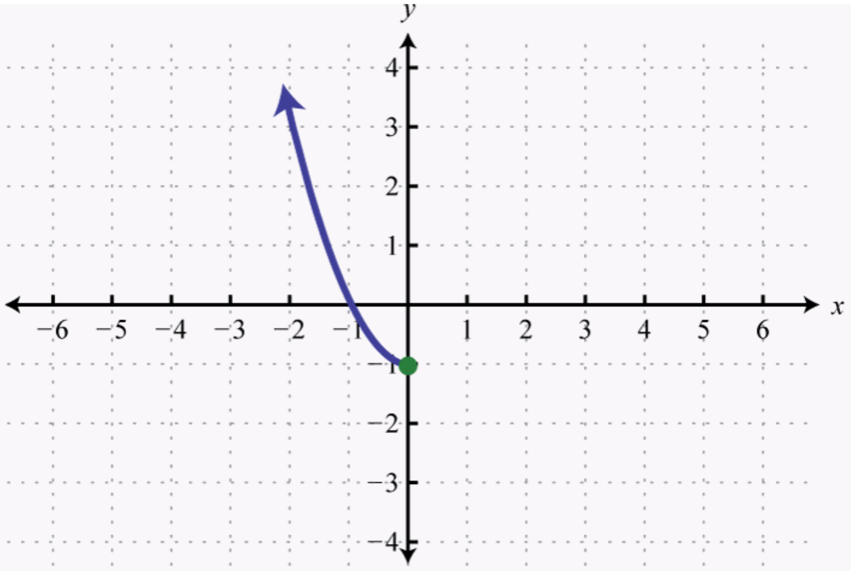

Figure 7.E.4

Answer

1. No, fails the HLT
3. Yes, passes the HLT

Exercise $7.E.3$

Verify algebraically that the two given functions are inverses. In other words, show that $\left(f\circ f^{-1}\right)(x)=x$ and $\left(f^{-1}\circ f\right)(x)=x$.

1. $f(x)=6x-5, f^{-1}(x)=\frac{1}{6}x+\frac{5}{6}$
2. $f(x)=\sqrt{2x+3}, f^{-1}(x)=\frac{x^2-3}{2}, x\geq 0$
3. $f(x)=\frac{x}{3x-2}, f^{-1}(x)=\frac{2x}{3x-1}$
4. $f(x)=\sqrt[3]{x+3}-4, f^{-1}(x)=(x+4)^3-3$

Answer

1. Proof
3. Proof

Exercise $7.E.4$

Find the inverses of each function defined as follows:

1. $f(x)=-7x+3$
2. $f(x)=\frac{2}{3}x-\frac{1}{2}$
3. $g(x)=x^2-12, x\geq 0$
4. $g(x)=(x-1)^3+5$
5. $g(x)=\frac{2}{x-1}$
6. $h(x)=\frac{x+5}{x-5}$
7. $h(x)=\frac{3x-1}{x}$
8. $p(x)=\sqrt[3]{5x}+3$
9. $h(x)=\sqrt[3]{2x-7}+2$
10. $h(x)=\sqrt[5]{x+2}-3$

Answer

1. $f^{-1}(x)=-\frac{1}{7}x+\frac{3}{7}$

3. $g^{-1}(x)=\sqrt{x+12}$

5. $g^{-1}(x)=\frac{x+2}{x}$

7. $h^{-1}(x)=-\frac{1}{x-3}$

9. $h^{-1}(x)=\frac{(x-2)^3+7}{2}$

Exercise $7.E.5$

Evaluate.

1. $f(x)=5^x$; find $f(-1)$, $f(0)$, and $f(3)$.
2. $f(x)=\left(\frac{1}{2}\right)^x$; find $f(-4)$, $f(0)$, and $f(-3)$.
3. $g(x)=10^{-x}$; find $g(-5)$, $g(0)$, and $g(2)$.
4. $g(x)=1-3^x$; find $g(-2)$, $g(0)$, and $g(3)$.

Answer

1. $f(-1)=\frac{1}{5}, f(0)=1, f(3)=125$

3. $g(-5)=100,000, g(0)=1, g(2)=\frac{1}{100}$

Exercise $7.E.6$

Sketch the exponential function. Draw the horizontal asymptote with a dashed line.

1. $f(x)=5^x+10$
2. $f(x)=5^{x-4}$
3. $f(x)=-3^x-9$
4. $f(x)=3^{x+2}+6$
5. $f(x)=\left(\frac{1}{3}\right)^x$
6. $f(x)=\left(\frac{1}{2}\right)^x-4$
7. $f(x)=2^{-x}+3$
8. $f(x)=1-3^{-x}$

Answer

1.

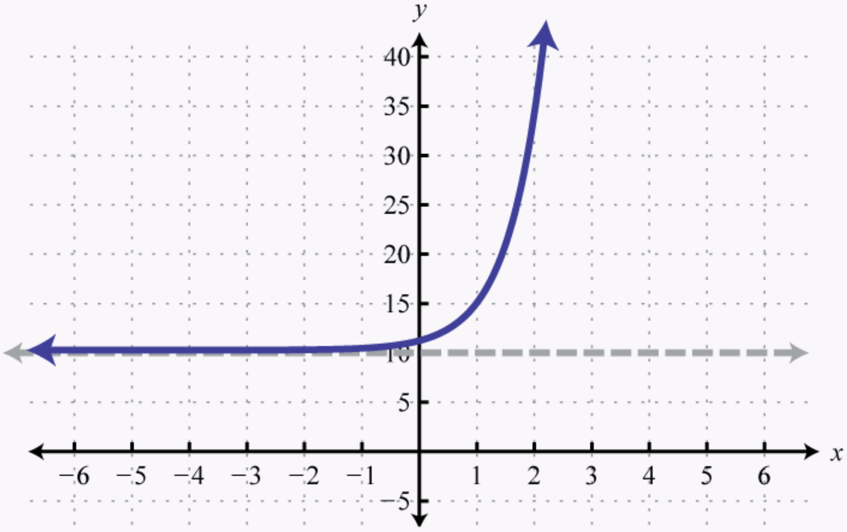

Figure 7.E.5

3.

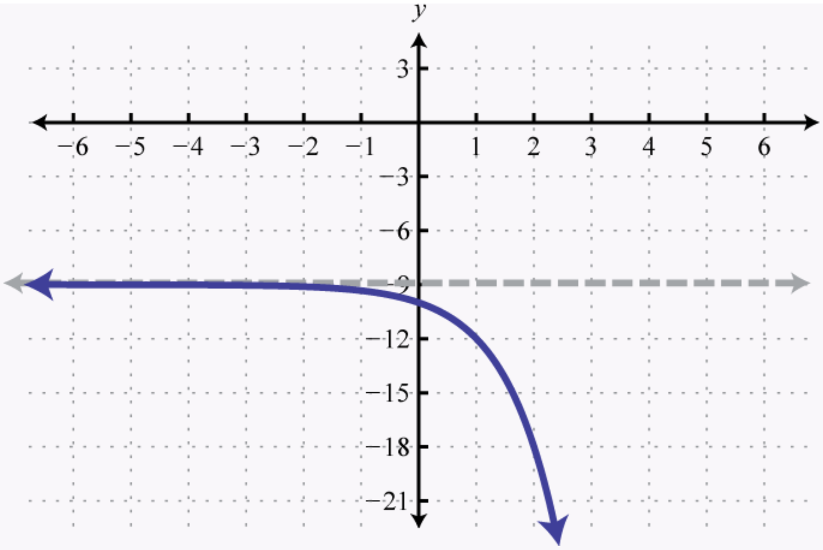

Figure 7.E.6

5.

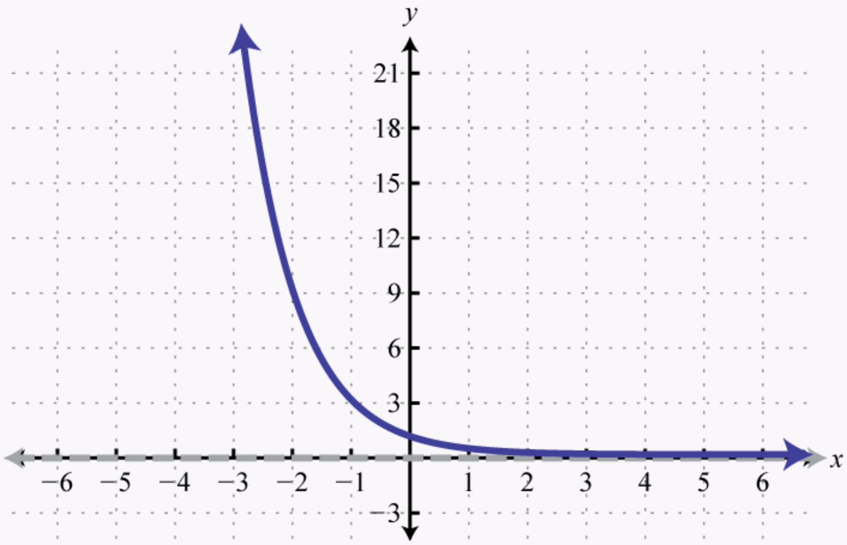

Figure 7.E.7

7.

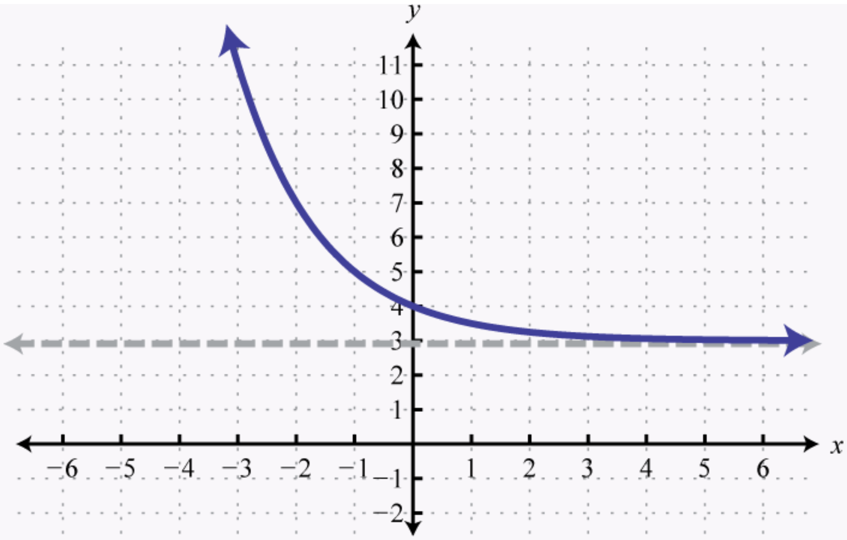

Figure 7.E.8

Exercise $7.E.7$

Use a calculator to evaluate the following. Round off to the nearest hundredth.

1. $f(x)=e^{x}+1$; find $f(-3), f(-1)$, and $f\left(\frac{1}{2}\right)$.
2. $g(x)=2-3e^{x}$; find $g(-1), g(0)$, and $g\left(\frac{2}{3}\right)$.
3. $p(x)=1-5e^{-x}$; find $p(-4), p\left(-\frac{1}{2}\right)$, and $p(0)$.
4. $r(x)=e^{-2x}-1$; find $r(-1), r\left(\frac{1}{4}\right)$, and $r(2)$.

Answer

1. $f(-3)\approx 1.05, f(-1)\approx 1.37, f\left(\frac{1}{2}\right)\approx 2.65$
3. $p(-4)\approx -271.99, p\left(-\frac{1}{2}\right)\approx -7.24, p(0)=-4$

Exercise $7.E.8$

Sketch the function. Draw the horizontal asymptote with a dashed line.

1. $f(x)=e^{x}+4$
2. $f(x)=e^{x-4}$
3. $f(x)=e^{x+3}+2$
4. $f(x)=e^{-x}+5$
5. Jerry invested $\$6,250$ in an account earning $3\frac{5}{8}\%$ annual interest that is compounded monthly. How much will be in the account after 4 years?
6. Jose invested $\$7,500$ in an account earning $4\frac{1}{4}\%$ annual interest that is compounded continuously. How much will be in the account after $3\frac{1}{2}$ years?
7. A 14-gram sample of radioactive iodine is accidently released into the atmosphere. The amount of the substance in grams is given by the formula $P(t)=14e^{-0.087t}$, where t represents the time in days after the sample was released. How much radioactive iodine will be present in the atmosphere 30 days after it was released?
8. The number of cells in a bacteria sample is given by the formula $N(t)=\frac{2.4\times 10^{5}}{1+9e^{-0.28t}}$, where t represents the time in hours since the initial placement of $24,000$ cells. Use the formula to calculate the number of cells in the sample 20 hours later.

Answer

1.

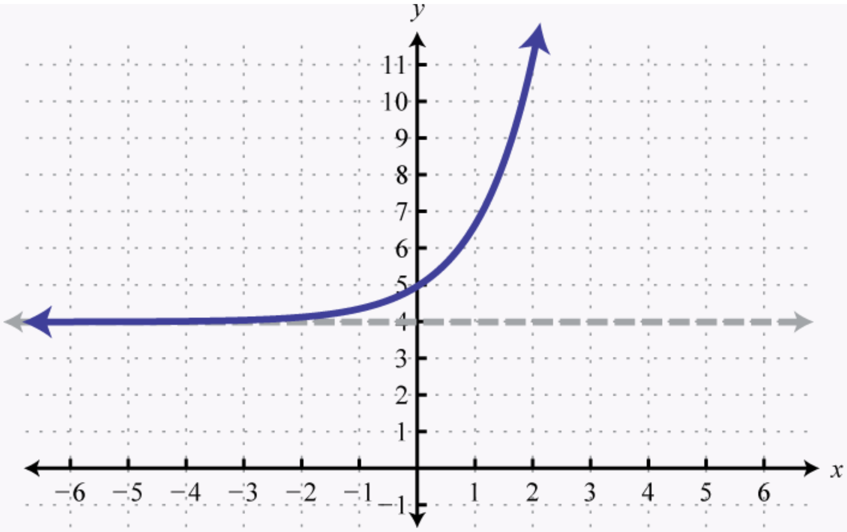

Figure 7.E.9

3.

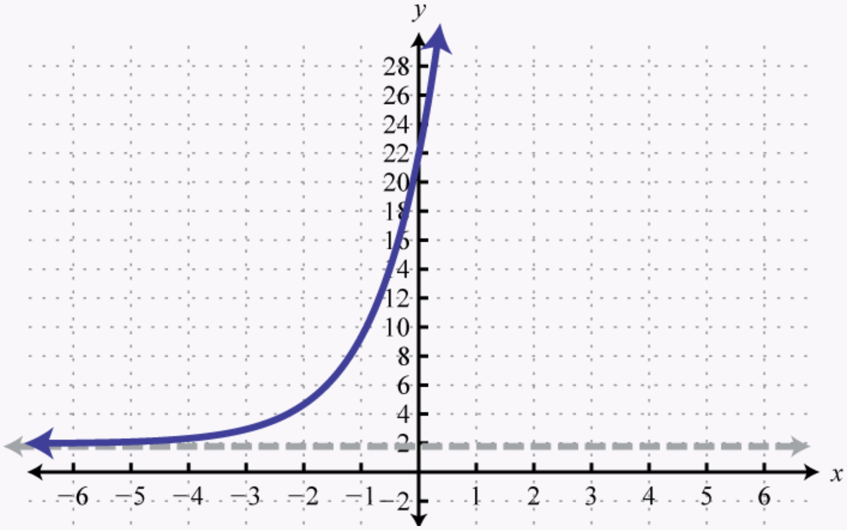

Figure 7.E.10

5. $7, 223.67

7. Approximately 1 gram

Exercise $7.E.9$

Evaluate.

1. $\log_4 16$
2. $\log_3 27$
3. $\log_2\left(\frac{1}{32}\right)$
4. $\log\left(\frac{1}{10}\right)$
5. $\log_{1/3} 9$
6. $\log_{3/4}\left(\frac{4}{3}\right)$
7. $\log_7 1$
8. $\log_3(-3)$
9. $\log_4 0$
10. $\log_3 81$

11. $\log_6 \sqrt{6}$
12. $\log_5 \sqrt[3]{25}$
13. $\ln e^8$
14. $\ln\left(\frac{1}{e^5}\right)$
15. $\log(0.00001)$
16. $\log 1,000,000$

Answer

1. 2
3. -5
5. -2
7. 0
9. Undefined
11. $\frac{1}{2}$
13. 8
15. -5

Exercise $7.E.10$

Find x.

1. $\log_5 x = 3$
2. $\log_3 x = -4$
3. $\log_{2/3} x = 3$
4. $\log_3 x = \frac{2}{5}$
5. $\log x = -3$
6. $\ln x = \frac{1}{2}$

Answer

1. 125
3. $\frac{8}{27}$
5. 0.001

Exercise $7.E.11$

Sketch the graph of the logarithmic function. Draw the vertical asymptote with a dashed line.

1. $f(x) = \log_2(x-5)$
2. $f(x) = \log_2 x - 5$
3. $g(x) = \log_3(x+5) + 15$
4. $g(x) = \log_3(x-5) - 5$
5. $h(x) = \log_4(-x) + 1$
6. $h(x) = 3 - \log_4 x$
7. $g(x) = \ln(x-2) + 3$
8. $g(x) = \ln(x+3) - 1$
9. The population of a certain small town is growing according to the function $P(t) = 89,000(1.035)^t$, where t represents time in years since the last census. Use the function to estimate the population $8\frac{1}{2}$ years after the census was taken.
10. The volume of sound L in decibels (dB) is given by the formula $L = 10\log\left(I/10^{-12}\right)$, where I represents the intensity of the sound in watts per square meter. Determine the volume of a sound with an intensity of 0.5 watts per square meter.

Answer

1.

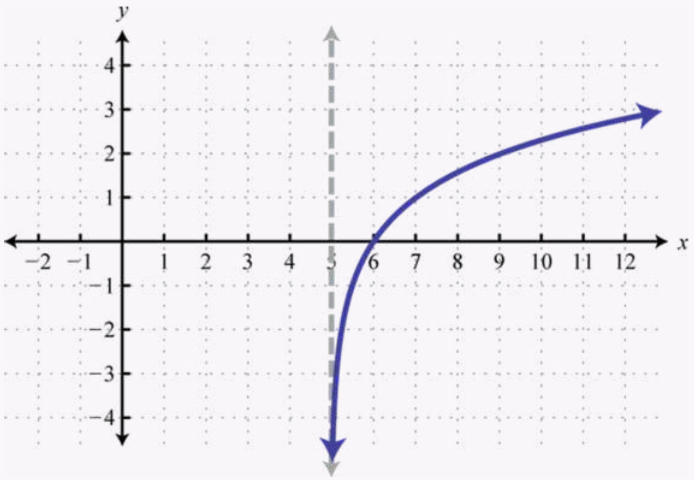

Figure 7.E.11

3.

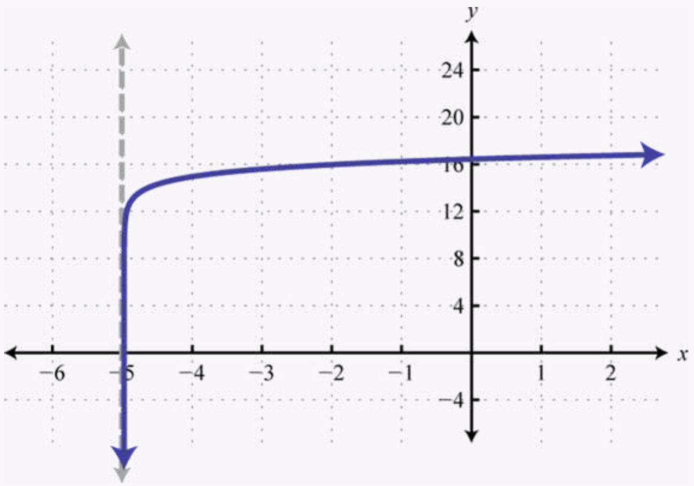

Figure 7.E.12

5.

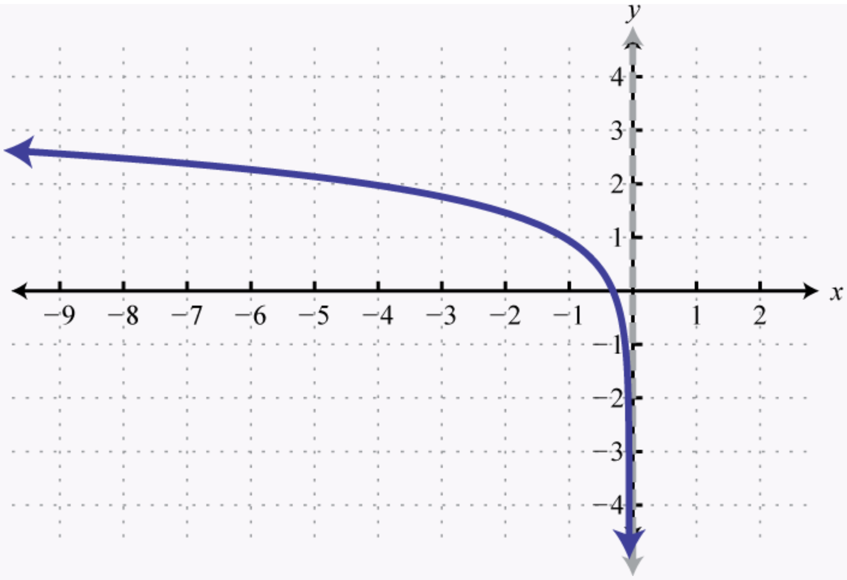

Figure 7.E.13

7.

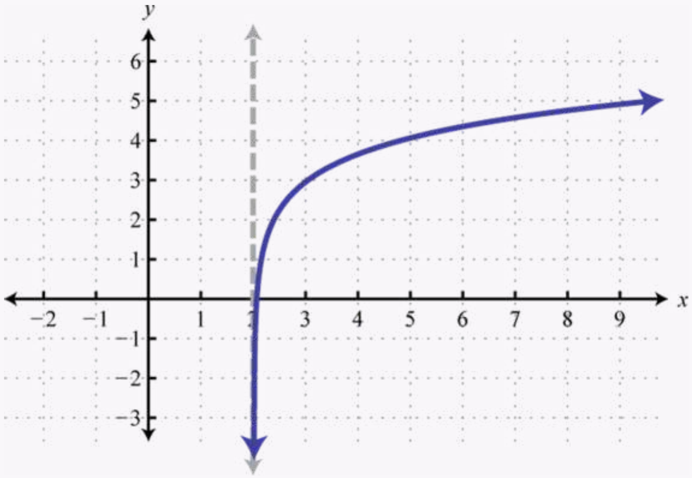

Figure 7.E.14

9. $119,229$ people

Exercise $7.E.12$

Evaluate without using a calculator.

1. $\log_9 9$
2. $\log_8 1$
3. $\log_{1/3} 3$
4. $\log\left(\frac{1}{10}\right)$
5. $e^{\ln 17}$
6. $10^{\log 27}$
7. $\ln e^{63}$
8. $\log 10^{33}$

Answer

1. 1

3. -1
5. 17
7. 63

Exercise $7.E.13$

Expand completely.

1. $\log\left(100x^2\right)$
2. $\log_5\left(5x^3\right)$
3. $\log_3\left(\frac{3x^5}{5}\right)$
4. $\ln\left(\frac{10}{3x^2}\right)$
5. $\log_2\left(\frac{8x^2}{y^2z}\right)$
6. $\log\left(\frac{x^{10}}{10y^3z^4}\right)$
7. $\ln\left(\frac{3b\sqrt{a}}{c^4}\right)$
8. $\log\left(\frac{20y^3}{\sqrt[3]{x^2}}\right)$

Answer

1. $2+2\log x$
3. $1+5\log_3 x-\log_3 5$
5. $3+2\log_2 x-2\log_2 y-\log_2 z$
7. $\ln 3+\ln b+\frac{1}{2}\ln a-4\ln c$

Exercise $7.E.14$

Write as a single logarithm with coefficient 1.

1. $\log x+2\log y-3\log z$
2. $\log_2 5-3\log_2 x+4\log_2 y$
3. $-2\log_5 x+\log_5 y-5\log_5(x-1)$
4. $\ln x-\ln(x-1)-\ln(x+1)$
5. $3\log_2 x+\frac{1}{2}\log_2 y-\frac{2}{3}\log_2 z$
6. $\frac{1}{3}\log x-3\log y-\frac{3}{5}\log z$
7. $\log_5 4+5\log_5 x-\frac{1}{3}\left(\log_5 y+2\log_5 z\right)$
8. $\ln x-\frac{1}{2}(\ln y-4\ln z)$

Answer

1. $\log\left(\frac{xy^2}{z^3}\right)$
3. $\log_5\left(\frac{y}{x^2(x-1)^5}\right)$
5. $\log_2\left(\frac{x^3\sqrt{y}}{\sqrt[3]{z^2}}\right)$
7. $\log_5\left(\frac{4x^5}{\sqrt[3]{yz^2}}\right)$

Exercise $7.E.15$

Solve. Give the exact answer and the approximate answer rounded to the nearest hundredth where appropriate.

1. $5^{2x+1}=125$
2. $10^{3x-2}=100$
3. $9^{x-3}=81$
4. $16^{2x+3}=8$
5. $5^x=7$
6. $3^{2x}=5$

7. $10^{x+2}-3=7$
8. $e^{2x-1}+2=3$
9. $7^{4x-1}-2=9$
10. $3^{5x-2}+5=7$
11. $3-e^{4x}=2$
12. $5+e^{3x}=4$
13. $\frac{4}{1+e^{5x}}=2$
14. $\frac{100}{1+e^{3x}}=\frac{1}{2}$

Answer

1. 1

3. 5

5. $\frac{\log(7)}{\log(5)}\approx 1.21$

7. -1

9. $\frac{\log 7+\log 11}{4\log 7}\approx 0.56$

11. 0

13. 0

Exercise $7.E.16$

Use the change of base formula to approximate the following to the nearest tenth.

1. $\log_5 13$
2. $\log_2 27$
3. $\log_4 5$
4. $\log_9 0.81$
5. $\log_{1/4} 21$
6. $\log_2 \sqrt[3]{5}$

Answer

1. 1.6

3. 1.2

5. -2.2

Exercise $7.E.17$

Solve.

1. $\log_2(3x-5)=\log_2(2x+7)$
2. $\ln(7x)=\ln(x+8)$
3. $\log_5 8-2\log_5 x=\log_5 2$
4. $\log_3(x+2)+\log_3(x)=\log_3 8$
5. $\log_5(2x-1)=2$
6. $2\log_4(3x-2)=4$
7. $2=\log_2\left(x^2-4\right)-\log_2 3$
8. $\log_2(x-1)+\log_2(x+1)=3$
9. $\log_2 x+\log_2(x-1)=1$
10. $\log_4(x+5)+\log_4(x+11)=2$
11. $\log(2x+5)-\log(x-1)=1$
12. $\ln x-\ln(2x-1)=1$
13. $2\log_2(x+4)=\log_2(x+2)+3$
14. $2\log_3 x=1+\log_3(x+6)$

15. $\log_3(x+1) - 2\log_3 x = 1$
16. $\log_5(2x) + \log_5(x-1) = 1$

Answer

1. 12

3. 2

5. 13

7. ± 4

9. 2

11. $\frac{15}{8}$

13. 0

15. $\frac{1+\sqrt{13}}{6}$

Exercise $7.E.18$

Solve.

1. An amount of $\$3,250$ is invested in an account that earns 4.6% annual interest that is compounded monthly. Estimate the number of years for the amount in the account to reach $\$4,000$.
2. An amount of $\$2,500$ is invested in an account that earns 5.5% annual interest that is compounded continuously. Estimate the number of years for the amount in the account to reach $\$3,000$.
3. How long does it take to double an investment made in an account that earns $6\frac{3}{4}\%$ annual interest that is compounded continuously?
4. How long does it take to double an investment made in an account that earns $6\frac{3}{4}\%$ annual interest that is compounded semi-annually?
5. In the year 2000 a certain small town had a population of $46,000$ people. In the year 2010 the population was estimated to have grown to $92,000$ people. If the population continues to grow exponentially at this rate, estimate the population in the year 2016.
6. A fleet van was purchased new for $\$28,000$ and 2 years later it was valued at $\$20,000$. If the value of the van continues to decrease exponentially at this rate, determine its value 7 years after it is purchased new.
7. A website that has been in decline registered $4,200$ unique visitors last month and $3,600$ unique visitors this month. If the number of unique visitors continues to decline exponentially, how many unique visitors would you expect next month?
8. An initial population of 18 rabbits was introduced into a wildlife preserve. The number of rabbits doubled in the first year. If the rabbit population continues to grow exponentially at this rate, how many rabbits will be present 5 years after they were introduced?
9. The half-life of sodium-24 is about 15 hours. How long will it take a 50-milligram sample to decay to 10 milligrams?
10. The half-life of radium-226 is about $1,600$ years. How long will it take an initial sample to decay to 30% of the original amount?
11. An archeologist discovered a bone tool artifact. After analysis, the artifact was found to contain 62% of the carbon-14 normally found in bone from the same animal. Given that carbon-14 has a half-life of $5,730 years$, estimate the age of the artifact.
12. The half-life of radioactive iodine-131 is about 8 days. What percentage of an initial sample accidentally released into the atmosphere do we expect to remain after 53 days?

Answer

1. 4.5 years

3. 10.27 years

5. About $139,446$ people

7. $3,086$ unique visitors

9. 35 hours

11. About $3,952$ years old

Sample Exam

Exercise $7.E.19$

1. Given $f(x)=x^{2}-x+3$ and $g(x)=3x-1$ find $(f\circ g)(x)$.
2. Show that $f(x)=\sqrt[3]{7x-2}$ and $g(x)=\frac{x^{3}+2}{7}$ are inverses.

Answer

1. $(f\circ g)(x)=9x^{2}-9x+5$

Exercise $7.E.20$

Find the inverse of the following functions:

1. $f(x)=\frac{1}{2}x-3$
2. $h(x)=x^{2}+3$ where $x\geq 0$

Answer

1. $f^{-1}(x)=2x+6$

Exercise $7.E.21$

Sketch the graph.

1. $f(x)=e^{x}-5$
2. $g(x)=10^{-x}$
3. Joe invested $\$5,200$ in an account earning 3.8% annual interest that is compounded monthly. How much will be in the account at the end of 4 years?
4. Mary has $\$3,500$ in a savings account earning $4\frac{1}{2}\%$ annual interest that is compounded continuously. How much will be in the account at the end of 3 years?

Answer

1.

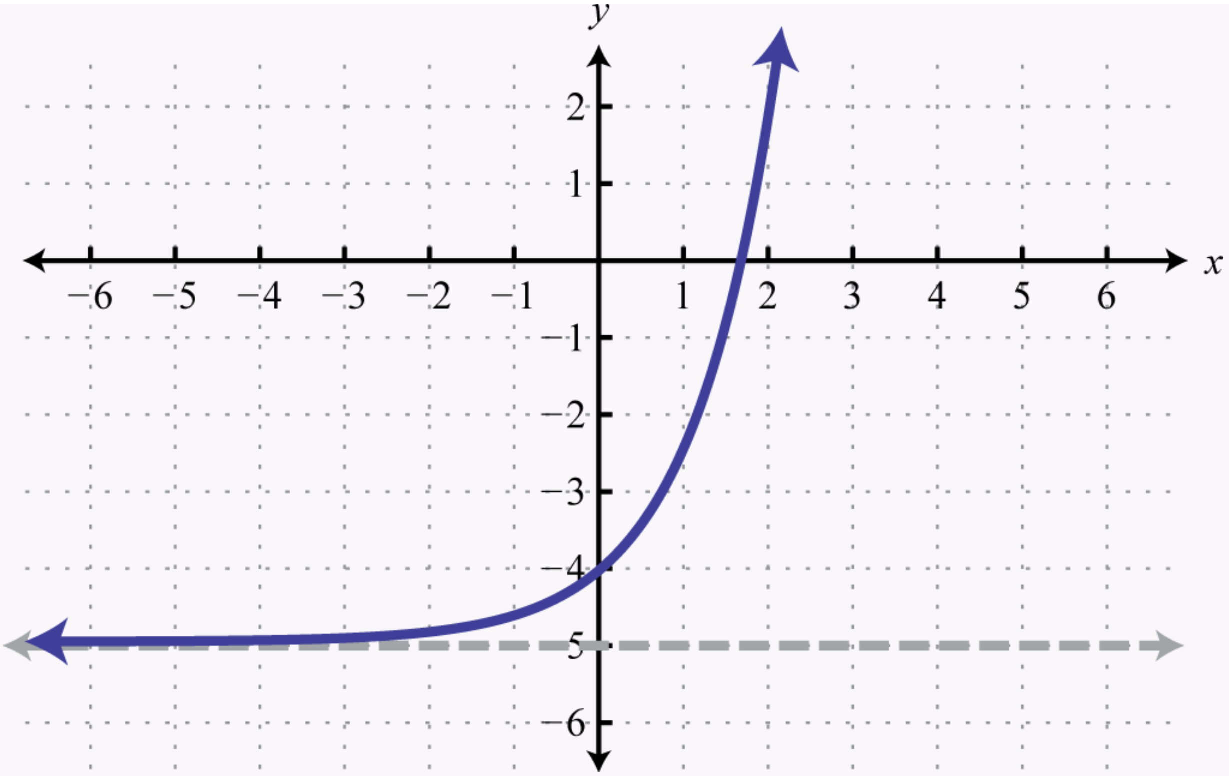

Figure 7.E.15

3. $\$6,052.18$

Exercise $7.E.22$

Evaluate.

1. $\log_3 81$
2. $\log_2\left(\frac{1}{4}\right)$
3. $\log 1,000$
4. $\ln e$
5. $\log_4 2$
6. $\log_9\left(\frac{1}{3}\right)$
7. $\ln e^3$
8. $\log_{1/5} 25$

Answer

1. 4
2. -2
3. 3
4. 1

Exercise $7.E.23$

Sketch the graph.

1. $f(x)=\log_4(x+5)+2$
2. $f(x)=-\ln(x-2)$

Answer

1.

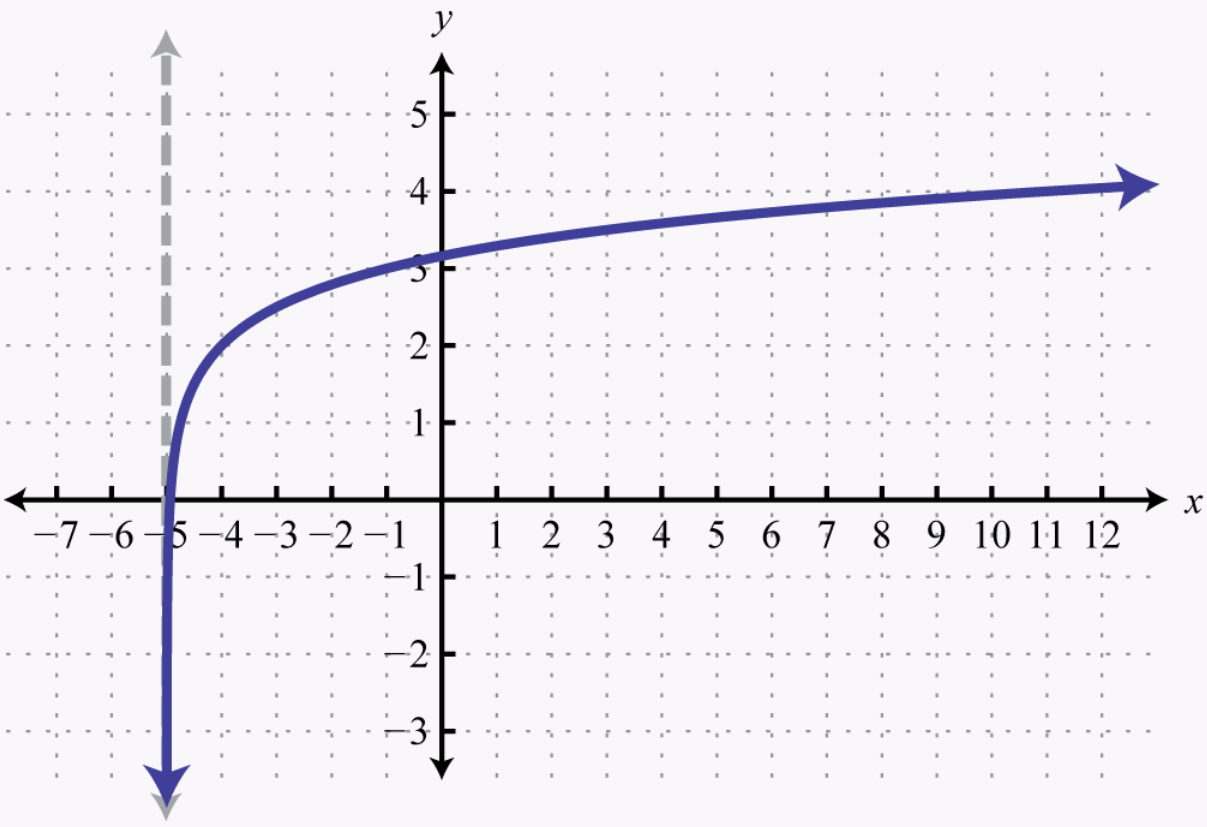

Figure 7.E.16

Exercise $7.E.24$

1. Expand: $\log\left(\frac{100x^2y}{\sqrt{z}}\right)$.
2. Write as a single logarithm with coefficient 1: $2\log_2 x+\frac{1}{3}\log_2 y-3\log_2 z$.

Answer

1. $2+2\log x+\log y-\frac{1}{2}\log z$

Exercise $7.E.25$

Evaluate. Round off to the nearest tenth.

1. $\log_2 10$
2. $\ln 1$
3. $\log_3\left(\frac{1}{5}\right)$

Answer

1. 3.3

2. 0

3. -1.5

Exercise $7.E.26$

Solve:

1. $2^{3x-1}=16$
2. $3^{7x+1}=5$

3. $\log_5(3x-4)=\log_5(2x+7)$
4. $\log_3\left(x^2+26\right)=3$
5. $\log_2 x+\log_2(2x+7)=2$
6. $\log(2x+3)=1+\log(x+1)$
7. Joe invested $\$5,200$ in an account earning 3.8% annual interest that is compounded monthly. How long will it take to accumulate a total of $\$6,200$ in the account?
8. Mary has $\$3,500$ in a savings account earning $4\frac{1}{2}\%$ annual interest that is compounded continuously. How long will it take to double the amount in the account?
9. During the exponential growth phase, certain bacteria can grow at a rate of 5.3% per hour. If $12,000$ cells are initially present in a sample, construct an exponential growth model and use it to:
   1. Estimate the population of bacteria in 3.5 hours.
   2. Estimate the time it will take the population to double.
10. The half-life of caesium-137 is about 30 years. Approximate the time it will take a 20-milligram sample of caesium-137 to decay to 8 milligrams.

Answer

2. $\frac{\log 5-\log 3}{7\log 3}$

4. ± 1

6. $-\frac{7}{8}$

8. 15.4 years

10. 40 years

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CHAPTER OVERVIEW

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8.3: Ellipses
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8.5: Solving Nonlinear Systems
8.E: Conic Sections (Exercises)

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8.1: Distance, Midpoint, and the Parabola

Learning Objectives

- Apply the distance and midpoint formulas.
- Graph a parabola using its equation given in standard from.
- Determine standard form for the equation of a parabola given general form.

Conic Sections

A **conic section**[1] is a curve obtained from the intersection of a right circular cone and a plane. The conic sections are the parabola, circle, ellipse, and hyperbola.

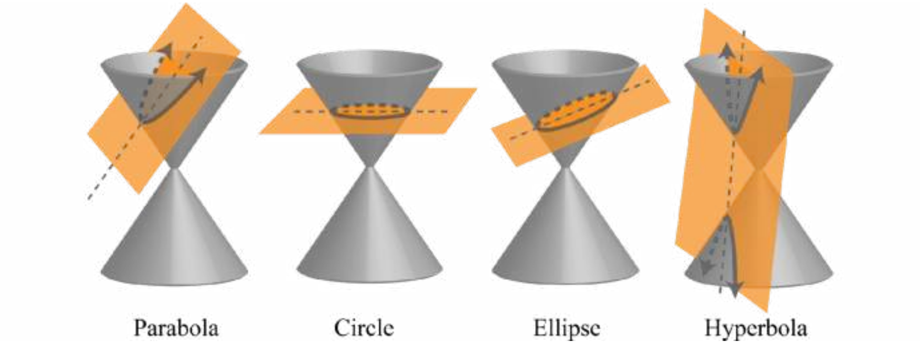

Figure 8.1.1

The goal is to sketch these graphs on a rectangular coordinate plane.

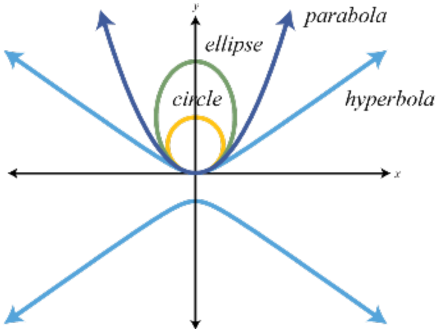

Figure 8.1.2

The Distance and Midpoint Formulas

We begin with a review of the **distance formula**[2]. Given two points $(x_1,\ y_1)$ and $(x_2,\ y_2)$ in a rectangular coordinate plane, the distance d between them is given by the distance formula,

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Furthermore, the point that bisects the line segment formed by these two points is called the **midpoint**[3] and is given by the formula,

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

The midpoint is an ordered pair formed by the average of the x-values and the average of the y-values.

Example 8.1.1:

Given $(-2,-5)$ and $(-4,-3)$ calculate the distance and midpoint between them.

Solution:

In this case, we will use the formulas with the following points:

$$\begin{array}{cc} (x_1, y_1) & (x_2, y_2) \\ (-2,-5) & (-4,-3) \end{array}$$

It is a good practice to include the formula in its general form before substituting values for the variables; this improves readability and reduces the probability of making errors.

$$\begin{aligned} d &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\ &= \sqrt{[-4-(-2)]^2+[-3-(-5)]^2} \\ &= \sqrt{(-4+2)^2+(-3+5)^2} \\ &= \sqrt{(-2)^2+(2)^2} \\ &= \sqrt{4+4} \\ &= \sqrt{8} \\ &= 2\sqrt{2} \end{aligned}$$

Next determine the midpoint.

$$\begin{aligned} \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) &= \left(\frac{-2+(-4)}{2}, \frac{-5+(-3)}{2}\right) \\ &= \left(\frac{-6}{2}, \frac{-8}{2}\right) \\ &= (-3,-4) \end{aligned}$$

Plotting these points on a graph we have,

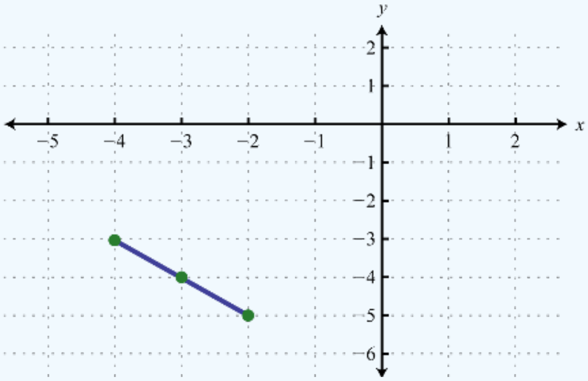

Figure $8.1.3$

Answer:

Distance: $2\sqrt{2}$ units; midpoint: $(-3,-4)$

Example $8.1.2$:

The diameter of a circle is defined by the two points $(-1,2)$ and $(1,-2)$. Determine the radius of the circle and use it to calculate its area.

Solution

Find the diameter using the distance formula.

$$\begin{aligned} d &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\ &= \sqrt{[1-(-1)]^2+(-2-2)^2} \\ &= \sqrt{[2]^2+(-4)^2} \\ &= \sqrt{4+16} \\ &= \sqrt{20} \\ &= 2\sqrt{5} \end{aligned}$$

Recall that the radius of a circle is one-half of the circle’s diameter. Therefore, if $d=2\sqrt{5}$ units, then

$$r=\frac{d}{2}=\frac{2\sqrt{5}}{2}=\sqrt{5}$$

The area of a circle is given by the formula $A=\pi r^2$ and we have

$$\begin{aligned} A &= \pi(\sqrt{5})^2 \\ &= \pi \cdot 5 \\ &= 5\pi \end{aligned}$$

Area is measured in square units.

Answer:

Radius: $\sqrt{5}$ units; area: 5π square units

Exercise $8.1.1$

Given $(0,0)$ and $(9,-3)$ calculate the distance and midpoint between them.

Answer

Distance: $3\sqrt{10}$ units; midpoint: $\left(\frac{9}{2},-\frac{3}{2}\right)$

www.youtube.com/v/6KQnAcoCPW0

The Parabola

A **parabola**[4] is the set of points in a plane equidistant from a given line, called the directrix, and a point not on the line, called the focus. In other words, if given a line L the directrix, and a point F the focus, then (x,y) is a point on the parabola if the shortest distance from it to the focus and from it to the line is equal as pictured below:

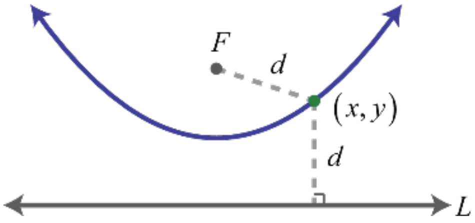

Figure $8.1.4$

The vertex of the parabola is the point where the shortest distance to the directrix is at a minimum. In addition, a parabola is formed by the intersection of a cone with an oblique plane that is parallel to the side of the cone:

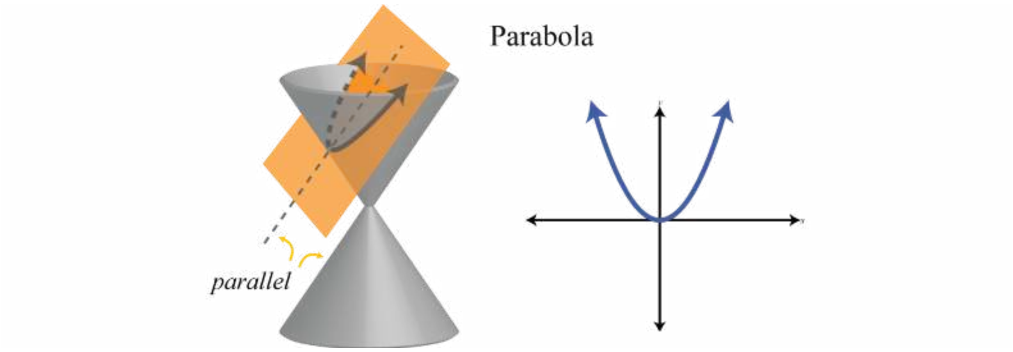

Figure $8.1.5$

Recall that the graph of a quadratic function, a polynomial function of degree 2, is parabolic. We can write the equation of a **parabola in general form**[5] or we can write the equation of a **parabola in standard form**[6]:

$$\begin{array}{cc} \textit{General Form} & \textit{Standard Form} \\ y=ax^2+bx+c & y=a(x-h)^2+k \end{array}$$

Both forms are useful in determining the general shape of the graph. However, in this section we will focus on obtaining standard form, which is often called **vertex form**[7]. Given a quadratic function in standard form, the vertex is (h,k). To see that this is the case, consider graphing $y=(x+3)^2+2$ using transformations.

$$\begin{aligned} y &= x^2 && \textit{Basic squaring function.} \\ y &= (x+3)^2 && \textit{Horizontal shift left 3 units.} \\ y &= (x+3)^2+2 && \textit{Vertical shift up 2 units.} \end{aligned}$$

Use these translations to sketch the graph,

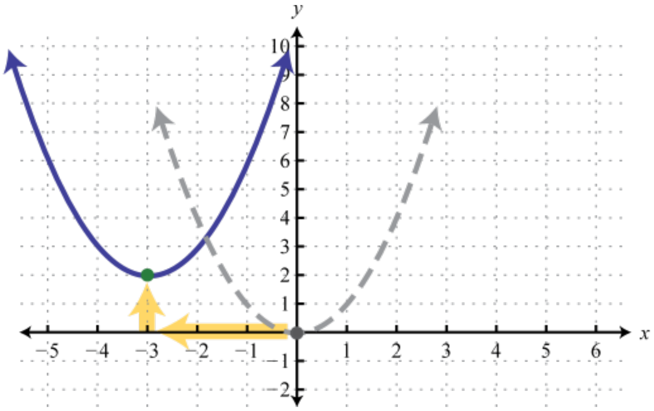

Figure 8.1.6

Here we can see that the vertex is $(-3,2)$. This can be determined directly from the equation in standard form,

$$\begin{aligned} y &= a(x-h)^2 \ + \ k \\ &\qquad\quad \downarrow \qquad \downarrow \\ y &= [x-(-3)]^2+2 \end{aligned}$$

Written in this form we can see that the vertex is $(-3,2)$. However, the equation is typically not given in standard form. Transforming general form to standard form, by completing the square, is the main process by which we will sketch all of the conic sections.

Example 8.1.3:

Rewrite the equation in standard form and determine the vertex of its graph: $y=x^2-8x+15$.

Solution

Begin by making room for the constant term that completes the square.

$$\begin{aligned} y &= x^2-8x+15 \\ &= x^2-8x+\_\_+15-\_\_ \end{aligned}$$

The idea is to add and subtract the value that completes the square, $\left(\frac{b}{2}\right)^2$, and then factor. In this case, add and subtract $\left(\frac{b}{2}\right)^2 = \left(\frac{-8}{2}\right)^2 = (-4)^2 = 16$.

$$\begin{aligned} y &= x^2-8x+15 && \textit{Add and subtract 16.} \\ &= \left(x^2-8x+16\right)+15-16 && \textit{Factor.} \\ &= (x-4)(x-4)-1 \\ &= (x-4)^2-1 \end{aligned}$$

Adding and subtracting the same value within an expression does not change it. Doing so is equivalent to adding 0. Once the equation is in this form, we can easily determine the vertex.

$$\begin{aligned} y &= a(x-h)^2 \ + \ k \\ &\qquad\quad \downarrow \qquad \downarrow \\ y &= (x \ - \ 4)^2+(-1) \end{aligned}$$

Here we have a translation to the right 4 units and down 1 unit. Hence, $h=4$ and $k=-1$.

Answer:

$y=(x-4)^2-1$; vertex: $(4,-1)$

If there is a leading coefficient other than 1, then begin by factoring out that leading coefficient from the first two terms of the trinomial.

Example 8.1.4:

Rewrite the equation in standard form and determine the vertex of the graph: $y=-2x^2+12x-16$.

Solution

Since $a=-2$, factor this out of the first two terms in order to complete the square. Leave room inside the parentheses to add and subtract the value that completes the square.

$$\begin{aligned} y &= -2x^2+12x-16 \\ &= -2\left(x^2-6x+\_\_-\_\_\right)-16 \end{aligned}$$

Now use -6 to determine the value that completes the square. In this case, $\left(\frac{b}{2}\right)^2=\left(\frac{-6}{2}\right)^2=(-3)^2=9$. Add and subtract 9 and factor as follows:

$$\begin{aligned} y &= -2x^2+12x-16 & \\ &= -2\left(x^2-6x+\_\_-\_\_\right)-16 & \textit{Add and subtract 9.} \\ &= -2\left(x^2-6x+9-9\right)-16 & \textit{Factor.} \\ &= -2[(x-3)(x-3)-9]-16 & \\ &= -2\left[(x-3)^2-9\right]-16 & \textit{Distribute the } -2. \\ &= -2(x-3)^2+18-16 & \\ &= -2(x-3)^2+2 & \end{aligned}$$

In this form, we can easily determine the vertex.

$$\begin{aligned} y &= a\,(x\;-\;h)^2+k \\ &\qquad\quad\;\;\downarrow\quad\downarrow \\ y &= -2(x-3)^2+2 \end{aligned}$$

Here $h=3$ and $k=2$.

Answer:

y=−2(x−3)2+2y=−2(x−3)2+2; vertex: (3,2)(3,2)

Make use of both general form and standard form when sketching the graph of a parabola.

Example 8.1.5:

Graph: $y=-2x^2+12x-16$.

Solution:

From the previous example we have two equivalent forms of this equation,

$$\begin{array}{cc} \textit{General Form} & \textit{Standard Form} \\ y=-2x^2+12x-16 & y=-2(x-3)^2+2 \end{array}$$

Recall that if the leading coefficient $a>0$ the parabola opens upward and if $a<0$ the parabola opens downward. In this case, $a=-2$ and we conclude the parabola opens downward. Use general form to determine the y-intercept. When $x=0$ we can see that the y-intercept is $(0,-16)$. From the equation in standard form, we can see that the vertex is $(3,2)$. To find the x-intercept we could use either form. In this case, we will use standard form to determine the x-values where $y=0$,

$$\begin{aligned} y &= -2(x-3)^2+2 & \textit{Set } y=0 \textit{ and solve.} \\ 0 &= -2(x-3)^2+2 & \\ -2 &= -2(x-3)^2 & \\ 1 &= (x-3)^2 & \textit{Apply the square root property.} \\ \pm 1 &= x-3 & \\ 3\pm 1 &= x & \end{aligned}$$

Here $x=3-1=2$ or $x=3+1=4$ and therefore the x-intercepts are $(2,0)$ and $(4,0)$. Use this information to sketch the graph.

Answer:

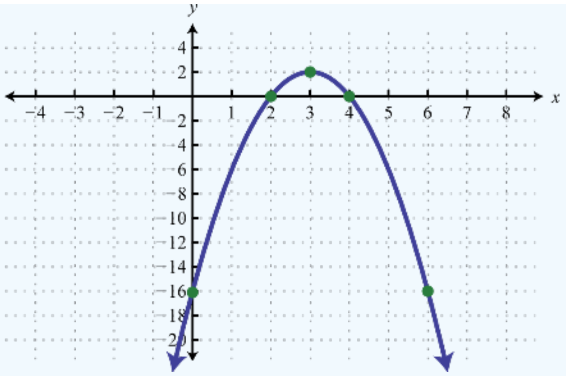

Figure $8.1.7$

So far we have been sketching parabolas that open upward or downward because these graphs represent functions. At this point we extend our study to include parabolas that open right or left. If we take the equation that defines the parabola in the previous example,

$$y=-2(x-3)^2+2$$

and switch the x and y values we obtain

$$x=-2(y-3)^2+2$$

This produces a new graph with symmetry about the line $y=x$.

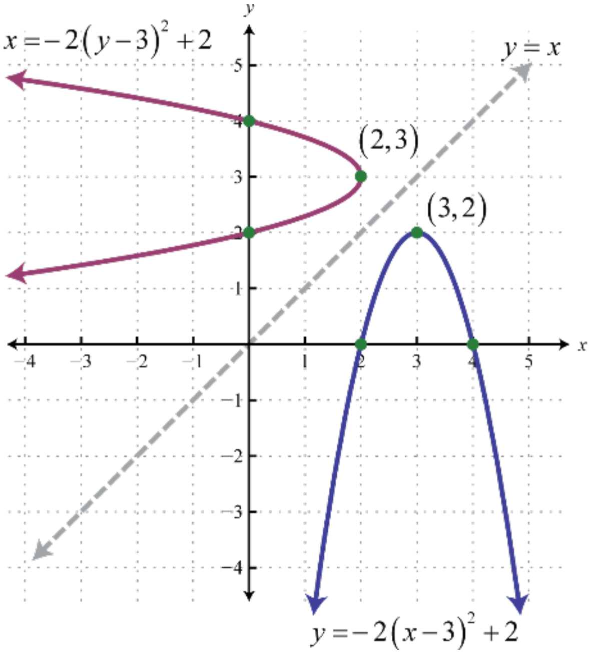

Figure $8.1.8$

Note that the resulting graph is not a function. However, it does have the same general parabolic shape that opens left. We can recognize equations of parabolas that open left or right by noticing that they are quadratic in y instead of x. Graphing parabolas that open left or right is similar to graphing parabolas that open upward and downward. In general, we have

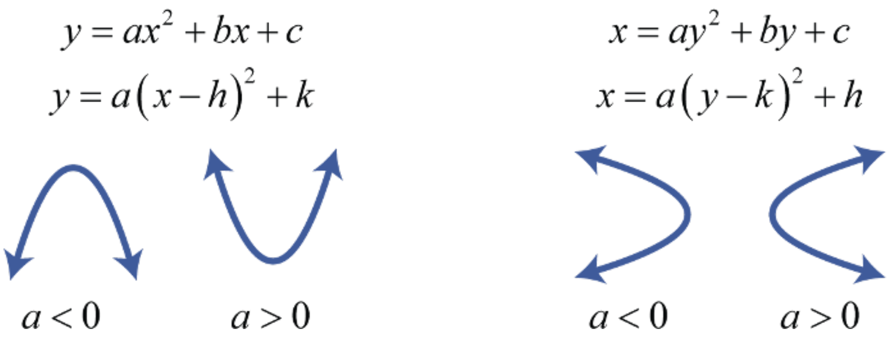

Figure $8.1.9$

In all cases, the vertex is (h, k). Take care to note the placement of h and k in each equation.

Example $8.1.6$:

Graph: $x = y^2 + 10y + 13$.

Solution

Because the coefficient of y^2 is positive, $a = 1$, we conclude that the graph is a parabola that opens to the right. Furthermore, when $y = 0$ it is clear that $x = 13$ and therefore the x-intercept is $(13, 0)$. Complete the square to obtain standard form. Here we will add and subtract $\left(\frac{b}{2}\right)^2 = \left(\frac{10}{2}\right)^2 = (5)^2 = 25$.

$$\begin{aligned} x &= y^2 + 10y + 13 \\ &= y^2 + 10y + 25 - 25 + 13 \\ &= (y+5)(y+5) - 12 \\ &= (y+5)^2 - 12 \end{aligned}$$

Therefore,

$$\begin{array}{l} x = a\,(y \;-\; k)^2 \;\;+\;\; h \\ \qquad\qquad\;\; \downarrow \qquad\;\; \downarrow \\ x = (y - (-5))^2 + (-12) \end{array}$$

From this we can see that the vertex $(h, k) = (-12, -5)$. Next use standard form to find the y-intercepts by setting $x = 0$.

$$\begin{aligned} x &= (y+5)^2 - 12 \\ 0 &= (y+5)^2 - 12 \\ 12 &= (y+5)^2 \\ \pm\sqrt{12} &= y + 5 \\ \pm 2\sqrt{3} &= y + 5 \\ -5 \pm 2\sqrt{3} &= y \end{aligned}$$

The y-intercepts are $(0, -5 - 2\sqrt{3})$ and $(0, -5 + 2\sqrt{3})$. Use this information to sketch the graph.

Answer

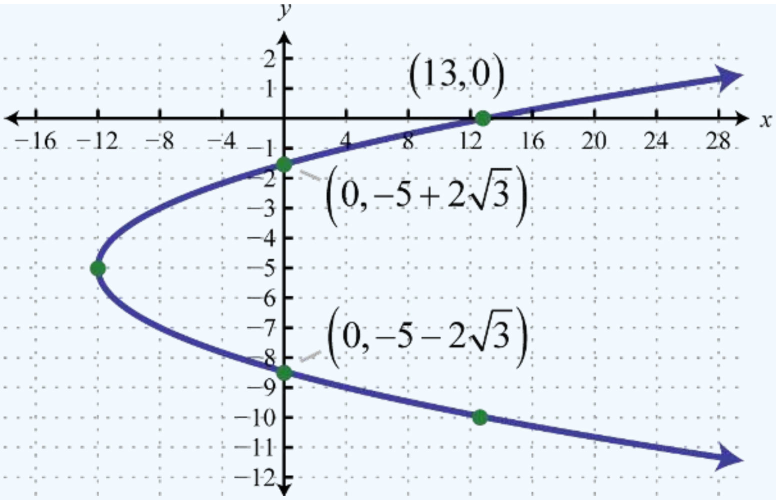

Figure 8.1.10

Example 8.1.7

Graph: $x=-2y^2+4y-5$.

Solution

Because the coefficient of y^2 is $a=-2$, we conclude that the graph is a parabola that opens to the left. Furthermore, when $y=0$ it is clear that $x=-5$ and therefore the x-intercept is $(-5,0)$. Begin by factoring out the leading coefficient as follows:

$$\begin{aligned} x &= -2y^2+4y-5 \\ &= -2\left(y^2-2y+\_\_-\_\_\right)-5 \end{aligned}$$

Here we will add and subtract $\left(\frac{b}{2}\right)^2=\left(\frac{-2}{2}\right)^2=(-1)^2=1$.

$$\begin{aligned} x &= -2y^2+4y-5 \\ &= -2\left(y^2-2y+1-1\right)-5 \\ &= -2\left[(y-1)^2-1\right]-5 \\ &= -2(y-1)^2+2-5 \\ &= -2(y-1)^2-3 \end{aligned}$$

Therefore, from vertex form, $x=-2(y-1)^2-3$, we can see that the vertex is $(h,k)=(-3,1)$. Because the vertex is at $(-3,1)$ and the parabola opens to the left, we can conclude that there are no y-intercepts. Since we only have two points, choose some y-values and find the corresponding x-values.

Table 8.1.1

x	y	$x=-2(y-1)^2-3$
-11	-1	$x=-2(-1-1)^2-3=-2(-2)^2-3=-11$
-5	2	$x=-2(2-1)^2-3=-2(1)^2-3=-5$
-11	3	$x=-2(3-1)^2-3=-2(2)^2-3=-11$

Answer:

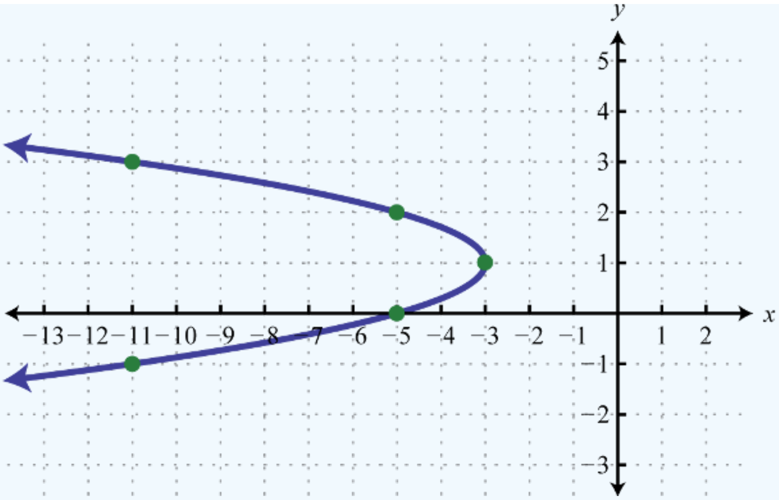

Figure $8.1.11$

Exercise $8.1.2$:

Graph: $x = y^2 - y - 6$.

Answer:

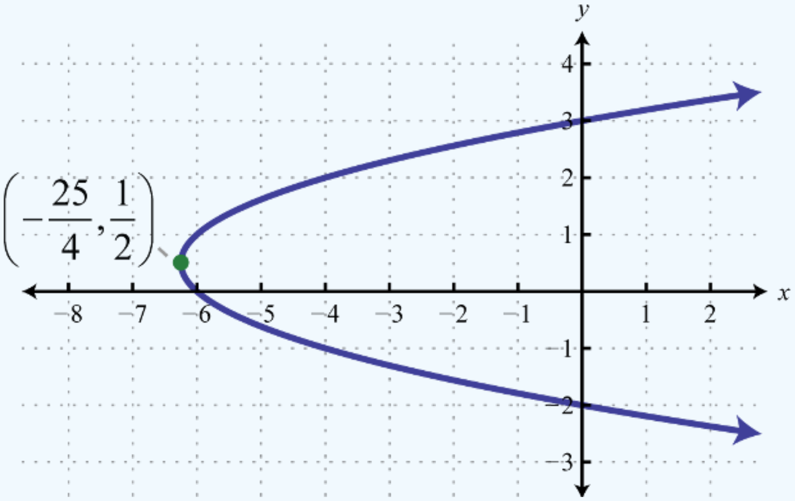

Figure $8.1.12$

www.youtube.com/v/pxgE96NhSCI

Key Takeaways

- Use the distance formula to determine the distance between any two given points. Use the midpoint formula to determine the midpoint between any two given points.
- A parabola can open upward or downward, in which case, it is a function. In this section, we extend our study of parabolas to include those that open left or right. Such graphs do not represent functions.
- The equation of a parabola that opens upward or downward is quadratic in $x, y = ax^2 + bx + c$. If $a > 0$, then the parabola opens upward and if a<0a<0, then the parabola opens downward.
- The equation of a parabola that opens left or right is quadratic in $y, x = ay^2 + by + c$. If $a > 0$, then the parabola opens to the right and if $a < 0$, then the parabola opens to the left.
- The equation of a parabola in general form $y = ax^2 + bx + c$ or $x = ay^2 + by + c$ can be transformed to standard form $y = a(x - h)^2 + k$ or $x = a(y - k)^2 + h$ by completing the square.
- When completing the square, ensure that the leading coefficient of the variable grouping is 1 before adding and subtracting the value that completes the square.

- Both general and standard forms are useful when graphing parabolas. Given standard form the vertex is apparent (h, k). To find the x-intercept set $y = 0$ and solve for x and to find the y-intercept set $x = 0$ and solve for y.

Exercise 8.1.3

Calculate the distance and midpoint between the given two points.

1. $(-1, -3)$ and $(5, -11)$
2. $(-3, 2)$ and $(1, -1)$
3. $(4, -2)$ and $(-2, -6)$
4. $(-5, -6)$ and $(-3, -4)$
5. $(10, -1)$ and $(9, 6)$
6. $(-6, -4)$ and $(-12, 1)$
7. $(0, 0)$ and $(\sqrt{2}, \sqrt{3})$
8. $(0, 0)$ and $(2\sqrt{2}, -\sqrt{3})$
9. $(\sqrt{5}, -\sqrt{3})$ and $(2\sqrt{5}, -\sqrt{3})$
10. $(3\sqrt{10}, \sqrt{6})$ and $(\sqrt{10}, -5\sqrt{6})$
11. $(\frac{1}{2}, -1)$ and $(-2, \frac{3}{2})$
12. $(-\frac{4}{3}, 2)$ and $(-\frac{1}{3}, -\frac{1}{2})$
13. $(\frac{1}{5}, -\frac{9}{5})$ and $(\frac{3}{10}, -\frac{5}{2})$
14. $(-\frac{1}{2}, \frac{4}{3})$ and $(-\frac{2}{3}, \frac{5}{6})$
15. (a, b) and $(0, 0)$
16. $(0, 0)$ and $(a\sqrt{2}, 2\sqrt{a})$

Answer

1. Distance: 10 units; midpoint: $(2, -7)$

3. Distance: $2\sqrt{13}$ units; midpoint: $(1, -4)$

5. Distance: $5\sqrt{2}$ units; midpoint: $(\frac{19}{2}, \frac{5}{2})$

7. Distance: $\sqrt{5}$ units; midpoint: $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\right)$

9. Distance: $\sqrt{5}$ units; midpoint: $\left(\frac{3\sqrt{5}}{2}, -\sqrt{3}\right)$

11. Distance: $\frac{5\sqrt{2}}{2}$ units; midpoint: $(-\frac{3}{4}, \frac{1}{4})$

13. Distance: $\frac{\sqrt{2}}{2}$ units; midpoint: $(\frac{1}{4}, -\frac{43}{20})$

15. Distance: $\sqrt{a^2 + b^2}$ units; midpoint: $(\frac{a}{2}, \frac{b}{2})$

Exercise 8.1.4

Determine the area of a circle whose diameter is defined by the given two points.

1. $(-8, 12)$ and $(-6, 8)$
2. $(9, 5)$ and $(9, -1)$
3. $(7, -8)$ and $(5, -10)$
4. $(0, -5)$ and $(6, 1)$
5. $(\sqrt{6}, 0)$ and $(0, 2\sqrt{3})$
6. $(0, \sqrt{7})$ and $(\sqrt{5}, 0)$

Answer

1. 5π square units

3. 2π square units

5. $\frac{9}{2}\pi$ square units

Exercise 8.1.5

Determine the perimeter of the triangle given the coordinates of the vertices.

1. $(5,3)$, $(2,-3)$, and $(8,-3)$
2. $(-3,2)$, $(-4,-1)$, and $(-1,0)$
3. $(3,3)$, $(5,3-2\sqrt{3})$, and $(7,3)$
4. $(0,0)$, $(0,2\sqrt{2})$, and $(\sqrt{2},0)$

Answer

1. $6+6\sqrt{5}$ units

3. 12 units

Exercise 8.1.6

Find a so that the distance d between the points is equal to the given quantity.

1. $(1,2)$ and $(4,a)$; $d=5$ units
2. $(-3,a)$ and $(5,6)$; $d=10$ units
3. $(3,1)$ and $(a,0)$; $d=\sqrt{2}$ units
4. $(a,1)$ and $(5,3)$; $d=\sqrt{13}$ units

Answer

1. $-2,6$

3. $2,4$

Exercise 8.1.7

Graph. Be sure to find the vertex and all intercepts.

1. $y=x^2+3$
2. $y=\frac{1}{2}(x-4)^2$
3. $y=-2(x+1)^2-1$
4. $y=-(x-2)^2+1$
5. $y=-x^2+3$
6. $y=-(x+1)^2+5$
7. $x=y^2+1$
8. $x=y^2-4$
9. $x=(y+2)^2$
10. $x=(y-3)^2$
11. $x=-y^2+2$
12. $x=-(y+1)^2$
13. $x=\frac{1}{3}(y-3)^2-1$
14. $x=-\frac{1}{3}(y+3)^2-1$

Answer

1.

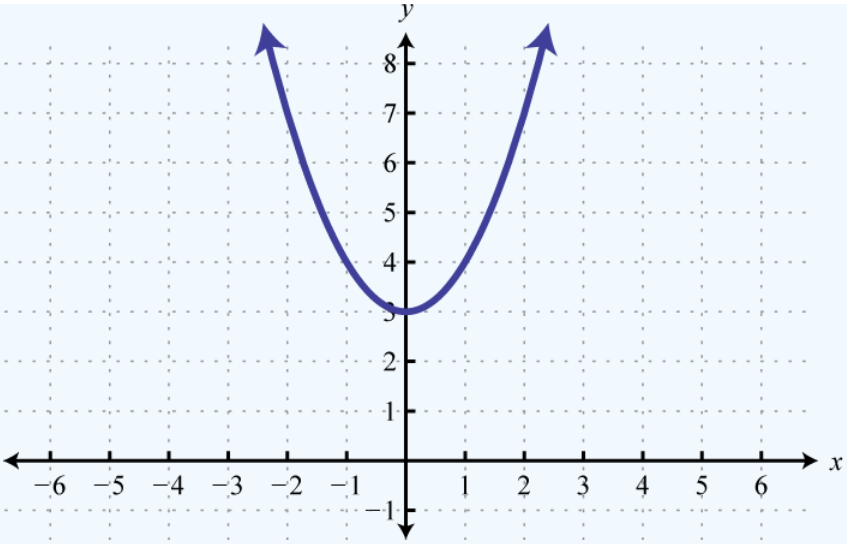

Figure 8.1.13

3.

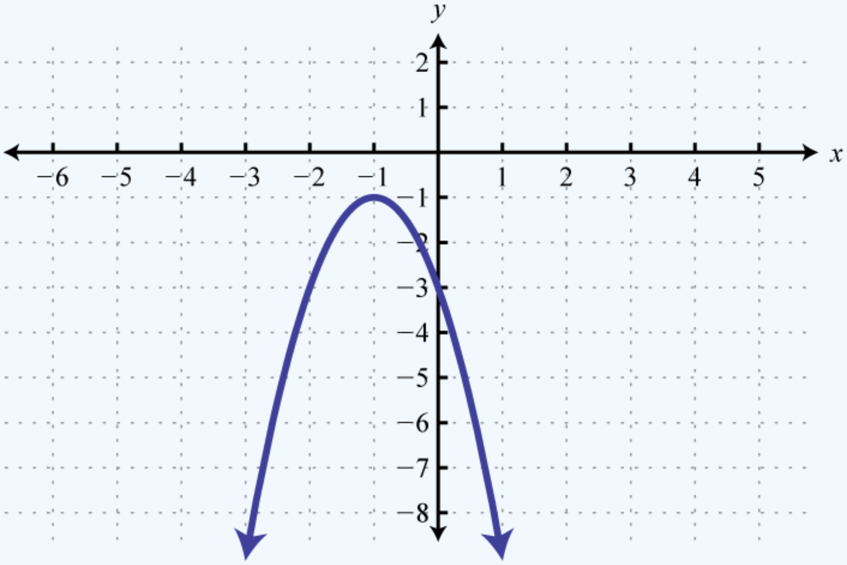

Figure 8.1.14

5.

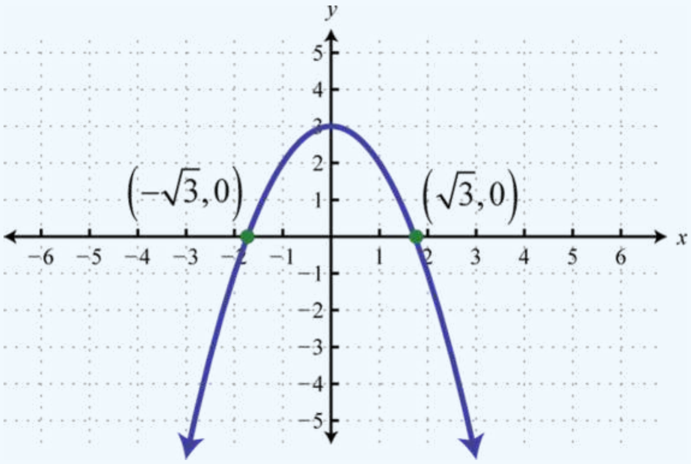

Figure 8.1.15

7.

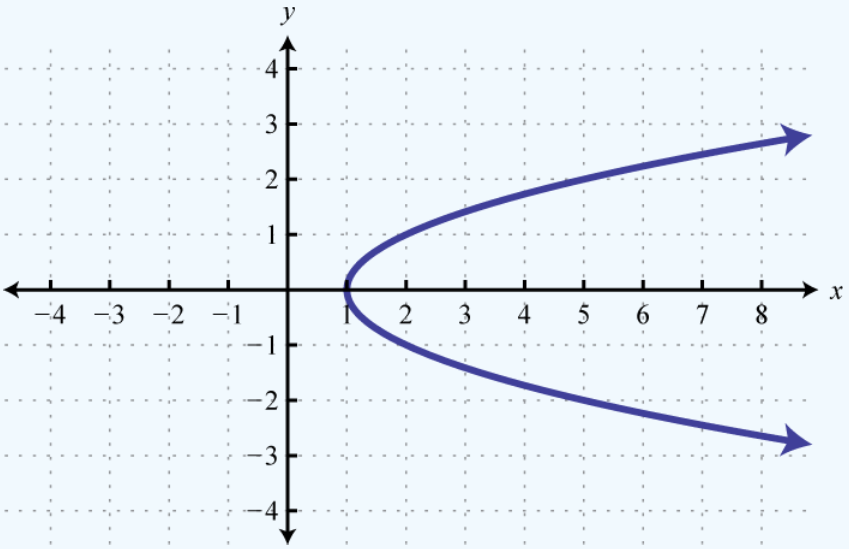

Figure 8.1.16

9.

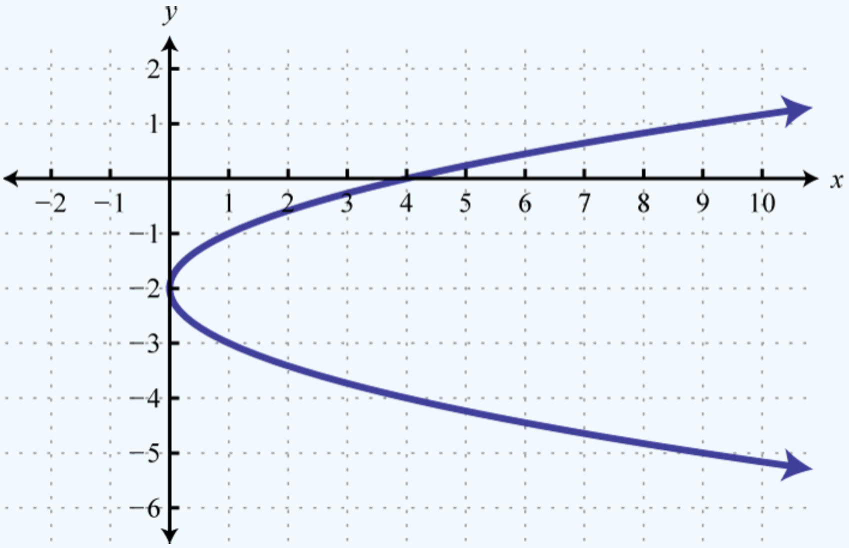

Figure 8.1.17

11.

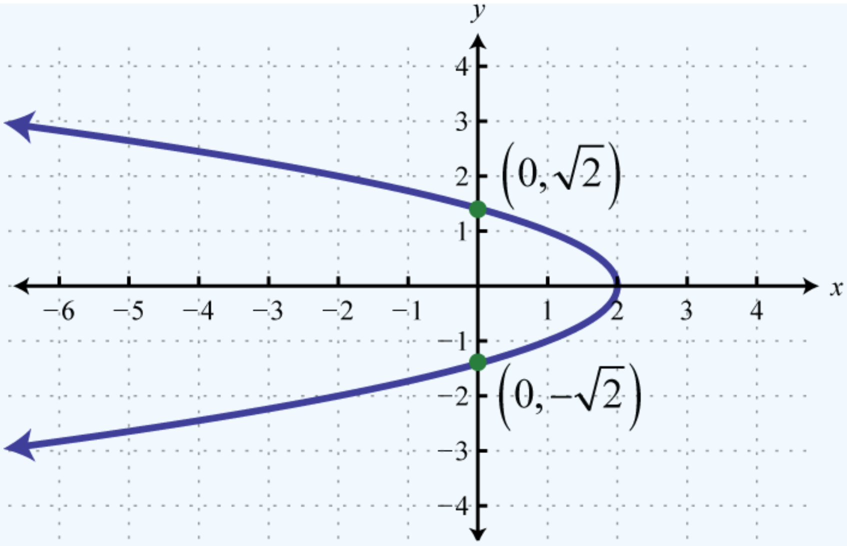

Figure 8.1.18

13.

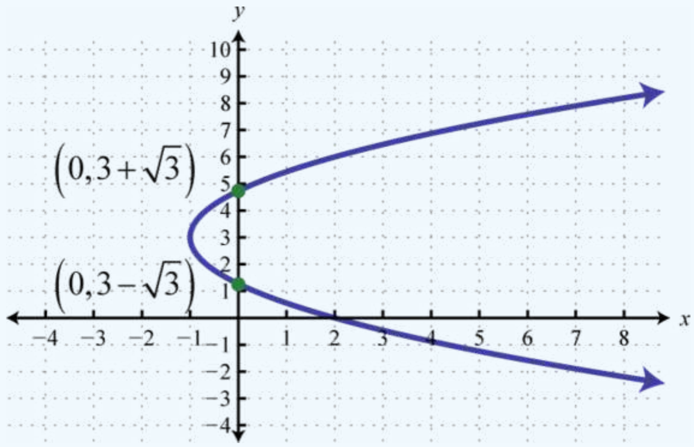

Figure 8.1.19

Exercise 8.1.8

Rewrite in standard form and give the vertex.

1. $y = x^2 - 6x + 18$
2. $y = x^2 + 8x + 36$
3. $x = y^2 + 20y + 87$
4. $x = y^2 - 10y + 21$
5. $y = x^2 - 14x + 49$
6. $x = y^2 + 16y + 64$
7. $x = 2y^2 - 4y + 5$
8. $y = 3x^2 - 30x + 67$
9. $y = 6x^2 + 36x + 54$
10. $x = 3y^2 + 6y - 1$
11. $y = 2x^2 - 2x - 1$
12. $x = 5y^2 + 15y + 9$
13. $x = -y^2 + 5y - 5$
14. $y = -x^2 + 9x - 20$

Answer

1. $y=(x-3)^2+9$; vertex: $(3,9)$

3. $x=(y+10)^2-13$; vertex: $(-13,-10)$

5. $y=(x-7)^2$; vertex: $(7,0)$

7. $x=2(y-1)^2+3$; vertex: $(3,1)$

9. $y=6(x+3)^2$; vertex: $(-3,0)$

11. $y=2\left(x-\frac{1}{2}\right)^2-\frac{3}{2}$; vertex: $\left(\frac{1}{2},-\frac{3}{2}\right)$

13. $x=-\left(y-\frac{5}{2}\right)^2+\frac{5}{4}$; vertex: $\left(\frac{5}{4},\frac{5}{2}\right)$

Exercise $8.1.9$

Rewrite in standard form and graph. Be sure to find the vertex and all intercepts.

1. $y=x^2-4x-5$
2. $y=x^2+6x-16$
3. $y=-x^2+12x-32$
4. $y=-x^2-10x$
5. $y=2x^2+4x+9$
6. $y=3x^2-6x+4$
7. $y=-5x^2+30x-45$
8. $y=-4x^2-16x-16$
9. $x=y^2-2y-8$
10. $x=y^2+4y+8$
11. $x=y^2-2y-3$
12. $x=y^2+6y-7$
13. $x=-y^2-10y-24$
14. $x=-y^2-12y-40$
15. $x=3y^2+12y+12$
16. $x=-2y^2+12y-18$
17. $x=y^2-4y-3$
18. $x=y^2+6y+1$
19. $x=-y^2+2y+5$
20. $y=2x^2-2x+1$
21. $y=-3x^2+2x+1$
22. $y=-x^2+3x+10$
23. $x=-4y^2-4y-5$
24. $x=y^2-y+2$
25. $y=x^2+5x-1$
26. $y=2x^2+6x+3$
27. $x=2y^2+10x+12$
28. $x=y^2+y-1$

Answer

1. $y=(x-2)^2-9$;

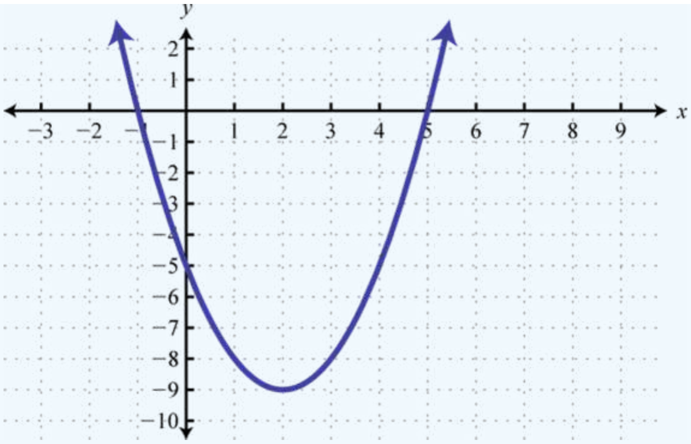

Figure 8.1.20

3. $y = -(x-6)^2 + 4$;

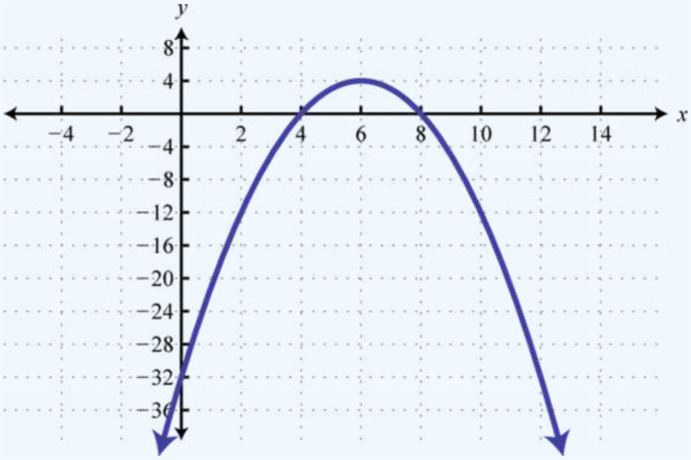

Figure 8.1.21

5. $y = 2(x+1)^2 + 7$;

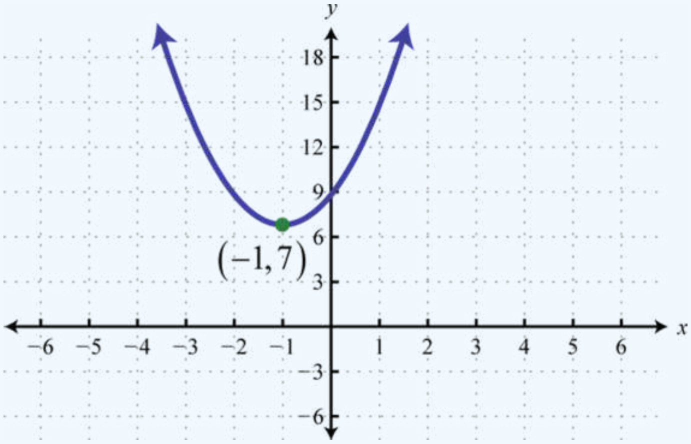

Figure 8.1.22

7. $y = -5(x-3)^2$;

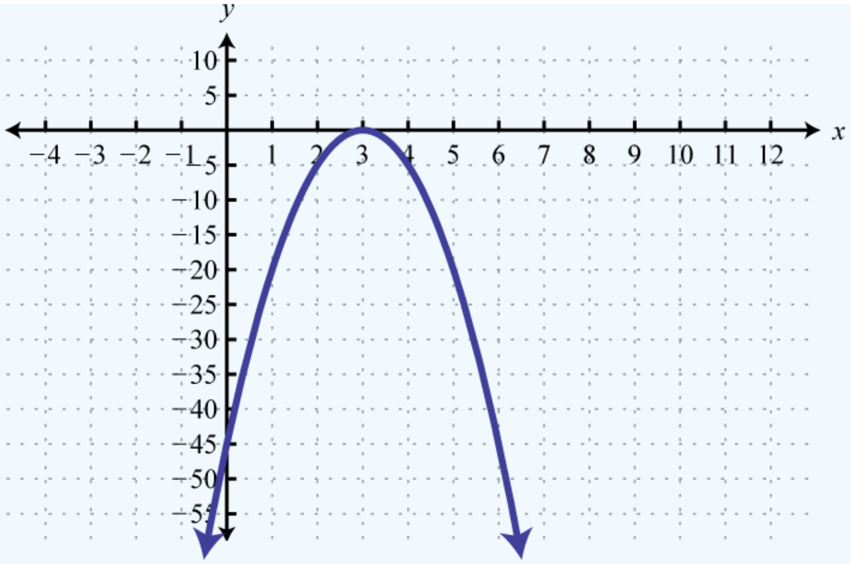

Figure 8.1.23

9. $x=(y-1)^2-9$;

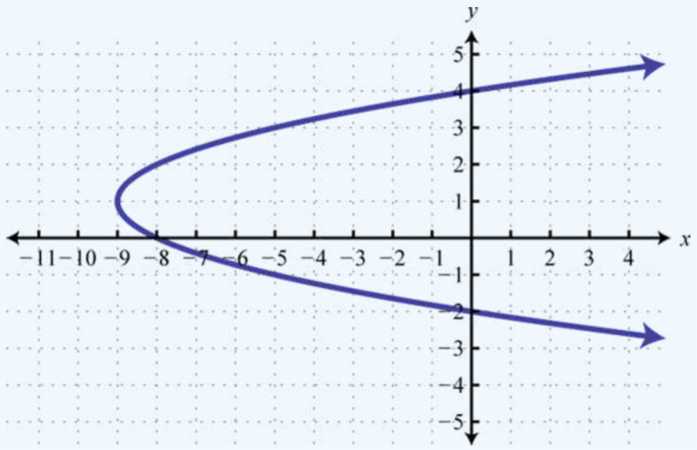

Figure 8.1.24

11. $x=(y-1)^2-4$;

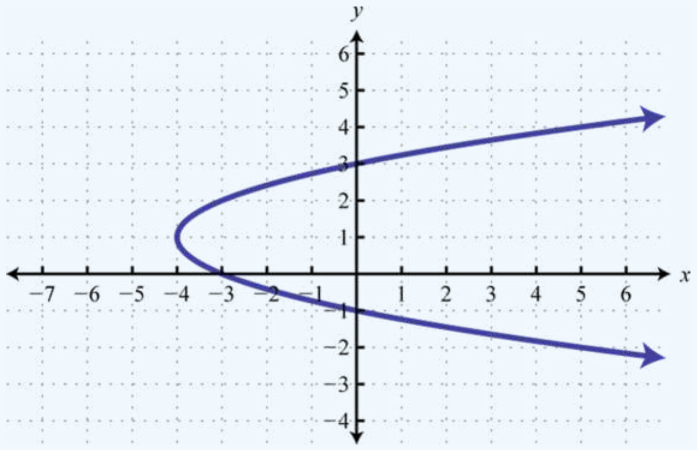

Figure 8.1.25

13. $x=-(y+5)^2+1$;

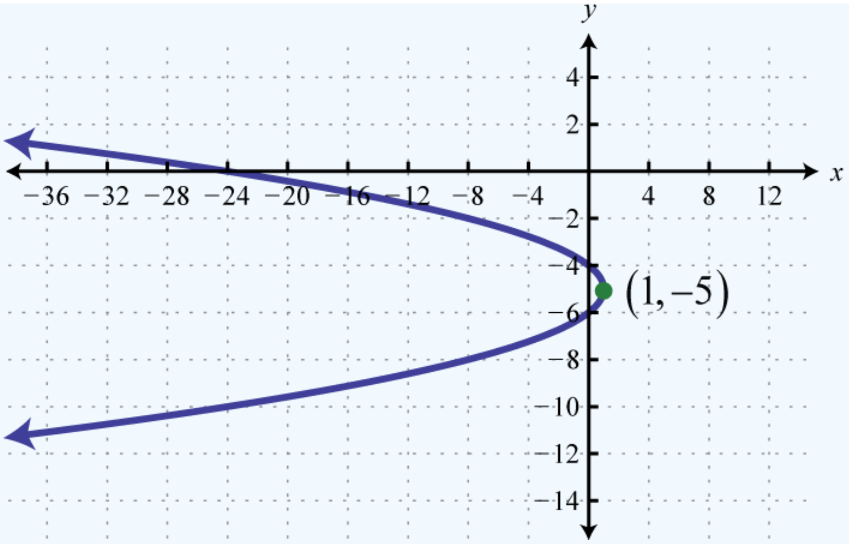

Figure 8.1.26

15. $x=3(y+2)^2$;

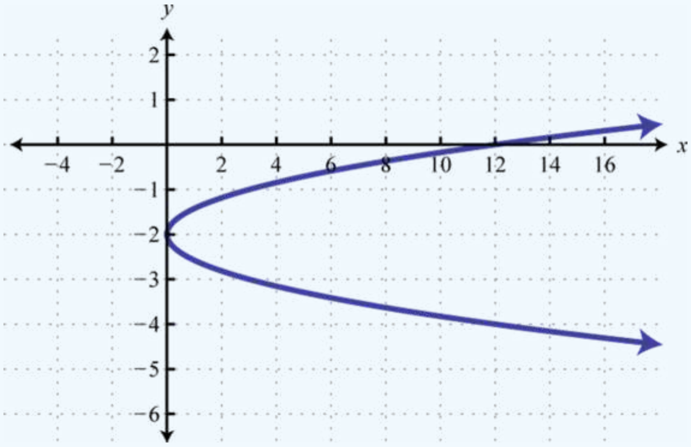

Figure 8.1.27

17. $x=(y-2)^2-7$;

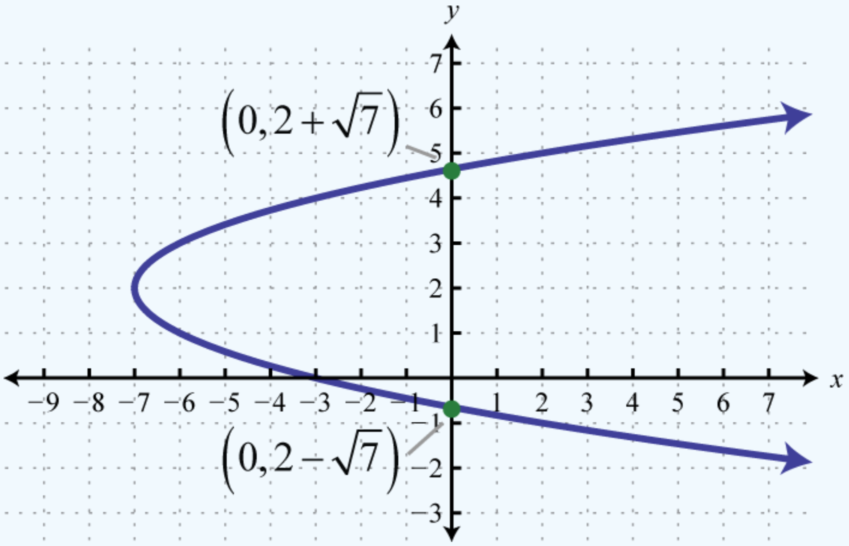

Figure 8.1.28

19. $x=-(y-1)^2+6$;

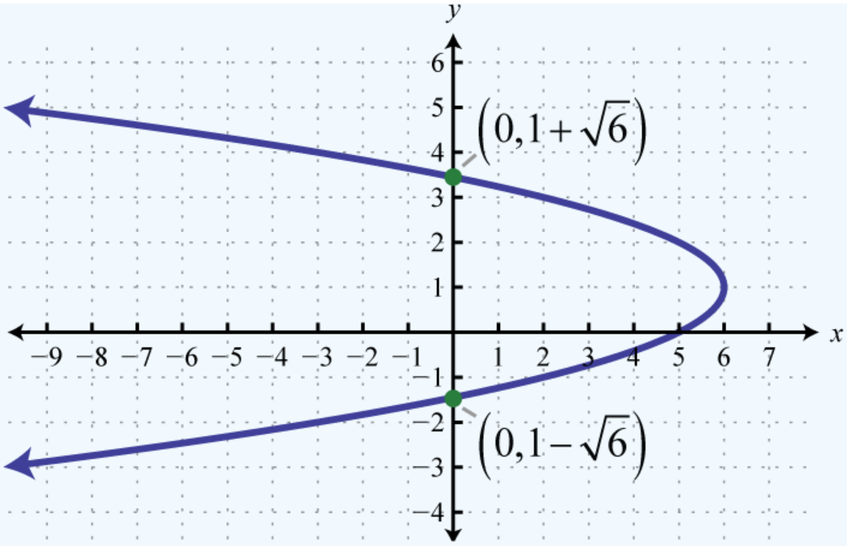

Figure 8.1.29

21. $y=-3\left(x-\frac{1}{3}\right)^{2}+\frac{4}{3}$;

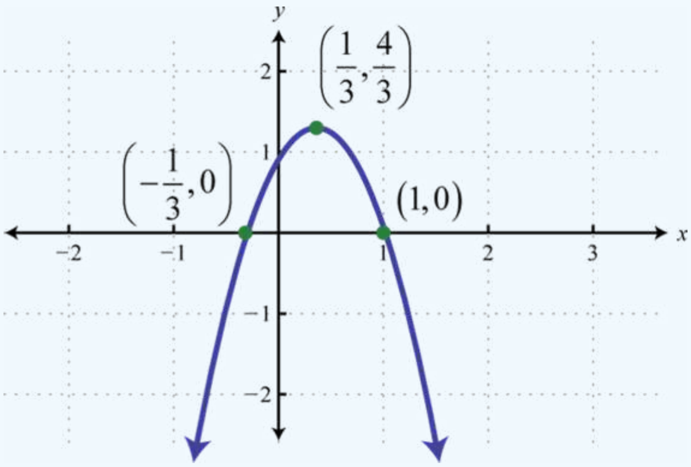

Figure 8.1.30

23. $x=-4\left(y+\frac{1}{2}\right)^{2}-4$;

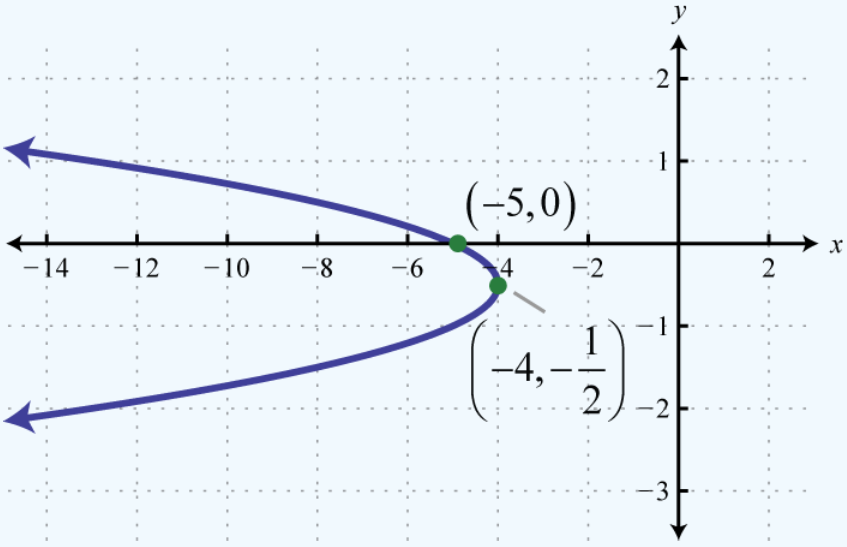

Figure 8.1.31

25. $y=\left(x+\frac{5}{2}\right)^{2}-\frac{29}{4}$;

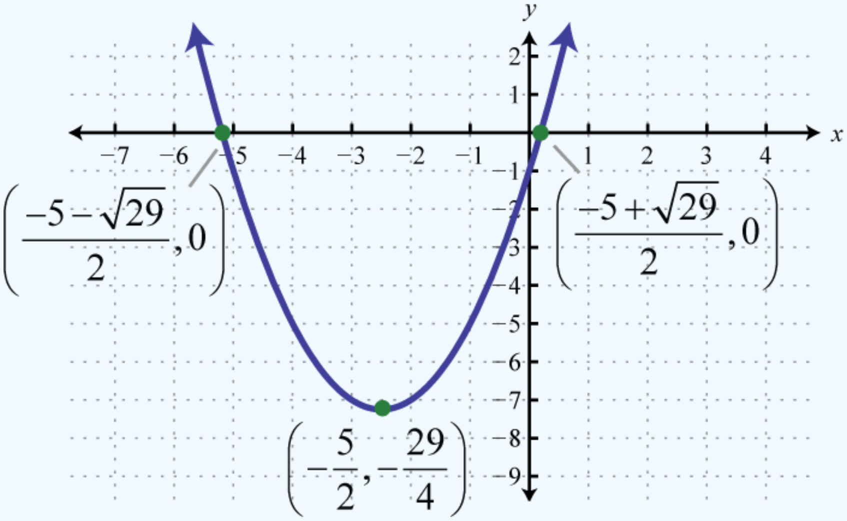

Figure 8.1.32

27. $x=2\left(y+\frac{5}{2}\right)^{2}-\frac{1}{2}$;

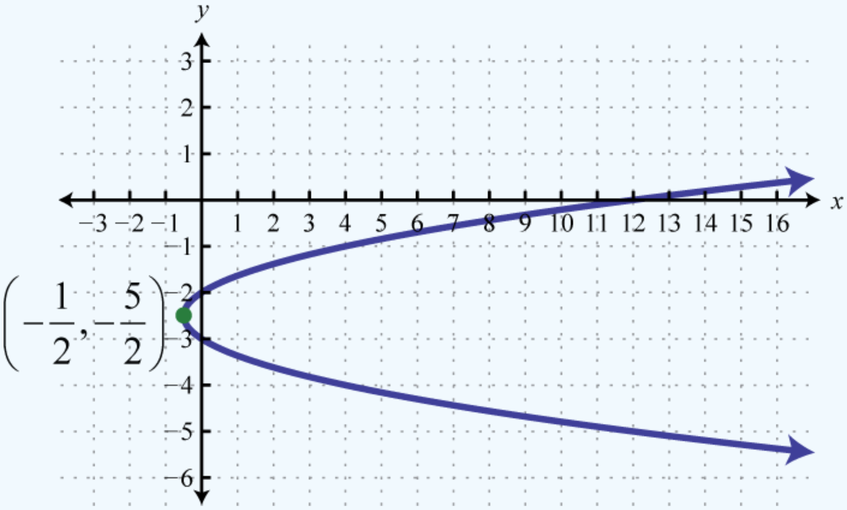

Figure 8.1.33

Exercise 8.1.10

1. Research and discuss real-world applications that involve a parabola.
2. Do all parabolas have x-intercepts? Explain.
3. Do all parabolas have y-intercepts? Explain.
4. Make up your own parabola that opens left or right, write it in general form, and graph it.

Answer

1. Answer may vary

3. Answer may vary

Footnotes

[1] A curve obtained from the intersection of a right circular cone and a plane.

[2] Given two points (x_1,y_1) and (x_2,y_2), the distance d between them is given by $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

[3] Given two points (x_1, y_1) and (x_2, y_2), the midpoint is an ordered pair given by $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

[4] The set of points in a plane equidistant from a given line, called the directrix, and a point not on the line, called the focus.

[5] The equation of a parabola written in the form $y = ax^2 + bx + c$ or $x = ay^2 + by + c$, where a, b, and c are real numbers and $a \neq 0$.

[6] The equation of a parabola written in the form $y = a(x-h)^2 + k$ or $x = a(y-k)^2 + h$.

[7] The equation of a parabola written in standard form is often called vertex form. In this form the vertex is apparent: (h, k).

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8.2: Circles

Learning Objectives

- Graph a circle in standard form.
- Determine the equation of a circle given its graph.
- Rewrite the equation of a circle in standard form.

The Circle in Standard Form

A **circle**[8] is the set of points in a plane that lie a fixed distance, called the **radius**[9], from any point, called the center. The **diameter**[10] is the length of a line segment passing through the center whose endpoints are on the circle. In addition, a circle can be formed by the intersection of a cone and a plane that is perpendicular to the axis of the cone:

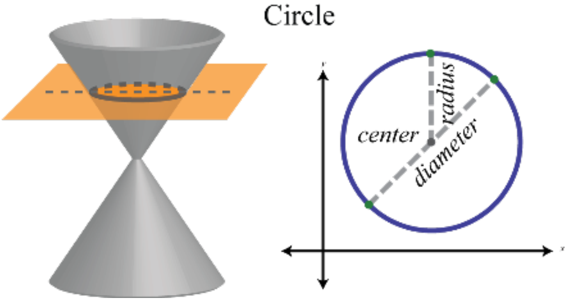


Figure 8.2.1

In a rectangular coordinate plane, where the center of a circle with radius r is (h, k), we have

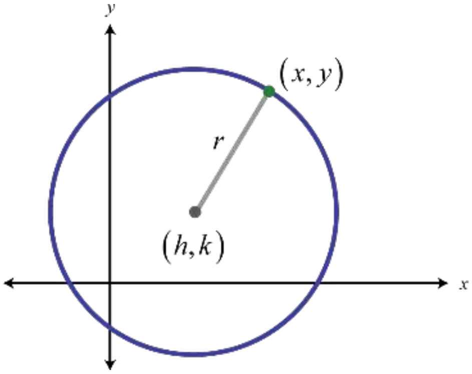


Figure 8.2.2

Calculate the distance between (h, k) and (x, y) using the distance formula,

$$\sqrt{(x-h)^2+(y-k)^2}=r$$

Squaring both sides leads us to the equation of a **circle in standard form**[11],

$$(x-h)^2+(y-k)^2=r^2$$

In this form, the center and radius are apparent. For example, given the equation $(x-2)^2+(y+5)^2=16$ we have,

$$\begin{array}{c} (x-h)^2 \;+\; (x\;-k)^2 = r^2 \\ \downarrow \qquad\qquad \downarrow \quad\;\; \downarrow \\ (x-2)^2+[y-(-5)]^2=4^2 \end{array}$$

In this case, the center is $(2, -5)$ and $r = 4$. More examples follow:

Table 8.2.1

Equation	Center	Radius
$(x-3)^2+(y-4)^2=25$	$(3, 4)$	$r=5$

Equation	Center	Radius
$(x-1)^2+(y+2)^2=7$	$(1,-2)$	$r=\sqrt{7}$
$(x+4)^2+(y-3)^2=1$	$(-4,3)$	$r=1$
$x^2+(y+6)^2=8$	$(0,-6)$	$r=2\sqrt{2}$

The graph of a circle is completely determined by its center and radius.

Example $8.2.1$:

Graph: $(x-2)^2+(y+5)^2=16$.

Solution

Written in this form we can see that the center is $(2,-5)$ and that the radius $r=4$ units. From the center mark points 4 units up and down as well as 4 units left and right.

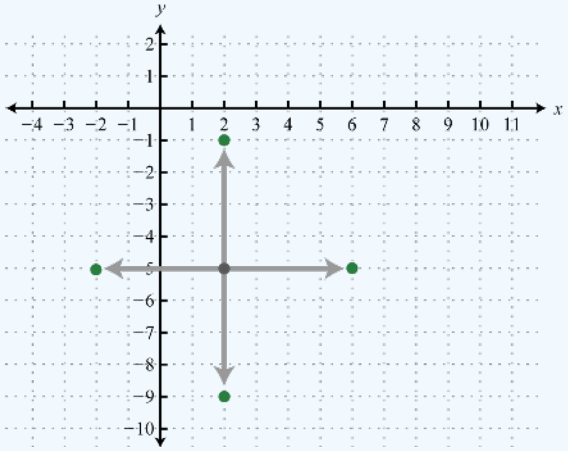


Figure $8.2.3$

Then draw in the circle through these four points.

Answer:

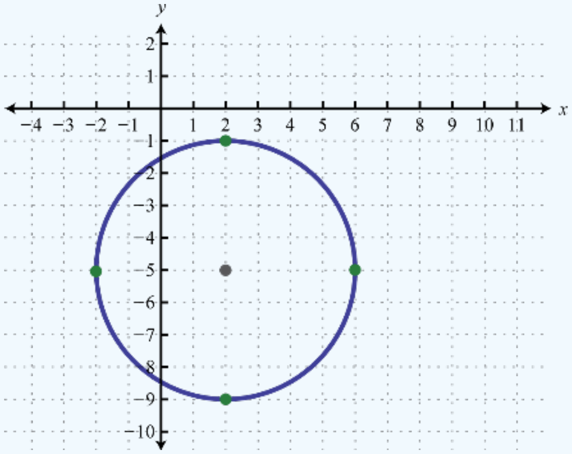


Figure $8.2.4$

As with any graph, we are interested in finding the x- and y-intercepts.

Example $8.2.2$:

Find the intercepts: $(x-2)^2+(y+5)^2=16$.

Solution

To find the y-intercepts set $x=0$:

$$\begin{aligned}(x-2)^2+(y+5)^2&=16\\(0-2)^2+(y+5)^2&=16\\4+(y+5)^2&=16\end{aligned}$$

For this equation, we can solve by extracting square roots.

$$\begin{aligned}(y+5)^2&=12\\y+5&=\pm\sqrt{12}\\y+5&=\pm 2\sqrt{3}\\y&=-5\pm 2\sqrt{3}\end{aligned}$$

Therefore, the y-intercepts are $(0,-5-2\sqrt{3})$ and $(0,-5+2\sqrt{3})$. To find the x-intercepts set $y=0$:

$$\begin{aligned}(x-2)^2+(y+5)^2&=16\\(x-2)^2+(0+5)^2&=16\\(x-2)^2+25&=16\\(x-2)^2&=-9\\x-2&=\pm\sqrt{-9}\\x&=2\pm 3i\end{aligned}$$

And because the solutions are complex we conclude that there are no real x-intercepts. Note that this does make sense given the graph.

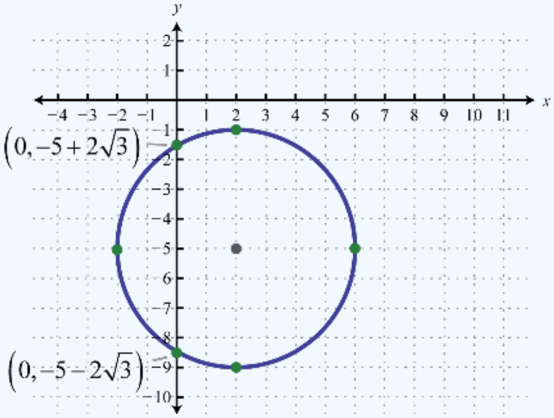


Figure $8.2.5$

Answer:

x-intercepts: none; y-intercepts: $(0,-5-2\sqrt{3})$ and $(0,-5+2\sqrt{3})$

Given the center and radius of a circle, we can find its equation.

Example $8.2.3$:

Graph the circle with radius $r=3$ units centered at $(-1,0)$. Give its equation in standard form and determine the intercepts.

Solution:

Given that the center is $(-1,0)$ and the radius is $r=3$ we sketch the graph as follows:

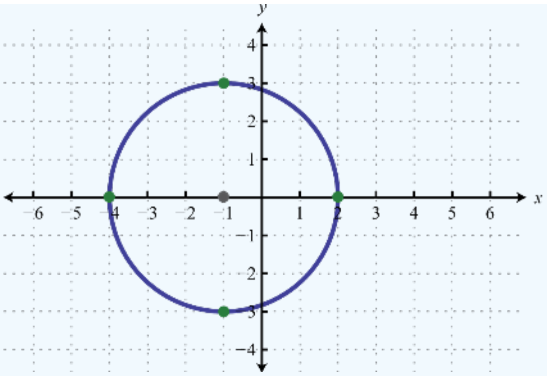


Figure 8.2.6

Substitute h, k, and r to find the equation in standard form. Since $(h, k) = (-1, 0)$ and $r = 3$ we have,

$$\begin{aligned}(x-h)^2+(y-k)^2&=r^2\\ [x-(-1)]^2+(y-0)^2&=3^2\\ (x+1)^2+y^2&=9\end{aligned}$$

The equation of the circle is $(x+1)^2+y^2=9$, use this to determine the y-intercepts.

$$\begin{aligned}(x+1)^2+y^2&=9 \quad \textit{Set } x=0 \textit{ to and solve for } y.\\ (0+1)^2+y^2&=9\\ 1+y^2&=9\\ y^2&=8\\ y&=\pm\sqrt{8}\\ y&=\pm 2\sqrt{2}\end{aligned}$$

Therefore, the y-intercepts are $(0, -2\sqrt{2})$ and $(0, 2\sqrt{2})$. To find the x-intercepts algebraically, set $y = 0$ and solve for x; this is left for the reader as an exercise.

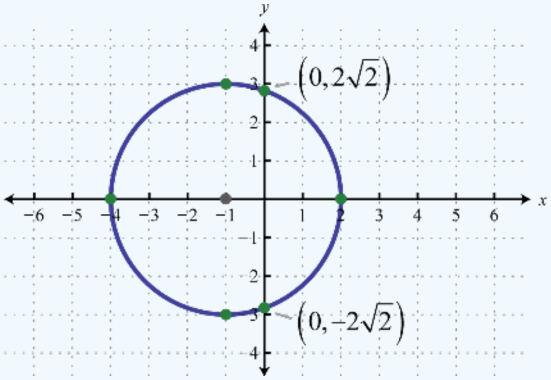


Figure 8.2.7

Answer:

Equation: $(x+1)^2+y^2=9$; y-intercepts: $(0, -2\sqrt{2})$ and $(0, 2\sqrt{2})$; x-intercepts: $(-4, 0)$ and $(2, 0)$

Of particular importance is the **unit circle**[12],

$$x^2+y^2=1$$

Or,

$$(x-0)^2+(y-0)^2=1^2$$

In this form, it should be clear that the center is $(0, 0)$ and that the radius is 1 unit. Furthermore, if we solve for y we obtain two functions:

$$\begin{aligned}x^2+y^2&=1\\ y^2&=1-x^2\\ y&=\pm\sqrt{1-x^2}\end{aligned}$$

The function defined by $y=\sqrt{1-x^2}$ is the top half of the circle and the function defined by $y=-\sqrt{1-x^2}$ is the bottom half of the unit circle:

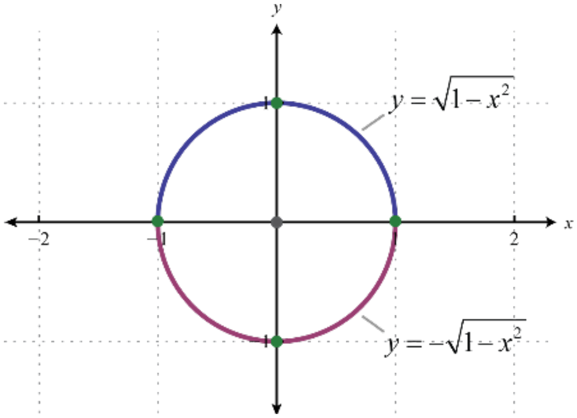


Figure 8.2.8

Exercise 8.2.1:

Graph and label the intercepts: $x^2+(y+2)^2=25$.

Answer:

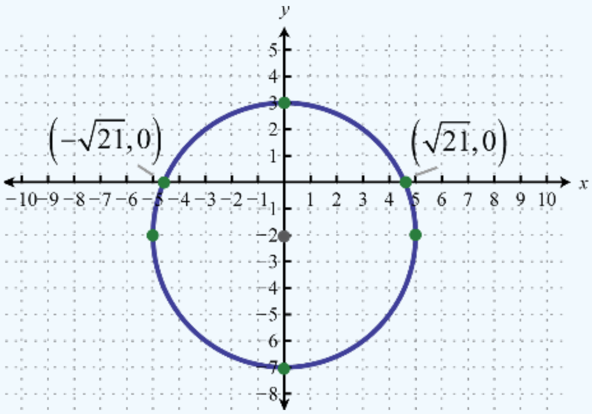


Figure 8.2.9

www.youtube.com/v/KeKt9k6IDCk

The Circle in General Form

We have seen that the graph of a circle is completely determined by the center and radius which can be read from its equation in standard form. However, the equation is not always given in standard form. The equation of a **circle in general form**[13] follows:

$$x^2+y^2+cx+dy+e=0$$

Here c, d, and e are real numbers. The steps for graphing a circle given its equation in general form follow.

Example 8.2.4:

Graph: $x^2+y^2+6x-8y+13=0$.

Solution

Begin by rewriting the equation in standard form.

Step 1: Group the terms with the same variables and move the constant to the right side. In this case, subtract 13 on both sides and group the terms involving x and the terms involving y as follows.

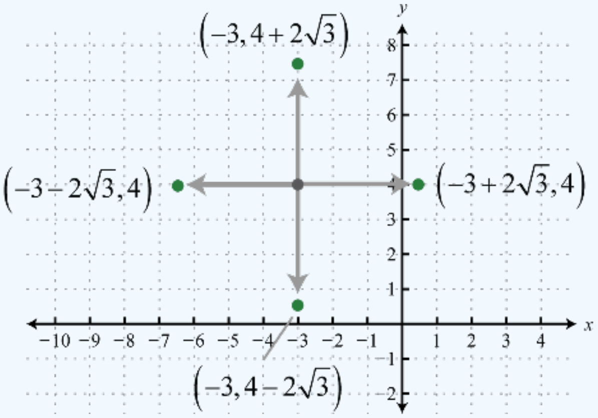
$$\begin{gathered}x^2+y^2+6x-8y+13=0\\ (x^2+6x+\_\_)+(y^2-8y+\_\_)=-13\end{gathered}$$

Step 2: Complete the square for each grouping. The idea is to add the value that completes the square, $\left(\frac{b}{2}\right)^2$, to both sides for both groupings, and then factor. For the terms involving x use $\left(\frac{6}{2}\right)^2 = 3^2 = 9$ and for the terms involving y use $\left(\frac{-8}{2}\right)^2 = (-4)^2 = 16$.

$$\begin{gathered}(x^2+6x+9)+(y^2-8y+16)=-13+9+16\\(x+3)^2+(y-4)^2=12\end{gathered}$$

Step 3: Determine the center and radius from the equation in standard form. In this case, the center is $(-3,4)$ and the radius $r=\sqrt{12}=2\sqrt{3}$.

Step 4: From the center, mark the radius vertically and horizontally and then sketch the circle through these points.



$\left(-3,4+2\sqrt{3}\right)$
$\left(-3-2\sqrt{3},4\right)$
$\left(-3+2\sqrt{3},4\right)$
$\left(-3,4-2\sqrt{3}\right)$
y
x
8
7
6
5
4
3
2
1
−1
−2
−10
−9
−8
−7
−6
−5
−4
−3
−2
−1
1
2
3
4

Figure $8.2.10$

Answer:

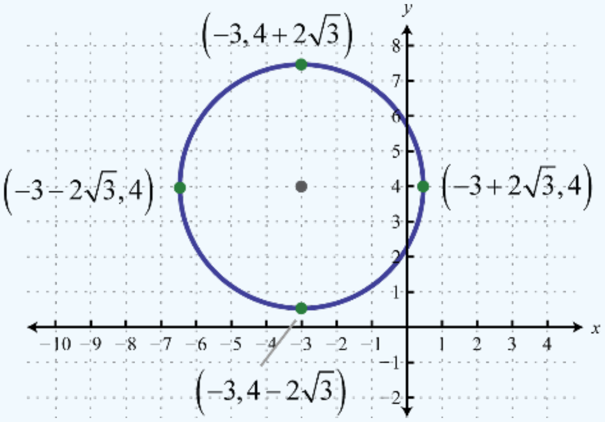


Figure $8.2.11$

Example $8.2.5$:

Determine the center and radius: $4x^2+4y^2-8x+12y-3=0$.

Solution

We can obtain the general form by first dividing both sides by 4.

$$\begin{gathered}\frac{4x^2+4y^2-8x+12y-3}{4}=\frac{0}{4}\\x^2+y^2-2x+3y-\frac{3}{4}=0\end{gathered}$$

Now that we have the general form for a circle, where both terms of degree two have a leading coefficient of 1, we can use the steps for rewriting it in standard form. Begin by adding $\frac{3}{4}$ to both sides and group variables that are the same.

$$(x^2-2x+\_\_)+(y^2+3y+\_\_)=\frac{3}{4}$$

Next complete the square for both groupings. Use $\left(\frac{-2}{2}\right)^2=(-1)^2=1$ for the first grouping and $\left(\frac{3}{2}\right)^2=\frac{9}{4}$ for the second grouping.

$$\begin{aligned}(x^2-2x+1)+\left(y^2+3y+\frac{9}{4}\right)&=\frac{3}{4}+1+\frac{9}{4}\\(x-1)^2+\left(y+\frac{3}{2}\right)^2&=\frac{16}{4}\\(x-1)^2+\left(y+\frac{3}{2}\right)^2&=4\end{aligned}$$

Answer:

Center: $(1,-\frac{3}{2})$; radius: $r=2$

In summary, to convert from standard form to general form we multiply, and to convert from general form to standard form we complete the square.

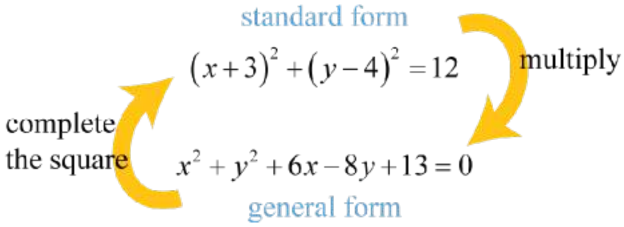


Figure 8.2.12

Exercise 8.2.2:

Graph: $x^2+y^2-10x+2y+21=0$.

Answer:

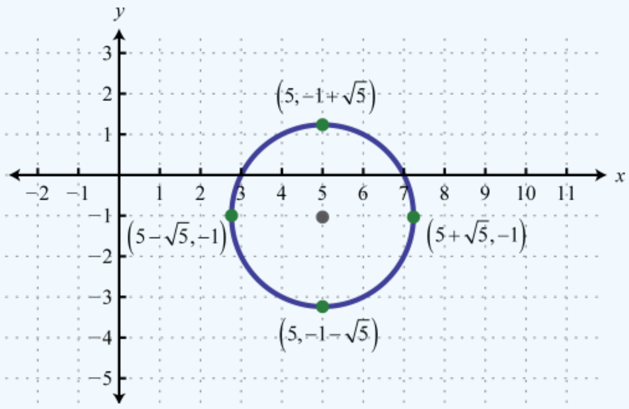


Figure 8.2.13

www.youtube.com/v/Ms8NESnqs6s

Key Takeaways

- The graph of a circle is completely determined by its center and radius.
- Standard form for the equation of a circle is $(x-h)^2+(y-k)^2=r^2$. The center is (h,k) and the radius measures r units.
- To graph a circle mark points r units up, down, left, and right from the center. Draw a circle through these four points.
- If the equation of a circle is given in general form $x^2+y^2+cx+dy+e=0$, group the terms with the same variables, and complete the square for both groupings. This will result in standard form, from which we can read the circle's center and radius.
- We recognize the equation of a circle if it is quadratic in both x and y where the coefficient of the squared terms are the same.

Exercise 8.2.3

Determine the center and radius given the equation of a circle in standard form.

1. $(x-5)^2+(y+4)^2=64$
2. $(x+9)^2+(y-7)^2=121$
3. $x^2+(y+6)^2=4$

4. $(x-1)^2+y^2=1$
5. $(x+1)^2+(y+1)^2=7$
6. $(x+2)^2+(y-7)^2=8$

Answer

1. Center: $(5,-4)$; radius: $r=8$
3. Center: $(0,-6)$; radius: $r=2$
5. Center: $(-1,-1)$; radius: $r=\sqrt{7}$

Exercise 8.2.4

Determine the standard form for the equation of the circle given its center and radius.

1. Center $(5,7)$ with radius $r=7$.
2. Center $(-2,8)$ with radius $r=5$.
3. Center $(6,-11)$ with radius $r=\sqrt{2}$.
4. Center $(-4,-5)$ with radius $r=\sqrt{6}$.
5. Center $(0,-1)$ with radius $r=2\sqrt{5}$.
6. Center $(0,0)$ with radius $r=3\sqrt{10}$.

Answer

1. $(x-5)^2+(y-7)^2=49$
3. $(x-6)^2+(y+11)^2=2$
5. $x^2+(y+1)^2=20$

Exercise 8.2.5

Graph.

1. $(x-1)^2+(y-2)^2=9$
2. $(x+3)^2+(y-3)^2=25$
3. $(x-2)^2+(y+6)^2=4$
4. $(x+6)^2+(y+4)^2=36$
5. $x^2+(y-4)^2=1$
6. $(x-3)^2+y^2=4$
7. $x^2+y^2=12$
8. $x^2+y^2=8$
9. $(x-7)^2+(y-6)^2=2$
10. $(x+2)^2+(y-5)^2=5$
11. $(x+3)^2+(y-1)^2=18$
12. $(x-3)^2+(y-2)^2=15$

Answer

1.

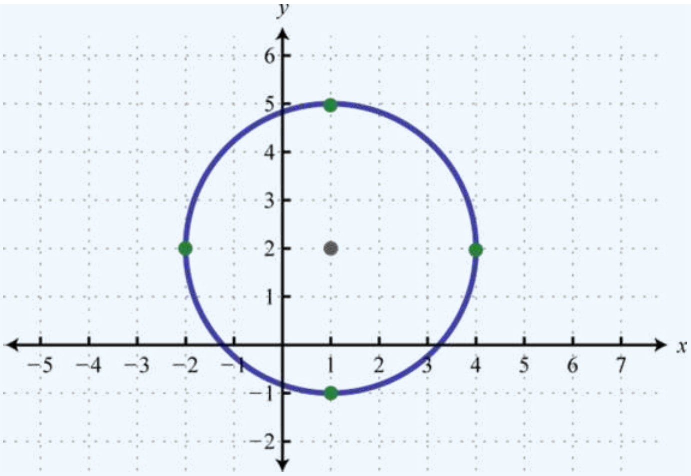


Figure 8.2.14

3\.

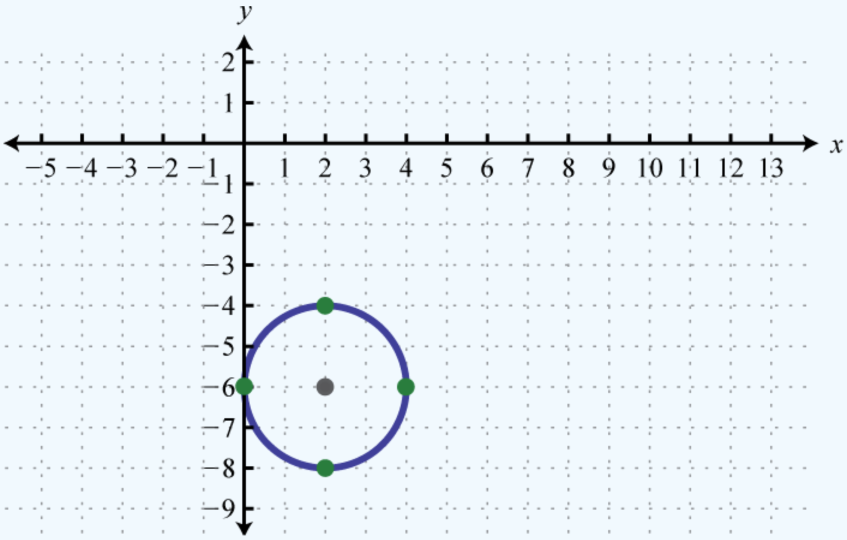


Figure 8.2.15

5\.

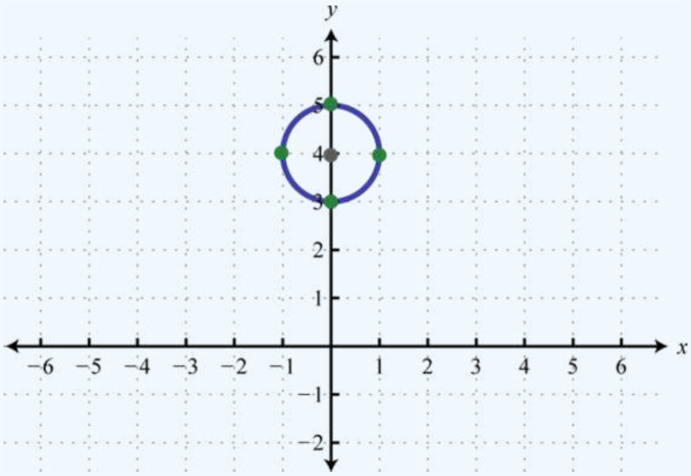
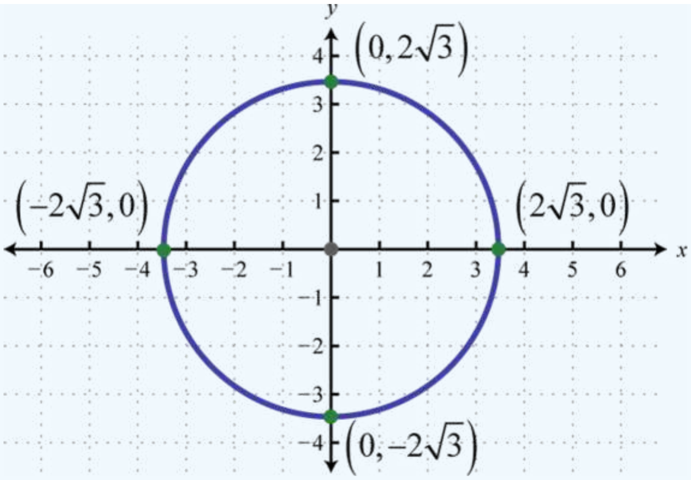


Figure 8.2.16

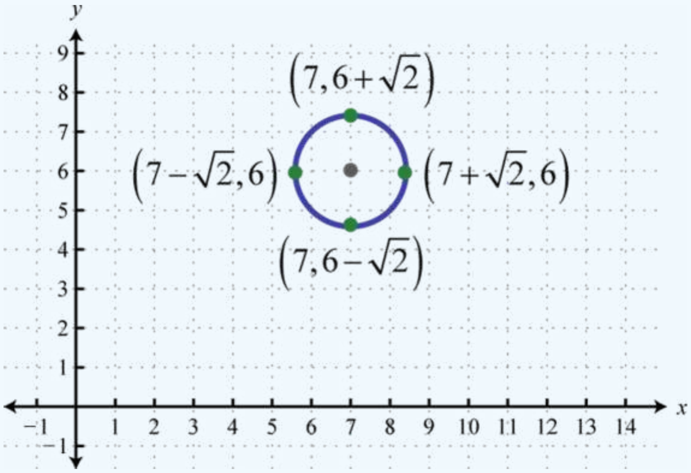
7\.



y
x
$(0,2\sqrt{3})$
$(-2\sqrt{3},0)$
$(2\sqrt{3},0)$
$(0,-2\sqrt{3})$

Figure 8.2.17

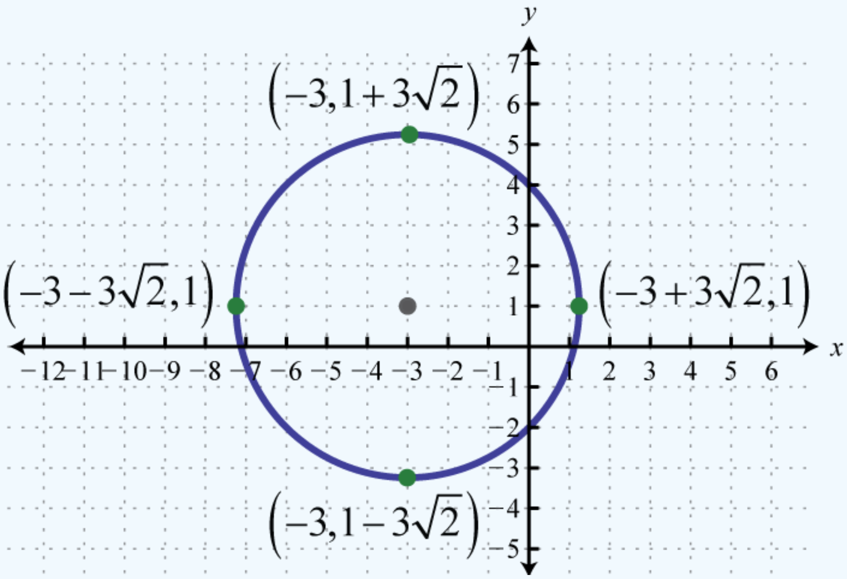
9.



y
x
$(7,6+\sqrt{2})$
$(7-\sqrt{2},6)$
$(7+\sqrt{2},6)$
$(7,6-\sqrt{2})$

Figure 8.2.18

11.



y
x
$(-3,1+3\sqrt{2})$
$(-3-3\sqrt{2},1)$
$(-3+3\sqrt{2},1)$
$(-3,1-3\sqrt{2})$

Figure 8.2.19

Exercise 8.2.6

Find the x- and y-intercepts.

1. $(x-1)^2+(y-2)^2=9$
2. $(x+5)^2+(y-3)^2=25$
3. $x^2+(y-4)^2=1$
4. $(x-3)^2+y^2=18$
5. $x^2+y^2=50$
6. $x^2+(y+9)^2=20$
7. $(x-4)^2+(y+5)^2=10$
8. $(x+10)^2+(y-20)^2=400$

Answer

1. x-intercepts: $(1\pm\sqrt{5},0)$; y-intercepts: $(0,2\pm2\sqrt{2})$
3. x-intercepts: none; y-intercepts: $(0,3),(0,5)$
5. x-intercepts: $(\pm5\sqrt{2},0)$; y-intercepts: $(0,\pm5\sqrt{2})$
7. x-intercepts: none; y-intercepts: none

Exercise 8.2.7

Find the equation of the circle.

1. Circle with center $(1,-2)$ passing through $(3,-4)$.
2. Circle with center $(-4,-1)$ passing through $(0,-3)$.
3. Circle whose diameter is defined by $(5,1)$ and $(-1,7)$.
4. Circle whose diameter is defined by $(-5,7)$ and $(-1,-5)$.
5. Circle with center $(5,-2)$ and area 9π square units.
6. Circle with center $(-8,-3)$ and circumference 12π square units.
7. Find the area of the circle with equation $(x+12)^2\pm(x-5)^2=7$.
8. Find the circumference of the circle with equation $(x+1)^2+(y+5)^2=8$.

Answer

1. $(x-1)^2+(y+2)^2=8$
3. $(x-2)^2+(y-4)^2=18$
5. $(x-5)^2+(y+2)^2=9$
7. 7π square units

Exercise 8.2.8

Rewrite in standard form and graph.

1. $x^2+y^2+4x-2y-4=0$
2. $x^2+y^2-10x+2y+10=0$
3. $x^2+y^2+2x+12y+36=0$
4. $x^2+y^2-14x-8y+40=0$
5. $x^2+y^2+6y+5=0$
6. $x^2+y^2-12x+20=0$
7. $x^2+y^2+8x+12y+16=0$
8. $x^2+y^2-20x-18y+172=0$
9. $4x^2+4y^2-4x+8y+1=0$
10. $9x^2+9y^2+18x+6y+1=0$
11. $x^2+y^2+4x+8y+14=0$

12. $x^2+y^2-2x-4y-15=0$
13. $x^2+y^2-x-2y+1=0$
14. $x^2+y^2-x+y-\frac{1}{2}=0$
15. $4x^2+4y^2+8x-12y+5=0$
16. $9x^2+9y^2+12x-36y+4=0$
17. $2x^2+2y^2+6x+10y+9=0$
18. $9x^2+9y^2-6x+12y+4=0$

Answer

1. $(x+2)^2+(y-1)^2=9$;

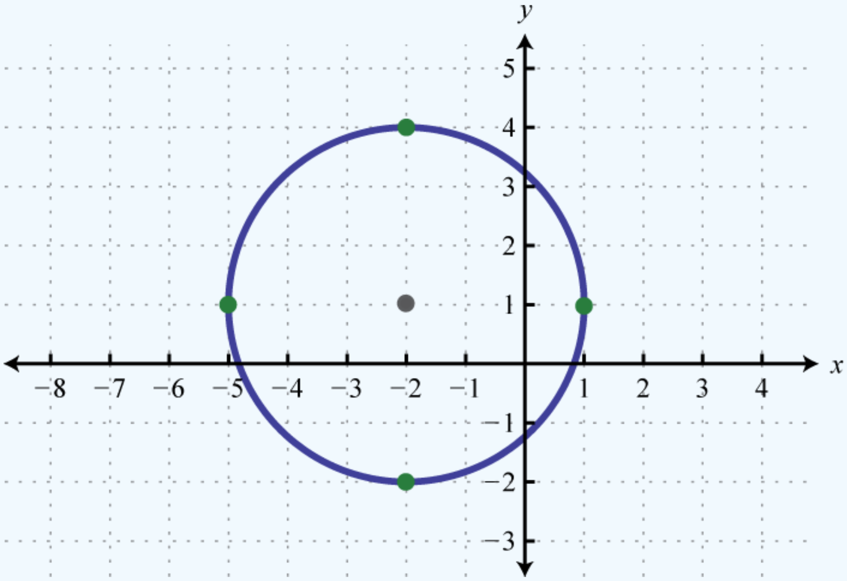


Figure 8.2.20

3. $(x+1)^2+(y+6)^2=1$;

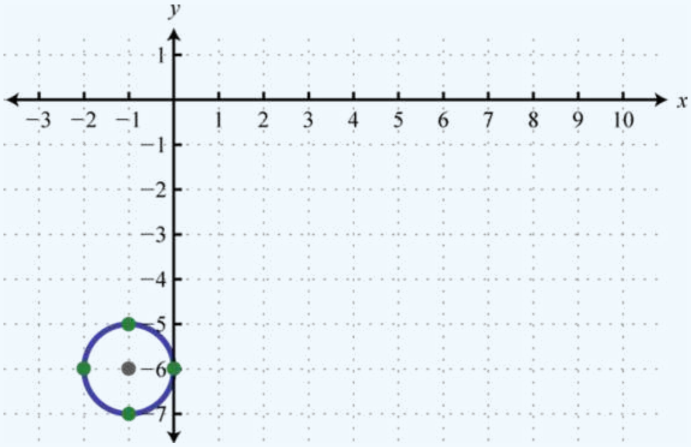


Figure 8.2.21

5. $x^2+(y+3)^2=4$;

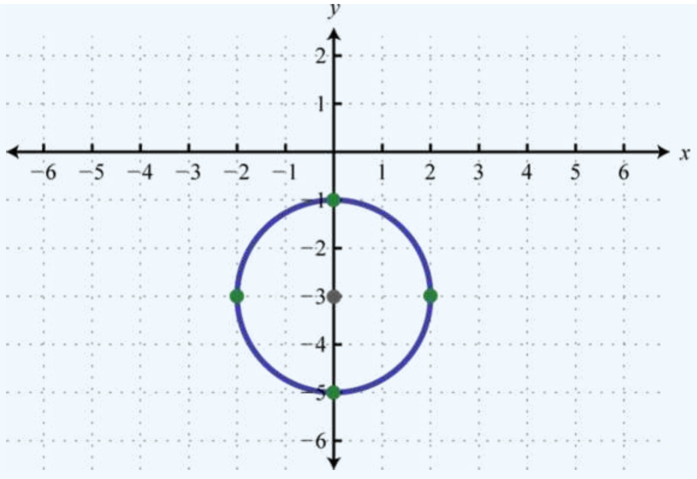


Figure 8.2.22

7. $(x+4)^2+(y+6)^2=36$;

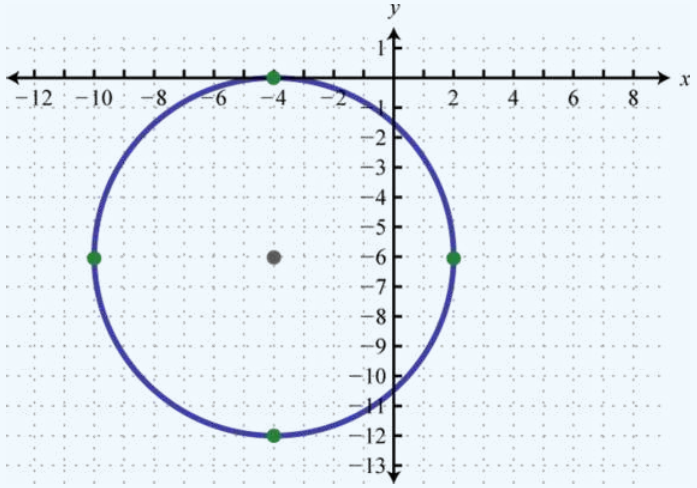


Figure 8.2.23

9. $\left(x-\frac{1}{2}\right)^2+(y+1)^2=1$;

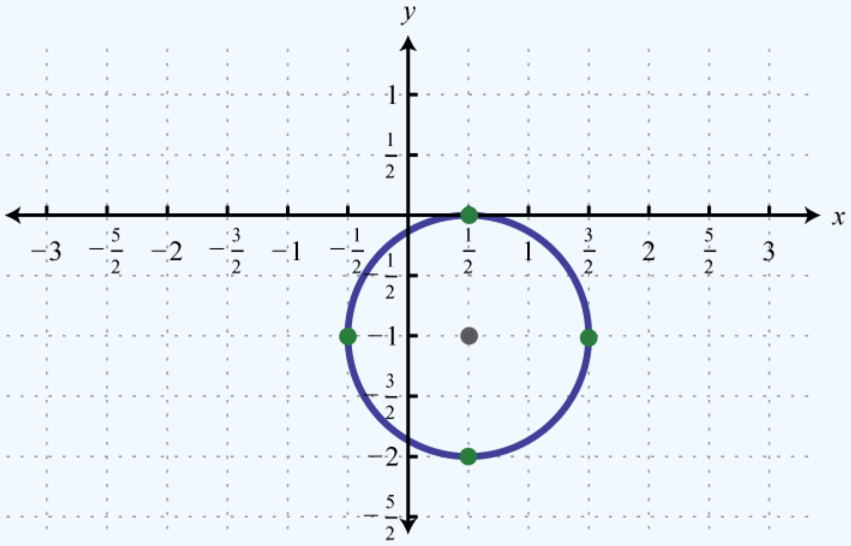


Figure 8.2.24

11. $(x+2)^2+(y-4)^2=6$;

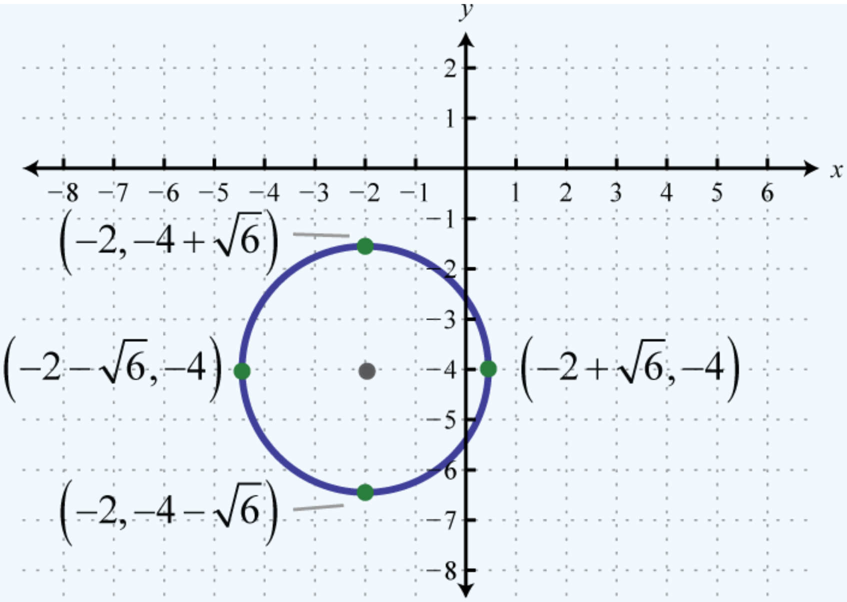


Figure 8.2.25

13. $\left(x-\frac{1}{2}\right)^2+(y-1)^2=\frac{1}{4}$;

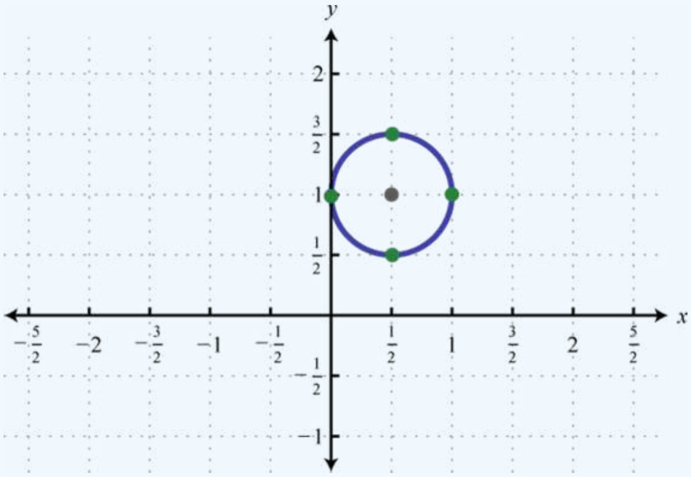


Figure 8.2.26

15. $(x+1)^2+\left(y-\frac{3}{2}\right)^2=2$;

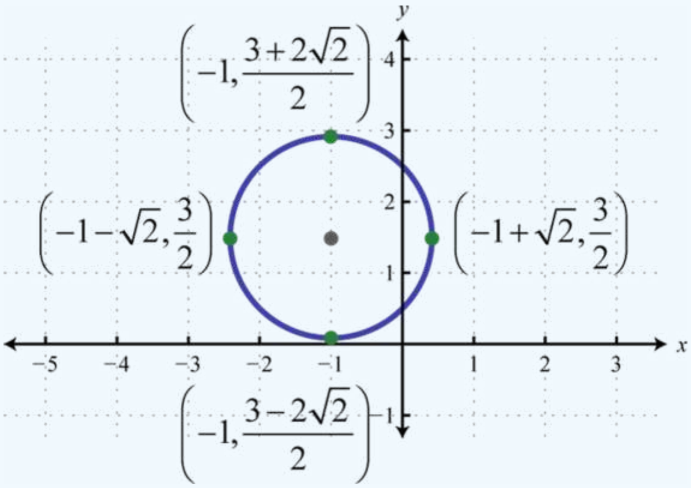


Figure 8.2.27

17. $\left(x+\frac{3}{2}\right)^2+\left(y+\frac{5}{2}\right)^2=4$;

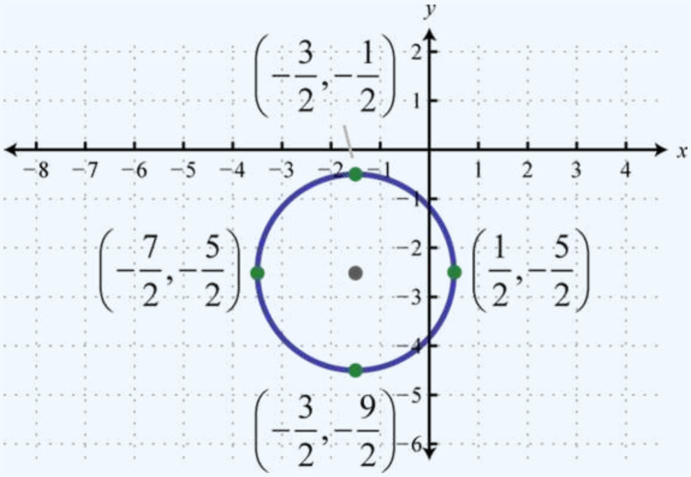


Figure 8.2.28

Exercise 8.2.9

Given a circle in general form, determine the intercepts.

1. $x^2+y^2-5x+3y+6=0$
2. $x^2+y^2+x-2y-7=0$
3. $x^2+y^2-6y+2=2$
4. $x^2+y^2-6x-8y+5=0$
5. $2x^2+2y^2-3x-9=0$
6. $3x^2+3y^2+8y-16=0$
7. Determine the area of the circle whose equation is $x^2+y^2-2x-6y-35=0$.
8. Determine the area of the circle whose equation is $4x^2+4y^2-12x-8y-59=0$.
9. Determine the circumference of a circle whose equation is $x^2+y^2-5x+1=0$.
10. Determine the circumference of a circle whose equation is $x^2+y^2+5x-2y+3=0$.
11. Find general form of the equation of a circle centered at $(-3,5)$ passing through $(1,-2)$.
12. Find general form of the equation of a circle centered at $(-2,-3)$ passing through $(-1,3)$.

Answer

1. x-intercepts: $(2,0),(3,0)$; y-intercepts: none

3. x-intercepts: $(0,0)$; y-intercepts: $(0,0),(0,6)$

5. x-intercepts: $(-\frac{3}{2},0),(3,0)$; y-intercepts: $\left(0,\pm\frac{3\sqrt{2}}{2}\right)$

7. 45π square units

9. $\pi\sqrt{21}$ units

11. $x^2+y^2+6x-10y-31=0$

Exercise 8.2.10

Given the graph of a circle, determine its equation in general form.

1.

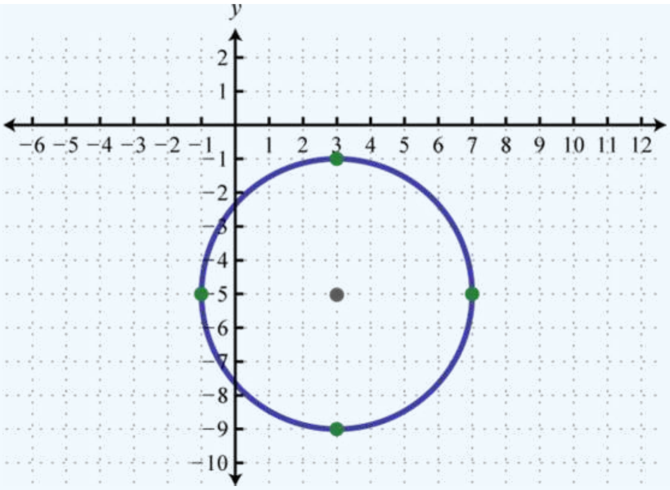


Figure 8.2.29

2.

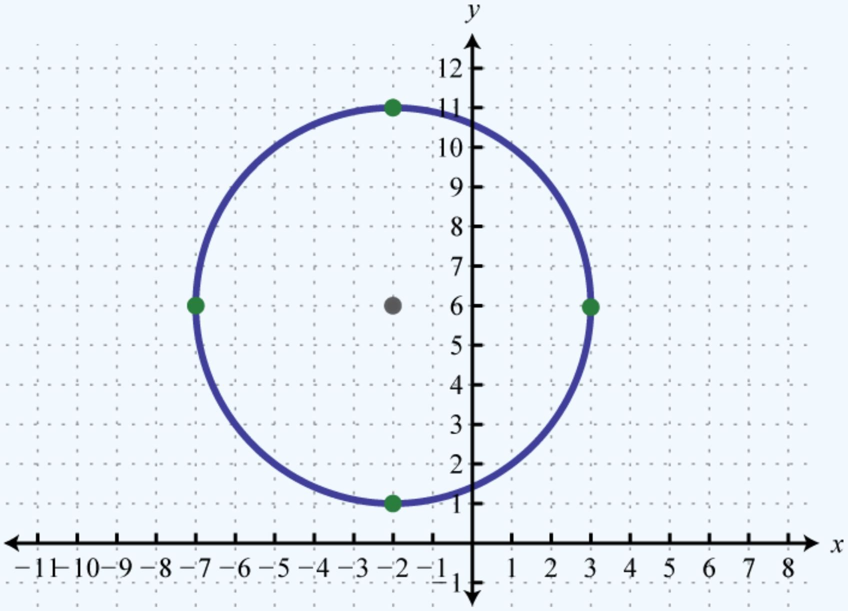


Figure 8.2.30

3.

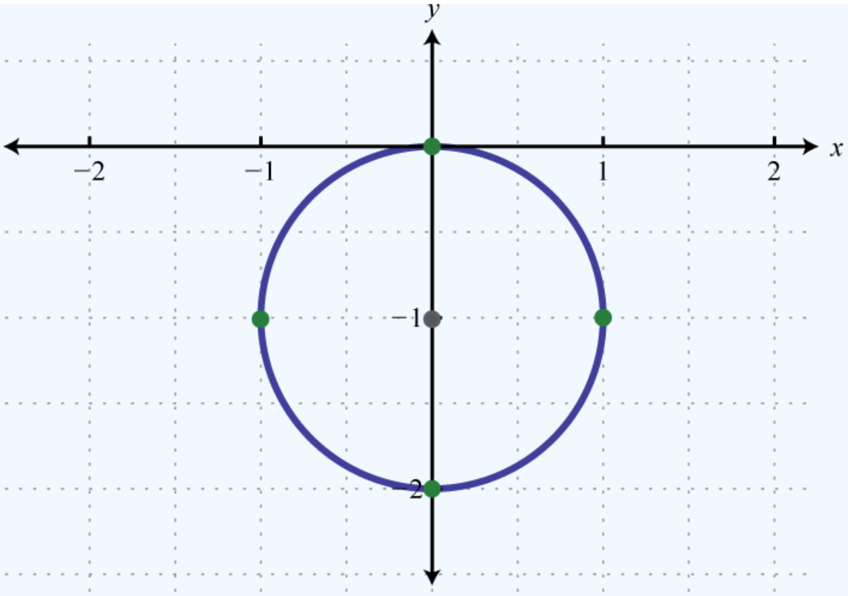


Figure 8.2.31

4.

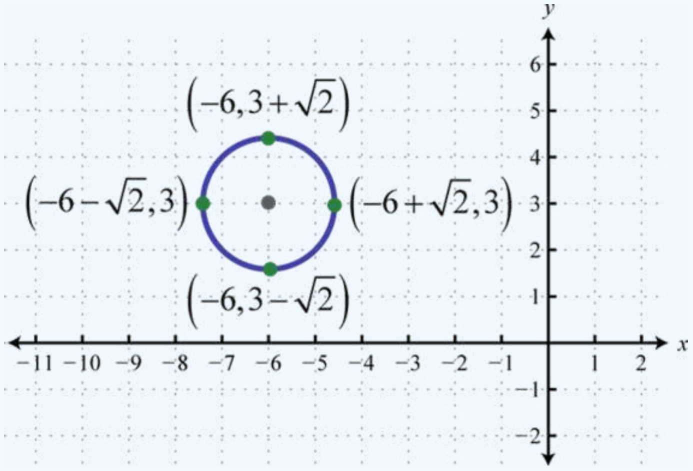


Figure 8.2.32

Answer

1. $x^2+y^2-6x+10y+18=0$
3. $x^2+y^2+2y=0$

Exercise 8.2.11

1. Is the center of a circle part of the graph? Explain.
2. Make up your own circle, write it in general form, and graph it.
3. Explain how we can tell the difference between the equation of a parabola in general form and the equation of a circle in general form. Give an example.
4. Do all circles have intercepts? What are the possible numbers of intercepts? Illustrate your explanation with graphs.

Answer

1. Answer may vary
3. Answer may vary

Footnotes

[8] A circle is the set of points in a plane that lie a fixed distance from a given point, called the center.

[9] The fixed distance from the center of a circle to any point on the circle.

[10] The length of a line segment passing through the center of a circle whose endpoints are on the circle.

[11] The equation of a circle written in the form $(x-h)^2+(y-k)^2=r^2$ where (h,k) is the center and r is the radius.

[12] The circle centered at the origin with radius 1; its equation is $x^2+y^2=1$.

[13] The equation of a circle written in the form $x^2+y^2+cx+dy+e=0$.

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8.3: Ellipses

Learning Objectives

- Graph an ellipse in standard form.
- Determine the equation of an ellipse given its graph.
- Rewrite the equation of an ellipse in standard form.

The Ellipse in Standard Form

An *ellipse*[14] is the set of points in a plane whose distances from two fixed points, called foci, have a sum that is equal to a positive constant. In other words, if points $F1$ and $F2$ are the foci (plural of focus) and d is some given positive constant then (x, y) is a point on the ellipse if $d = d_1 + d_2$ as pictured below:

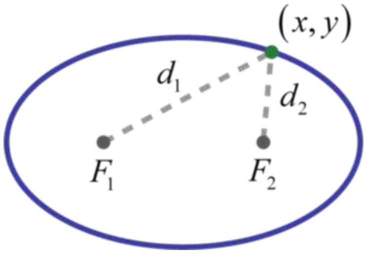

Figure $8.3.1$

In addition, an ellipse can be formed by the intersection of a cone with an oblique plane that is not parallel to the side of the cone and does not intersect the base of the cone. Points on this oval shape where the distance between them is at a maximum are called *vertices*[15] and define the *major axis*[16]. The center of an ellipse is the midpoint between the vertices. The *minor axis*[17] is the line segment through the center of an ellipse defined by two points on the ellipse where the distance between them is at a minimum. The endpoints of the minor axis are called *co-vertices*[18].

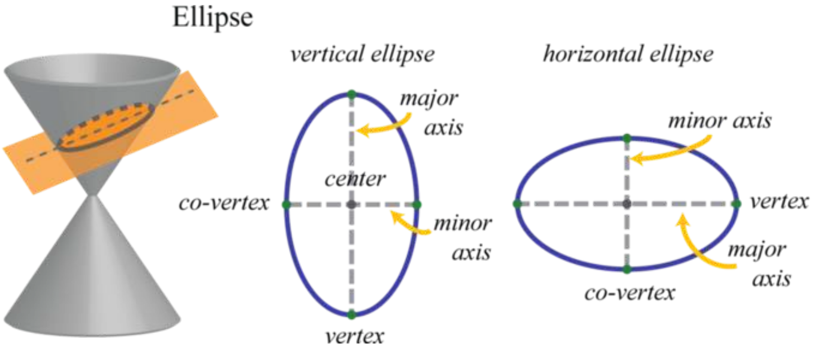

Figure $8.3.2$

If the major axis of an ellipse is parallel to the x-axis in a rectangular coordinate plane, we say that the ellipse is horizontal. If the major axis is parallel to the y-axis, we say that the ellipse is vertical. In this section, we are only concerned with sketching these two types of ellipses. However, the ellipse has many real-world applications and further research on this rich subject is encouraged. In a rectangular coordinate plane, where the center of a horizontal ellipse is (h, k), we have

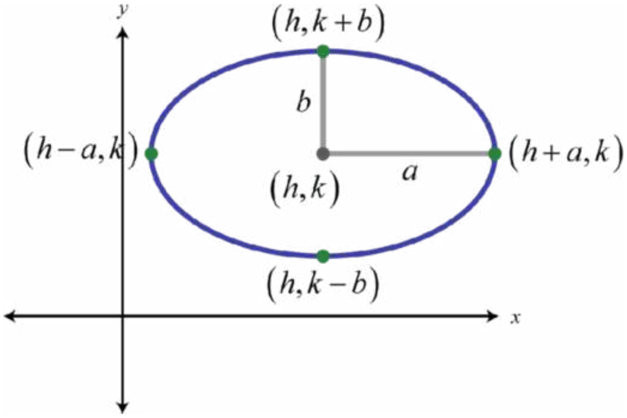

Figure 8.3.3

As pictured $a > b$ where a, one-half of the length of the major axis, is called the *major radius*[19]. And b, one-half of the length of the minor axis, is called the *minor radius*[20]. The equation of an *ellipse in standard form*[21] follows:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

The vertices are $(h \pm a, k)$ and $(h, k \pm b)$ and the orientation depends on a and b. If $a > b$, then the ellipse is horizontal as shown above and if $a < b$, then the ellipse is vertical and b becomes the major radius. What do you think happens when $a = b$?

Table 8.3.1

Equation	Center	a	b	Orientation
$\frac{(x-1)^2}{4}+\frac{(y-8)^2}{9}=1$	$(1,8)$	$a=2$	$b=3$	Vertical
$\frac{(x-3)^2}{2}+\frac{(y+5)^2}{16}=1$	$(3,-5)$	$a=\sqrt{2}$	$b=4$	Vertical
$\frac{(x+1)^2}{1}+\frac{(y-7)^2}{8}=1$	$(-1,7)$	$a=1$	$b=2\sqrt{2}$	Vertical
$\frac{x^2}{25}+\frac{(y+6)^2}{10}=1$	$(0,-6)$	$a=5$	$b=\sqrt{10}$	Horizontal

The graph of an ellipse is completely determined by its center, orientation, major radius, and minor radius, all of which can be determined from its equation written in standard from.

Example 8.3.1:

Graph: $\frac{(x+3)^2}{4}+\frac{(y-2)^2}{25}=1$.

Solution

Written in this form we can see that the center of the ellipse is $(-3,2)$, $a=\sqrt{4}=2$, and $b=\sqrt{25}=5$. From the center mark points 2 units to the left and right and 5 units up and down.

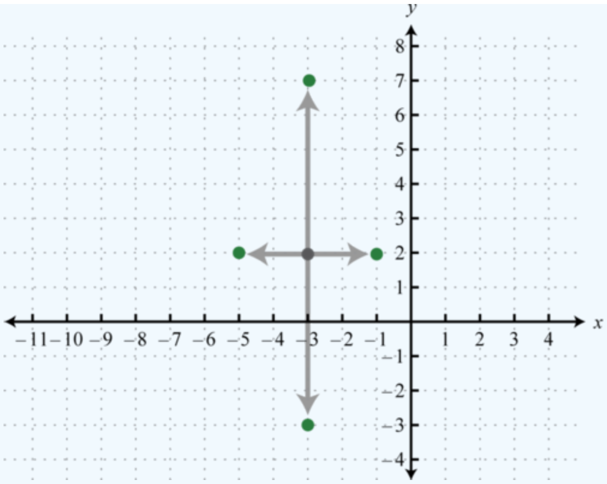

Figure 8.3.4

Then draw an ellipse through these four points.

Answer

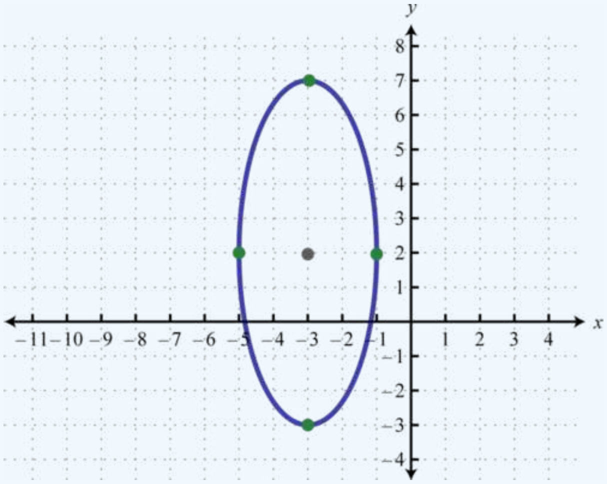

Figure 8.3.5

As with any graph, we are interested in finding the x- and y-intercepts.

Example 8.3.2

Find the intercepts: $\frac{(x+3)^2}{4}+\frac{(y-2)^2}{25}=1$.

Solution

To find the x-intercepts set $y=0$:

$$\begin{aligned}\frac{(x+3)^2}{4}+\frac{(0-2)^2}{25}&=1\\ \frac{(x+3)^2}{4}+\frac{4}{25}&=1\\ \frac{(x+3)^2}{4}&=1-\frac{4}{25}\\ \frac{(x+3)^2}{4}&=\frac{21}{25}\end{aligned}$$

At this point we extract the root by applying the square root property.

$$\begin{aligned}\frac{x+3}{2} &= \pm\sqrt{\frac{21}{25}}\\ x+3 &= \pm\frac{2\sqrt{21}}{5}\\ x &= -3 \pm \frac{2\sqrt{21}}{5} = \frac{-15\pm 2\sqrt{21}}{5}\end{aligned}$$

Setting $x=0$ and solving for y leads to complex solutions, therefore, there are no y-intercepts. This is left as an exercise.

Answer:

x-intercepts: $\left(\frac{-15\pm 2\sqrt{21}}{5}, 0\right)$; y-intercepts: none.

Unlike a circle, standard form for an ellipse requires a 1 on one side of its equation.

Example 8.3.3:

Graph and label the intercepts: $(x-2)^2+9(y-1)^2=9$.

Solution

To obtain standard form, with 1 on the right side, divide both sides by 9.

$$\begin{aligned}\frac{(x-2)^2+9(y-1)^2}{9} &= \frac{9}{9}\\ \frac{(x-2)^2}{9}+\frac{9(y-1)^2}{9} &= \frac{9}{9}\\ \frac{(x-2)^2}{9}+\frac{(y-1)^2}{1} &= 1\end{aligned}$$

Therefore, the center of the ellipse is $(2,1)$, $a=\sqrt{9}=3$, and $b=\sqrt{1}=1$. The graph follows:

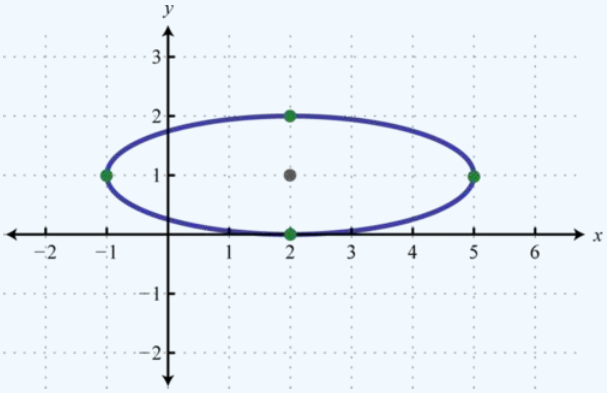

Figure $8.3.6$

To find the intercepts we can use the standard form $\frac{(x-2)^2}{9}+(y-1)^2=1$:

Table $8.3.2$

x-intercepts set $y=0$	y-intercepts set $x=0$
$\frac{(x-2)^2}{9}+(0-1)^2=1$ $\frac{(x-2)^2}{9}+1=1$ $(x-2)^2=0$ $x-2=0$ $x=2$	$\frac{(0-2)^2}{9}+(y-1)^2=1$ $\frac{4}{9}+(y-1)^2=1$ $(y-1)^2=\frac{5}{9}$ $y-1=\pm\sqrt{\frac{5}{9}}$ $y=1\pm\frac{\sqrt{5}}{3}=\frac{3\pm\sqrt{5}}{3}$

Therefore the x-intercept is $(2,0)$ and the y-intercepts are $\left(0,\frac{3+\sqrt{5}}{3}\right)$ and $\left(0,\frac{3-\sqrt{5}}{3}\right)$.

Answer:

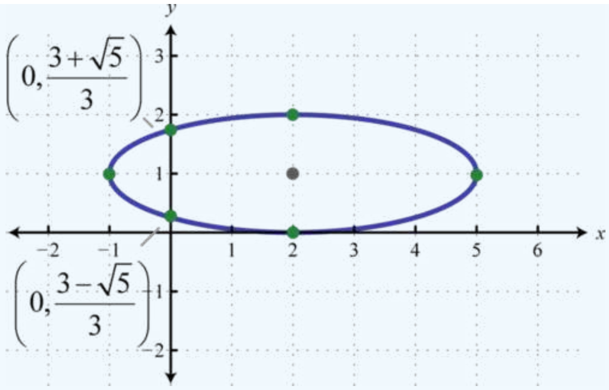

Figure 8.3.7

Consider the ellipse centered at the origin,

$$x^2 + \frac{y^2}{4} = 1$$

Given this equation we can write,

$$\frac{(x-0)^2}{1^2} + \frac{(y-0)^2}{2^2} = 1$$

In this form, it is clear that the center is $(0, 0)$, $a = 1$, and $b = 2$. Furthermore, if we solve for y we obtain two functions:

$$\begin{aligned} x^2 + \frac{y^2}{4} &= 1 \\ \frac{y^2}{4} &= 1 - x^2 \\ y^2 &= 4\left(1 - x^2\right) \\ y &= \pm\sqrt{4\left(1 - x^2\right)} \\ y &= \pm 2\sqrt{1 - x^2} \end{aligned}$$

The function defined by $y = 2\sqrt{1 - x^2}$ is the top half of the ellipse and the function defined by $y = -2\sqrt{1 - x^2}$ is the bottom half.

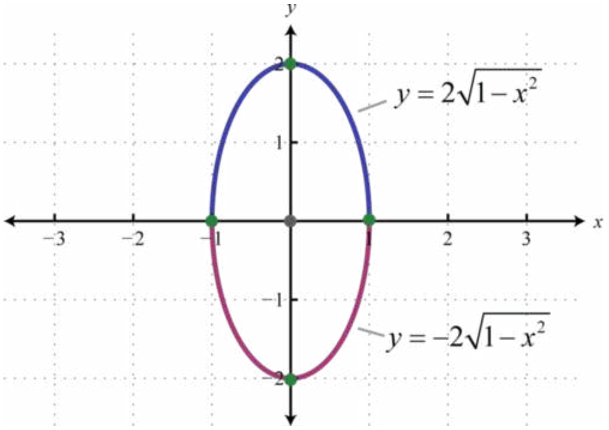

Figure 8.3.8

Exercise 8.3.1

Graph: $9(x-3)^2 + 4(y+2)^2 = 36$.

Answer

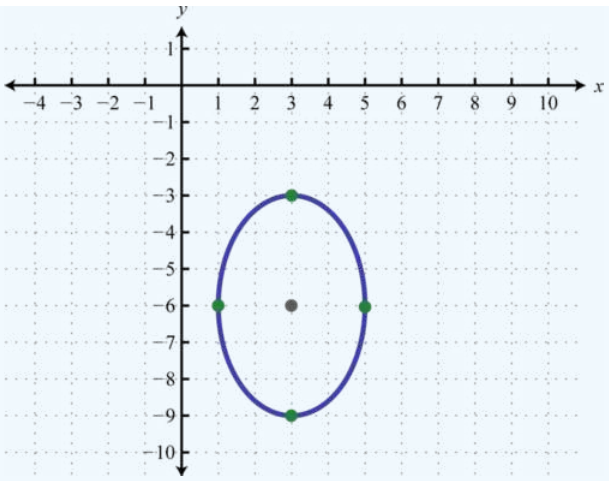

Figure $8.3.9$

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Watch on YouTube

The Ellipse in General Form

We have seen that the graph of an ellipse is completely determined by its center, orientation, major radius, and minor radius; which can be read from its equation in standard form. However, the equation is not always given in standard form. The equation of an *ellipse in general form*[22] follows,

$$px^2+qy^2+cx+dy+e=0$$

where $p, q>0$. The steps for graphing an ellipse given its equation in general form are outlined in the following example.

Example $8.3.4$:

Graph $2x^2+9y^2+16x-90y+239=0$.

Solution

Begin by rewriting the equation in standard form.

Step 1: Group the terms with the same variables and move the constant to the right side. Factor so that the leading coefficient of each grouping is 1.

$$\begin{aligned} 2x^2+9y^2+16x-90y+239&=0 \\ (2x^2+16x+\_\_)+(9y^2-90y+\_\_)&=-239 \\ 2\left(x^2+8x+\_\_\right)+9\left(y^2-10y+\_\_\right)&=-239 \end{aligned}$$

Step 2: Complete the square for each grouping. In this case, for the terms involving x use $\left(\frac{8}{2}\right)^2=4^2=16$ and for the terms involving y use $\left(\frac{-10}{2}\right)^2=(-5)^2=25$. The factor in front of the grouping affects the value used to balance the equation on the right side:

$$2(x^2+8x+16)+9(y^2-10y+25)=-239+32+225$$

Because of the distributive property, adding 16 inside of the first grouping is equivalent to adding $2 \cdot 16 = 32$. Similarly, adding 25 inside of the second grouping is equivalent to adding $9 \cdot 25 = 225$. Now factor and then divide to obtain 1 on the right side.

$$\begin{gathered}
2(x+4)^2 + 9(y-5)^2 = 18 \\
\frac{2(x+4)^2 + 9(y-5)^2}{18} = \frac{18}{18} \\
\frac{2(x+4)^2}{18} + \frac{9(y-5)^2}{18} = \frac{18}{18} \\
\frac{(x+4)^2}{9} + \frac{(y-5)^2}{2} = 1
\end{gathered}$$

Step 3: Determine the center, a, and b. In this case, the center is $(-4, 5)$, $a = \sqrt{9} = 3$, and $b = \sqrt{2}$.

Step 4: Use a to mark the vertices left and right of the center, use b to mark the vertices up and down from the center, and then sketch the graph. In this case, the vertices along the minor axes $(-4, 5 \pm \sqrt{2})$ are not apparent and should be labeled.

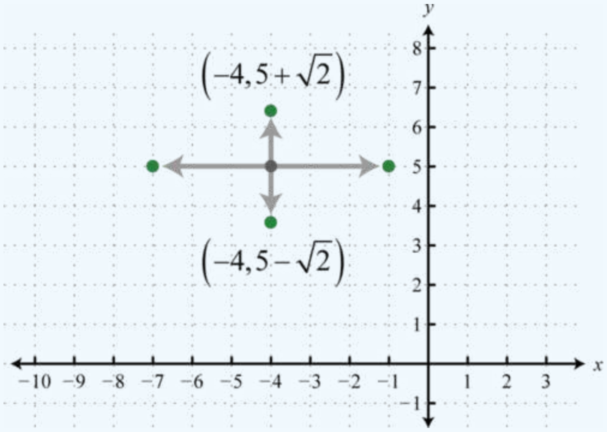

Figure $8.3.10$

Answer:

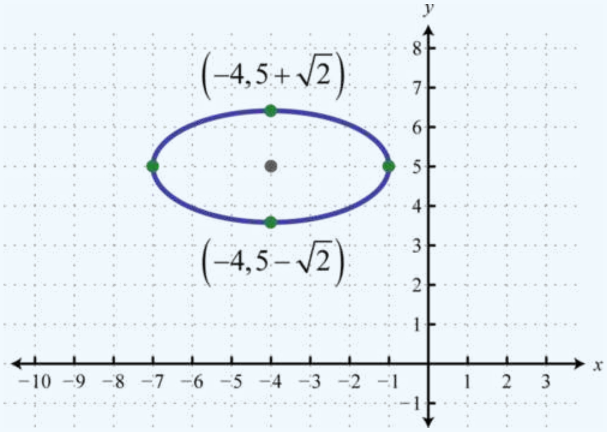

Figure $8.3.11$

Example $8.3.5$:

Determine the center of the ellipse as well as the lengths of the major and minor axes: $5x^2 + y^2 - 3x + 40 = 0$.

Solution

In this example, we only need to complete the square for the terms involving x.

$$\begin{gathered}
5x^2 + y^2 - 30x + 40 = 0 \\
(5x^2 - 30x + \_\_) + y^2 = -40 \\
5\left(x^2 - 6x + \_\_\right) + y^2 = -40
\end{gathered}$$

Use $\left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$ for the first grouping to be balanced by $5 \cdot 9 = 45$ on the right side.

$$\begin{aligned} 5(x^2 - 6x + 9) + y^2 &= -40 + 45 \\ 5(x-3)^2 + y^2 &= 5 \\ \frac{5(x-3)^2 + y^2}{5} &= \frac{5}{5} \\ \frac{(x-3)^2}{1} + \frac{y^2}{5} &= 1 \end{aligned}$$

Here, the center is $(3, 0)$, $a = \sqrt{1} = 1$, and $b = \sqrt{5}$. Because b is larger than a, the length of the major axis is $2b$ and the length of the minor axis is $2a$.

Answer

Center: $(3, 0)$; major axis: $2\sqrt{5}$ units; minor axis: 2 units.

Exercise $8.3.2$

Graph $x^2 + 4y^2 + 10x - 16y + 25 = 0$.

Answer

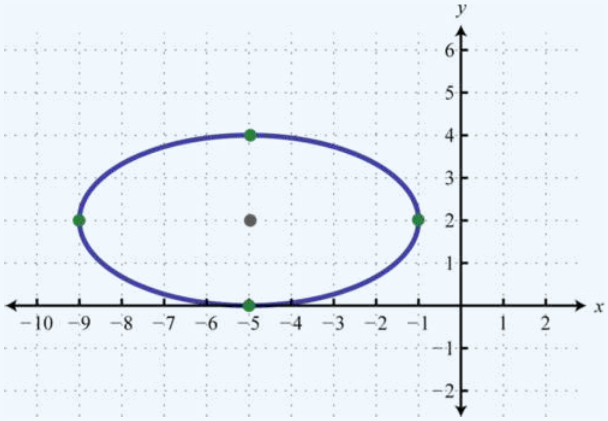

Figure $8.3.12$

www.youtube.com/v/Mhp7S5H2820

Key Takeaways

- The graph of an ellipse is completely determined by its center, orientation, major radius, and minor radius.
- The center, orientation, major radius, and minor radius are apparent if the equation of an ellipse is given in standard form: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.
- To graph an ellipse, mark points a units left and right from the center and points b units up and down from the center. Draw an ellipse through these points.
- The orientation of an ellipse is determined by a and b. If $a > b$ then the ellipse is wider than it is tall and is considered to be a horizontal ellipse. If a<ba<b then the ellipse is taller than it is wide and is considered to be a vertical ellipse.
- If the equation of an ellipse is given in general form $px^2 + qy^2 + cx + dy + e = 0$ where $p, q > 0$, group the terms with the same variables, and complete the square for both groupings.
- We recognize the equation of an ellipse if it is quadratic in both x and y and the coefficients of each square term have the same sign.

Exercise $8.3.3$

Given the equation of an ellipse in standard form, determine its center, orientation, major radius, and minor radius.

1. $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{49} = 1$
2. $\frac{(x+3)^2}{64} + \frac{(y-2)^2}{9} = 1$
3. $\frac{x^2}{3} + (y+9)^2 = 1$

4. $\frac{(x-1)^2}{8}+y^2=1$
5. $4(x+5)^2+9(y+5)^2=36$
6. $16(x-1)^2+3(y+10)^2=48$

Answer

1. Center: $(1,-2)$; orientation: vertical; major radius: 7 units; minor radius: 2 units; $a=2; b=7$
3. Center: $(0,-9)$; orientation: horizontal; major radius: $\sqrt{3}$ units; minor radius: 1 unit; $a=\sqrt{3}; b=1$
5. Center: $(-5,-5)$; orientation: horizontal; major radius: 3 units; minor radius: 2 units; $a=3; b=2$

Exercise 8.3.4

Determine the standard form for the equation of an ellipse given the following information.

1. Center $(3,4)$ with $a=5$ and $b=2$.
2. Center $(-1,9)$ with $a=7$ and $b=3$.
3. Center $(5,-1)$ with $a=\sqrt{6}$ and $b=2\sqrt{3}$.
4. Center $(-7,-2)$ with $a=5\sqrt{2}$ and $b=\sqrt{7}$.
5. Center $(0,-3)$ with $a=1$ and $b=\sqrt{5}$.
6. Center $(0,0)$ with $a=\sqrt{2}$ and $b=4$.

Answer

1. $\frac{(x-3)^2}{25}+\frac{(y-4)^2}{4}=1$
3. $\frac{(x-5)^2}{6}+\frac{(y+1)^2}{12}=1$
5. $x^2+\frac{(y+3)^2}{5}=1$

Exercise 8.3.5

Graph.

1. $\frac{(x-4)^2}{4}+\frac{(y+2)^2}{9}=1$
2. $\frac{(x+1)^2}{25}+\frac{(y-2)^2}{4}=1$
3. $\frac{(x-5)^2}{16}+\frac{(y+6)^2}{1}=1$
4. $\frac{(x+4)^2}{4}+\frac{(y+3)^2}{36}=1$
5. $\frac{(x-2)^2}{9}+\frac{(y-1)^2}{64}=1$
6. $\frac{(x+1)^2}{49}+(y+3)^2=1$
7. $4(x+3)^2+9(y-3)^2=36$
8. $16x^2+(y-1)^2=16$
9. $4(x-2)^2+25y^2=100$
10. $81x^2+y^2=81$
11. $\frac{(x-2)^2}{8}+\frac{(y-4)^2}{9}=1$
12. $\frac{(x+1)^2}{4}+\frac{(y-1)^2}{12}=1$
13. $\frac{(x-6)^2}{2}+\frac{(y+2)^2}{5}=1$
14. $\frac{(x+3)^2}{18}+\frac{(y-5)^2}{3}=1$
15. $3x^2+2(y-3)^2=6$
16. $5(x+1)^2+3y^2=15$
17. $4x^2+6y^2=24$
18. $5x^2+10y^2=50$

Answer

1.

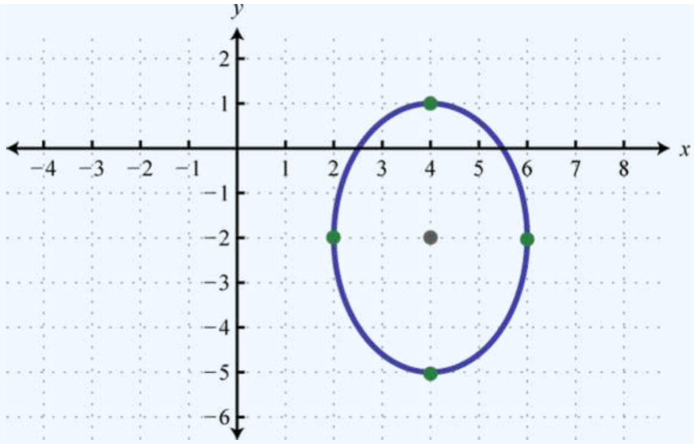

Figure 8.3.13

3.

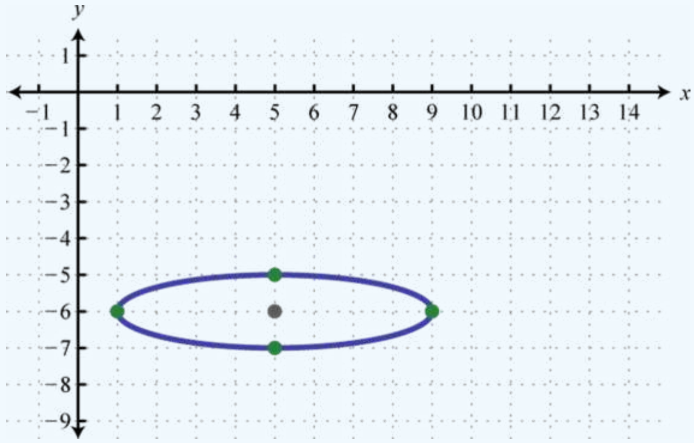

Figure 8.3.14

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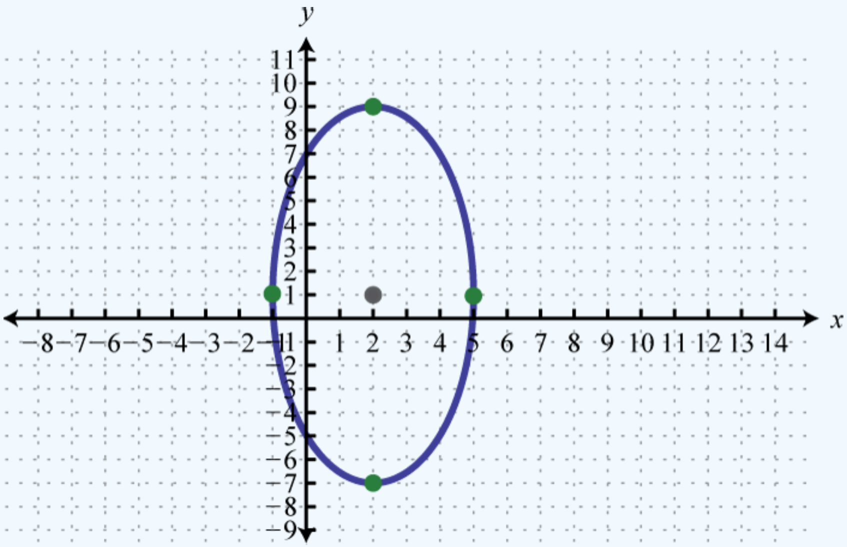

Figure 8.3.15

7.

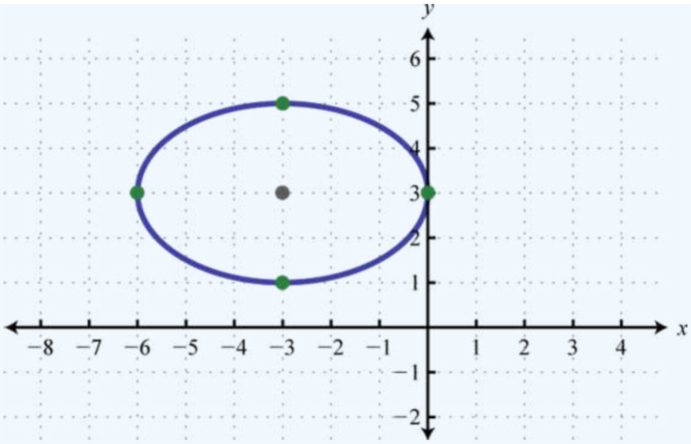

Figure 8.3.16

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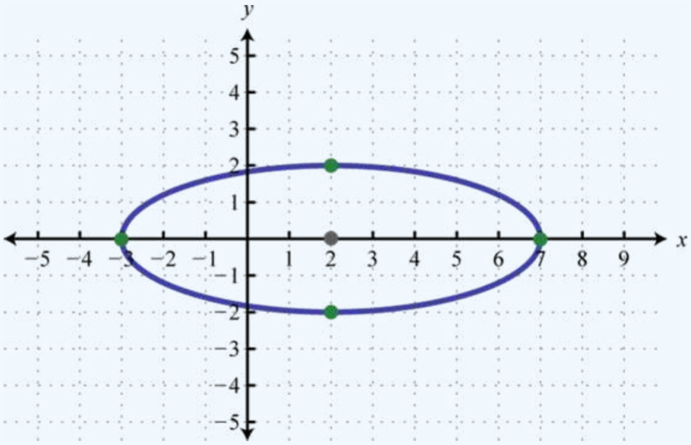

Figure 8.3.17

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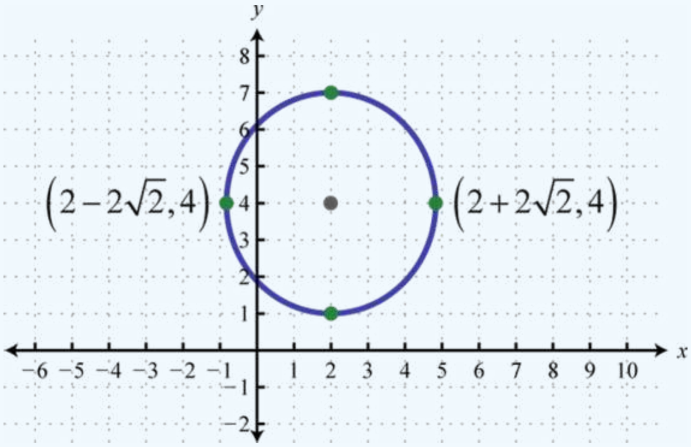

Figure 8.3.18

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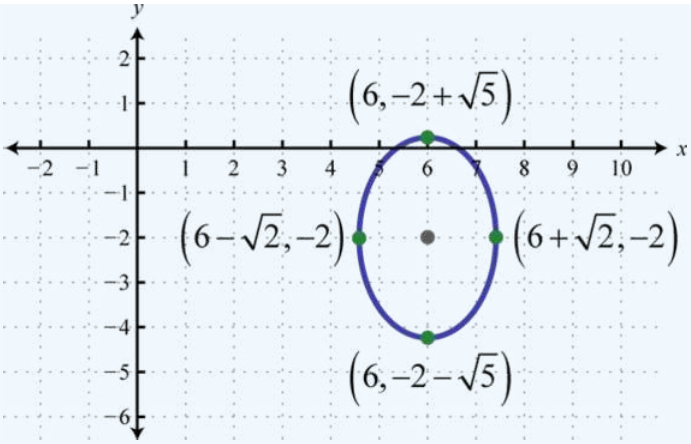

Figure 8.3.19

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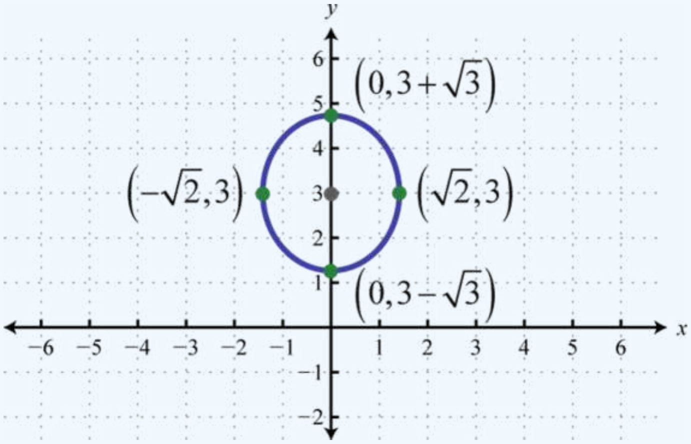

Figure 8.3.20

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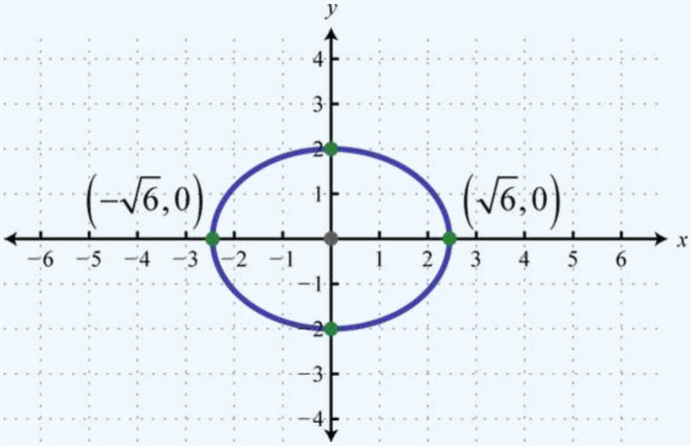

Figure 8.3.21

Exercise 8.3.6

Find the x- and y-intercepts.

1. $\frac{(x-3)^2}{4}+\frac{(y-2)^2}{9}=1$
2. $\frac{(x+3)^2}{16}+\frac{(y-7)^2}{9}=1$

3. $\frac{(x-2)^2}{4}+\frac{(y+6)^2}{36}=1$
4. $\frac{(x+1)^2}{25}+\frac{(y-1)^2}{9}=1$
5. $5x^2+2(y-4)^2=20$
6. $4(x-3)^2+9y^2=72$
7. $5x^2+2y^2=10$
8. $3x^2+4y^2=24$

Answer

1. x-intercepts: $\left(\frac{9\pm2\sqrt{5}}{3},0\right)$;$y$-intercepts: none
3. x-intercepts: $(2,0)$;y-intercepts: $(0,-6$
5. x-intercepts: none;y-intercepts: $(0,4\pm\sqrt{10})$
7. x-intercepts: $(\pm\sqrt{2},0)$;y-intercepts: $(0,\pm\sqrt{5})$

Exercise 8.3.7

Find the equation of the ellipse.

1. Ellipse with vertices $(\pm5,0)$ and $(0,\pm6)$.
2. Ellipse whose major axis has vertices $(2,9)$ and $(2,-1)$ and minor axis has vertices $(-2,4)$ and $(6,4)$.
3. Ellipse whose major axis has vertices $(-8,-2)$ and $(0,-2)$ and minor axis has a length of 4 units.
4. Ellipse whose major axis has vertices $(-2,2)$ and $(-2,8)$ and minor axis has a length of 2 units.

Answer

1. $\frac{x^2}{25}+\frac{y^2}{36}=1$
3. $\frac{(x+4)^2}{16}+\frac{(y+2)^2}{4}=1$

Exercise 8.3.8

Rewrite in standard form and graph.

1. $4x^2+9y^2+8x-36y+4=0$
2. $9x^2+25y^2-18x+100y-116=0$
3. $4x^2+49y^2+24x+98y-111=0$
4. $9x^2+4y^2-72x+24y+144=0$
5. $x^2+64y^2-12x+128y+36=0$
6. $16x^2+y^2-96x-4y+132=0$
7. $36x^2+4y^2-40y-44=0$
8. $x^2+9y^2-2x-8=0$
9. $x^2+9y^2-4x-36y-41=0$
10. $16x^2+y^2+160x-10y+361=0$
11. $4x^2+5y^2+32x-20y+64=0$
12. $2x^2+3y^2-8x-30y+65=0$
13. $8x^2+5y^2-16x+10y-27=0$
14. $7x^2+2y^2+28x-16y+46=0$
15. $36x^2+16y^2-36x-32y-119=0$
16. $16x^2+100y^2+64x-300y-111=0$
17. $x^2+4y^2-20y+21=0$
18. $9x^2+y^2+12x-2y-4=0$

Answer

1. $\frac{(x+1)^2}{9}+\frac{(y-2)^2}{4}=1$;

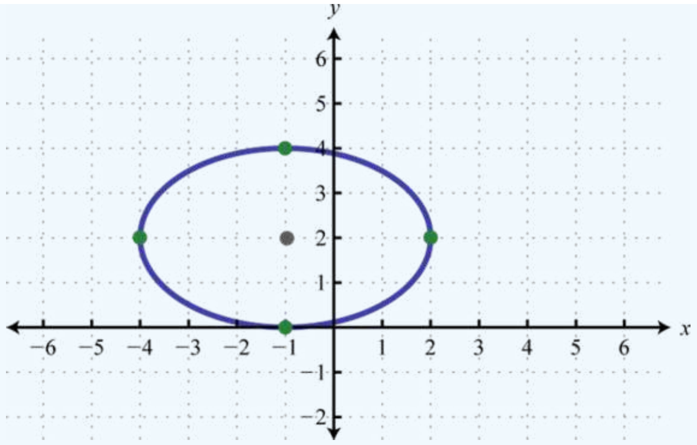

Figure 8.3.22

3. $\frac{(x+3)^2}{49}+\frac{(y+1)^2}{4}=1$;

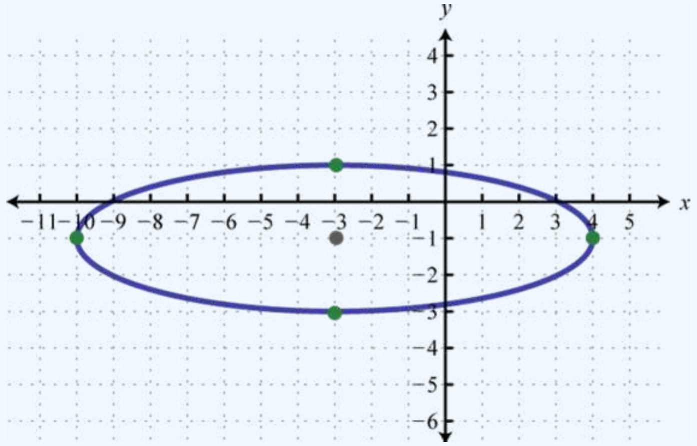

Figure 8.3.23

5. $\frac{(x-6)^2}{64}+(y+1)^2=1$;

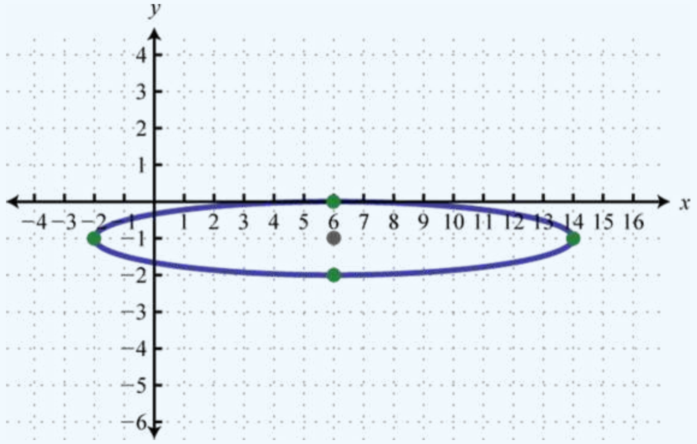

Figure 8.3.24

7. $\frac{x^2}{4}+\frac{(y-5)^2}{36}=1$;

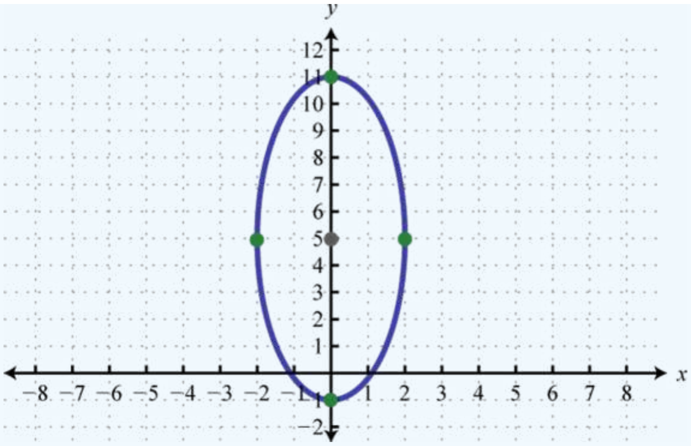

Figure 8.3.25

9. $\frac{(x-2)^2}{81}+\frac{(y-2)^2}{9}=1$;

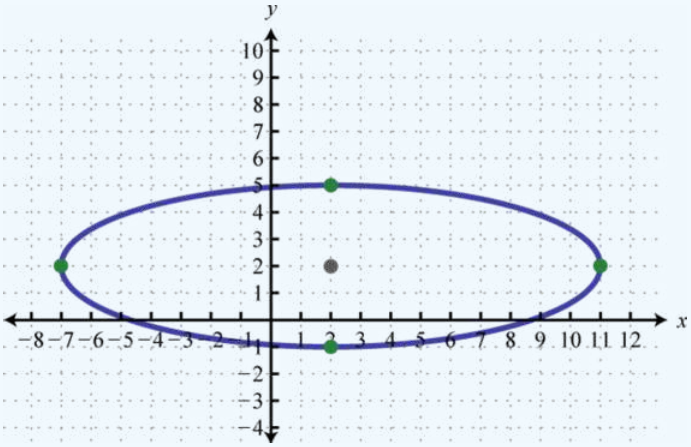

Figure 8.3.26

11. $\frac{(x+4)^2}{5}+\frac{(y-2)^2}{4}=1$;

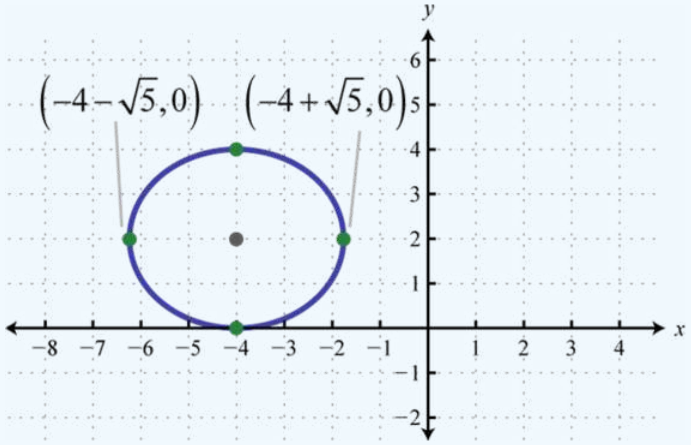

Figure 8.3.27

13. $\frac{(x-1)^2}{5}+\frac{(y+1)^2}{8}=1$;

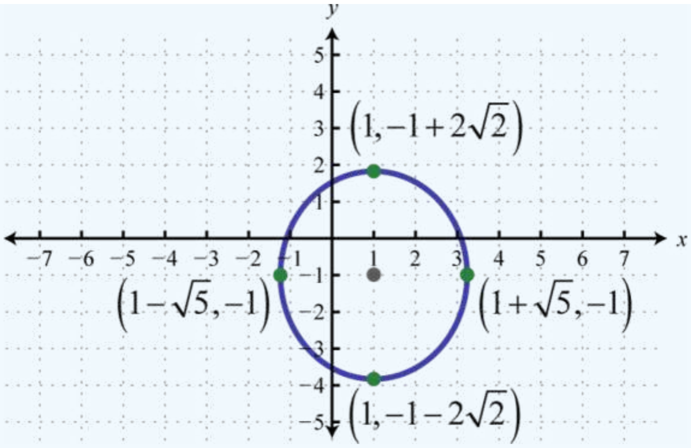

Figure 8.3.28

15. $\frac{\left(x-\frac{1}{2}\right)^2}{4}+\frac{(y-1)^2}{9}=1$;

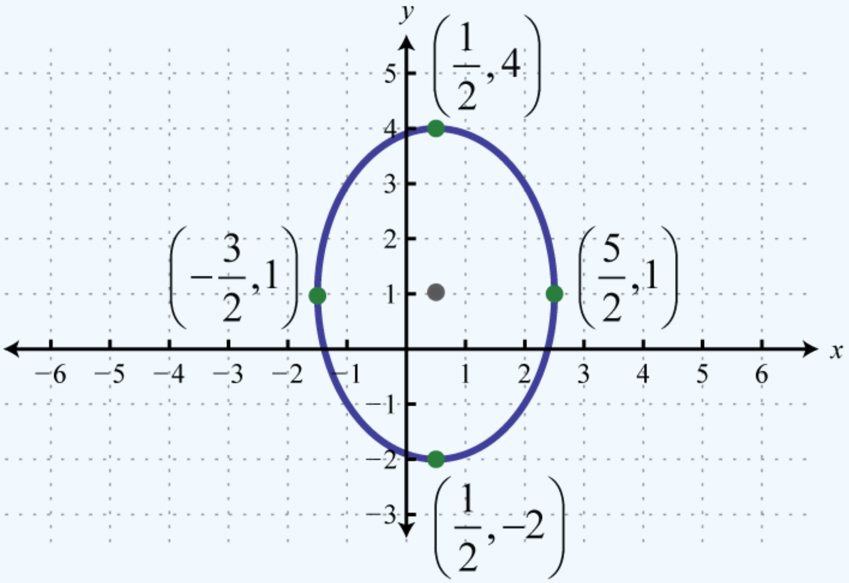

Figure 8.3.29

17. $\frac{x^2}{4}+\left(y-\frac{5}{2}\right)^2=1$;

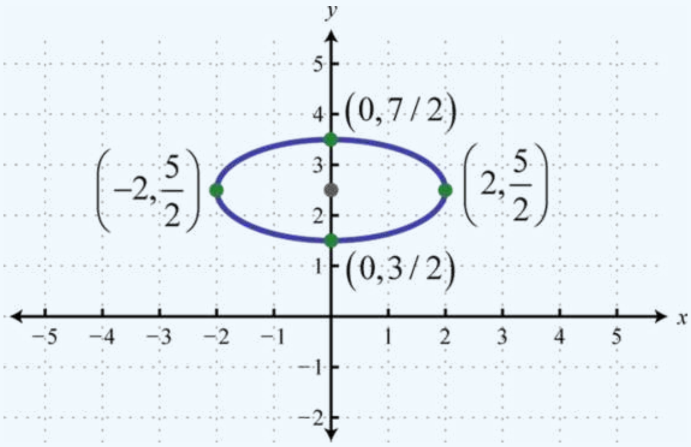

Figure 8.3.30

Exercise 8.3.9

Given general form determine the intercepts.

1. $5x^2+4y^2-20x+24y+36=0$
2. $4x^2+3y^2-8x+6y-5=0$
3. $6x^2+y^2-12x+4y+4=0$
4. $8x^2+y^2-6y-7=0$
5. $5x^2+2y^2-20x-8y+18=0$
6. $2x^2+3y^2-4x-5y+1=0$

Answer

1. x-intercepts: none; y-intercepts: $(0,-3)$
3. x-intercepts: $\left(\frac{3\pm\sqrt{3}}{3},0\right)$; y-intercepts: $(0,-2)$
5. x-intercepts: $\left(\frac{10\pm\sqrt{10}}{5},0\right)$; y-intercepts: none

Exercise 8.3.10

Determine the area of the ellipse. (The area of an ellipse is given by the formula $A=\pi ab$, where a and b are the lengths of the major radius and the minor radius.)

1. $\frac{(x-10)^2}{25}+\frac{(y+3)^2}{5}=1$
2. $\frac{(x+1)^2}{18}+\frac{y^2}{36}=1$
3. $7x^2+3y^2-14x+36y+94=0$
4. $4x^2+8y^2+20x-8y+11=0$

Answer

1. $5\pi\sqrt{5}$ square units
3. $\pi\sqrt{21}$ square units

Exercise 8.3.11

Given the graph of an ellipse, determine its equation in general form.

1.

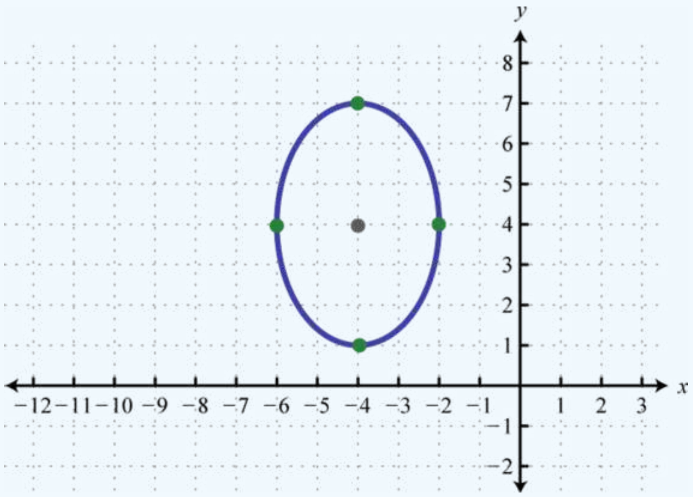

Figure 8.3.31

2.

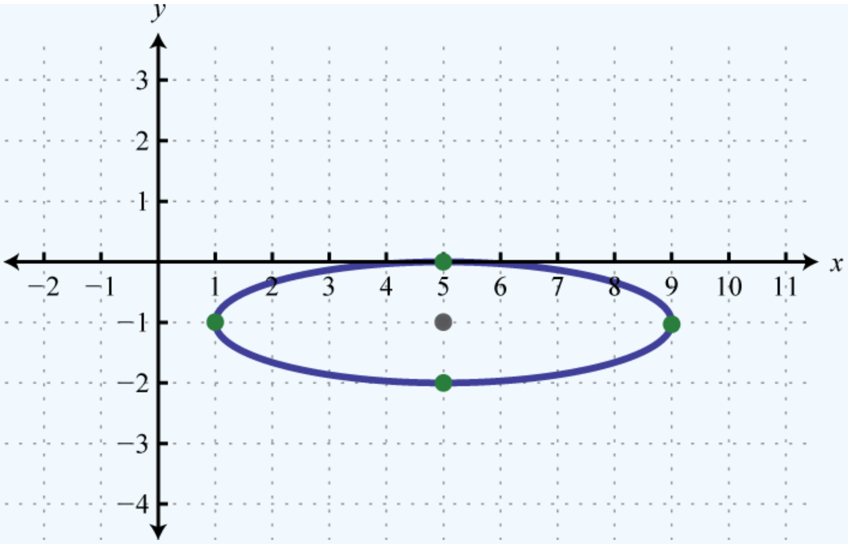

Figure 8.3.32

3.

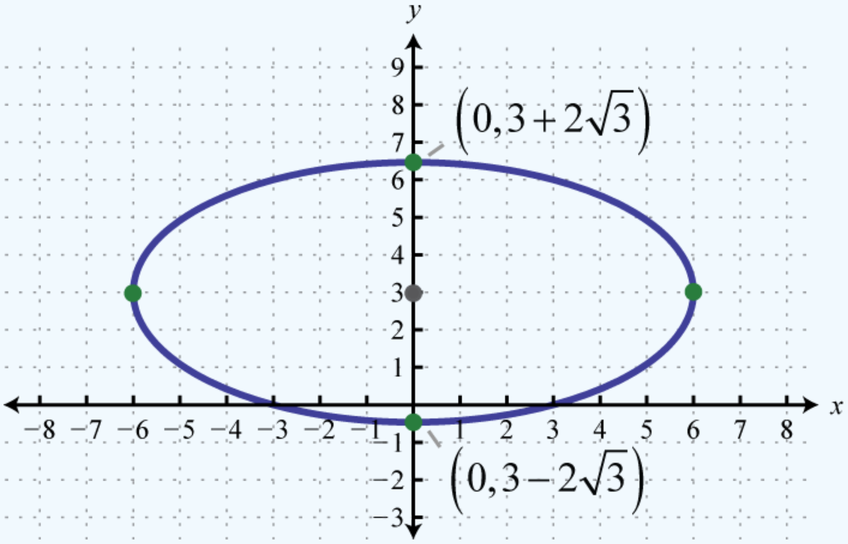

Figure 8.3.33

4.

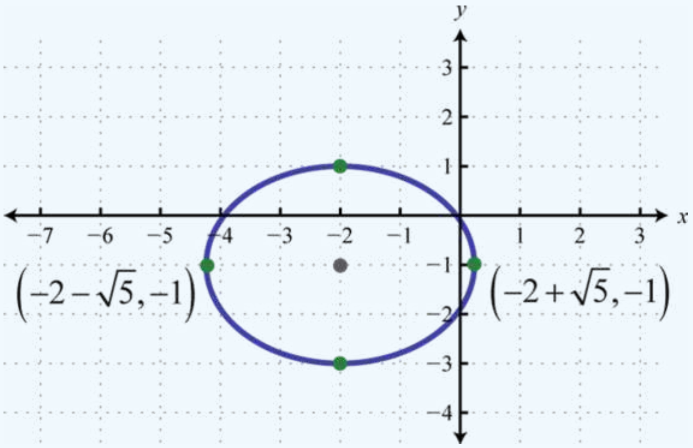

Figure 8.3.34

Answer

1. $9x^2+4y^2+72x-32y+172=0$

3. $x^2+3y^2-18y-9=0$

Exercise $8.3.12$

1. Explain why a circle can be thought of as a very special ellipse.
2. Make up your own equation of an ellipse, write it in general form and graph it.
3. Do all ellipses have intercepts? What are the possible numbers of intercepts for an ellipse? Explain.
4. Research and discuss real-world examples of ellipses.

Answer

1. Answer may vary

3. Answer may vary

Footnotes

[14] The set of points in a plane whose distances from two fixed points have a sum that is equal to a positive constant.

[15] Points on the ellipse that mark the endpoints of the major axis.

[16] The line segment through the center of an ellipse defined by two points on the ellipse where the distance between them is a maximum.

[17] The line segment through the center of an ellipse defined by two points on the ellipse where the distance between them is a minimum.

[18] Points on the ellipse that mark the endpoints of the minor axis.

[19] One-half of the length of the major axis.

[20] One-half of the length of the minor axis.

[21] The equation of an ellipse written in the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.The center is (h,k) and the larger of a and b is the major radius and the smaller is the minor radius.

[22] The equation of an ellipse written in the form $px^2+qy^2+cx+dy+e=0$ where $p,q>0$.

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8.4: Hyperbolas

Learning Objectives

- Graph a hyperbola in standard form.
- Determine the equation of a hyperbola given its graph.
- Rewrite the equation of a hyperbola in standard form.
- Identify a conic section given its equation.

The Hyperbola in Standard Form

A **hyperbola**[23] is the set of points in a plane whose distances from two fixed points, called foci, has an absolute difference that is equal to a positive constant. In other words, if points F_1 and F_2 are the foci and d is some given positive constant then (x, y) is a point on the hyperbola if $d = |d_1 - d_2|$ as pictured below:

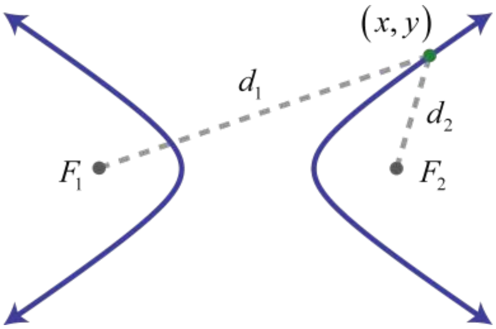

Figure 8.4.1

In addition, a hyperbola is formed by the intersection of a cone with an oblique plane that intersects the base. It consists of two separate curves, called **branches**[24]. Points on the separate branches of the graph where the distance is at a minimum are called **vertices**[25]. The midpoint between a hyperbola's vertices is its center. Unlike a parabola, a hyperbola is asymptotic to certain lines drawn through the center. In this section, we will focus on graphing hyperbolas that open left and right or upward and downward.

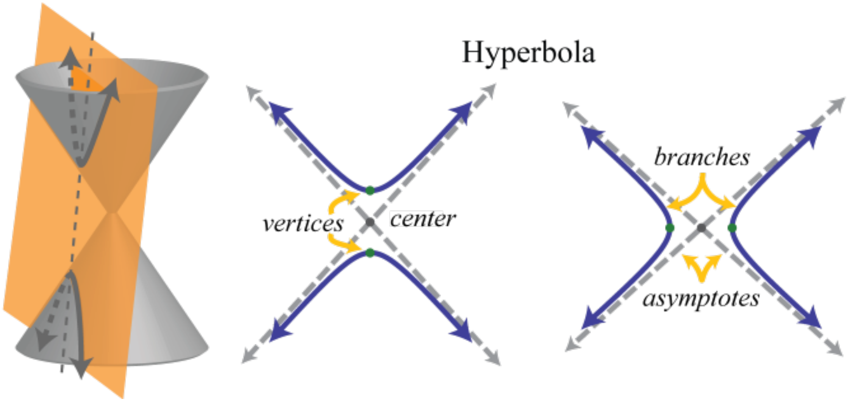

Figure 8.4.2

The asymptotes are drawn dashed as they are not part of the graph; they simply indicate the end behavior of the graph. The equation of a **hyperbola opening left and right in standard form**[26] follows:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

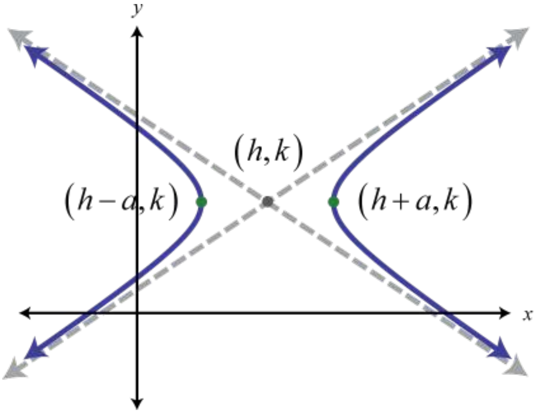

Figure $8.4.3$

Here the center is (h,k) and the vertices are $(h\pm a,k)$. The equation of a **hyperbola opening upward and downward in standard form**[27] follows:

$$\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$$

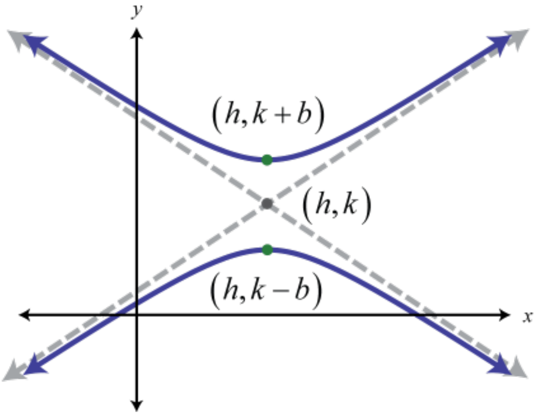

Figure $8.4.4$

Here the center is (h,k) and the vertices are $(h,k\pm b)$.

The asymptotes are essential for determining the shape of any hyperbola. Given standard form, the asymptotes are lines passing through the center (h,k) with slope $m=\pm\frac{b}{a}$. To easily sketch the asymptotes we make use of two special line segments through the center using a and b. Given any hyperbola, the **transverse axis**[28] is the line segment formed by its vertices. The **conjugate axis**[29] is the line segment through the center perpendicular to the transverse axis as pictured below:

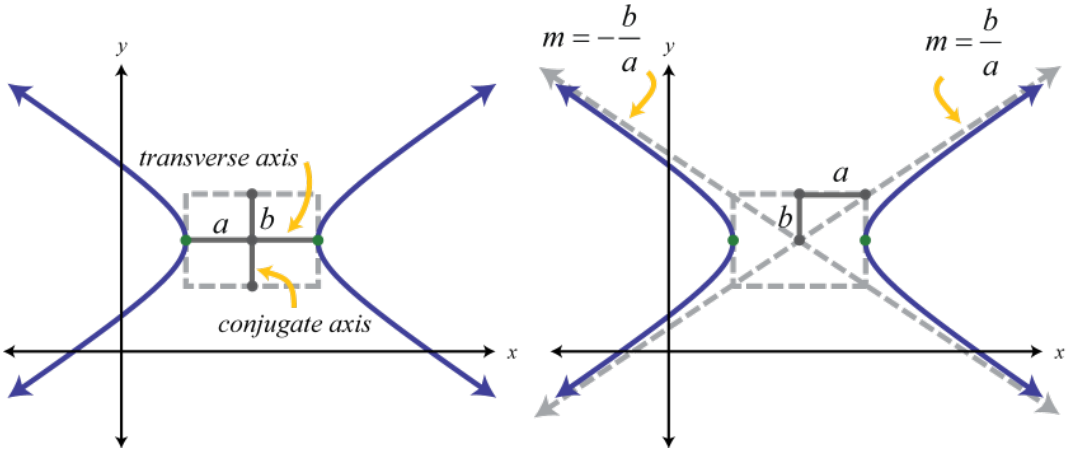

Figure $8.4.5$

The rectangle defined by the transverse and conjugate axes is called the **fundamental rectangle**[30]. The lines through the corners of this rectangle have slopes $m = \pm \frac{b}{a}$. These lines are the asymptotes that define the shape of the hyperbola. Therefore, given standard form, many of the properties of a hyperbola are apparent.

Table $8.4.1$

Equation	Center	a	b	Opens
$\frac{(x-3)^2}{25} - \frac{(y-5)^2}{16} = 1$	$(3,5)$	$a=5$	$b=4$	Left and right
$\frac{(y-2)^2}{36} - \frac{(x+1)^2}{9} = 1$	$(-1,2)$	$a=3$	$b=6$	Upward and downward
$\frac{(y+2)^2}{3} - (x-5)^2 = 1$	$(5,-2)$	$a=1$	$b=\sqrt{3}$	Upward and downward
$\frac{x^2}{49} - \frac{(y+4)^2}{8} = 1$	$(0,-4)$	$a=7$	$b=2\sqrt{2}$	Left and right

The graph of a hyperbola is completely determined by its center, vertices, and asymptotes.

Example $8.4.1$:

Graph: $\frac{(x-5)^2}{9} - \frac{(y-4)^2}{4} = 1$.

Solution:

In this case, the expression involving x has a positive leading coefficient; therefore, the hyperbola opens left and right. Here $a = \sqrt{9} = 3$ and $b = \sqrt{4} = 2$. From the center $(5,4)$, mark points 3 units left and right as well as 2 units up and down. Connect these points with a rectangle as follows:

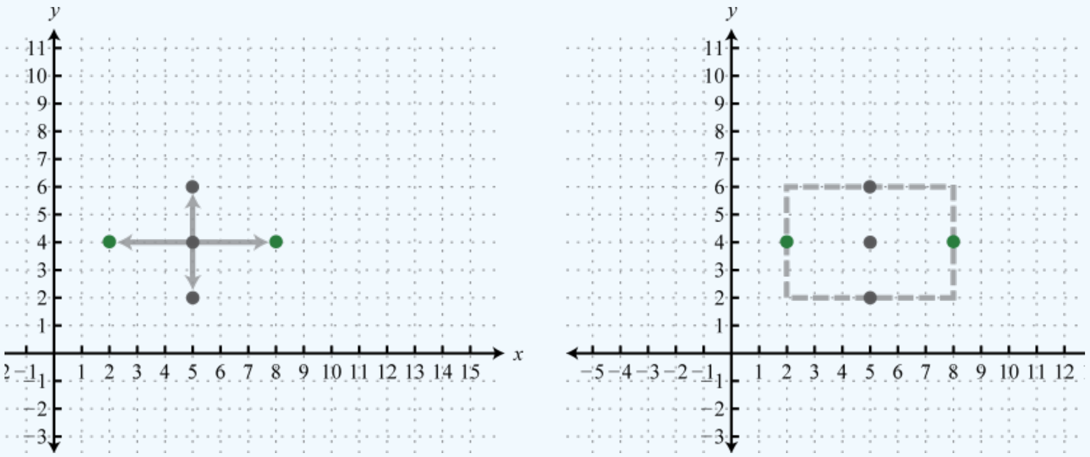

Figure $8.4.6$

The lines through the corners of this rectangle define the asymptotes.

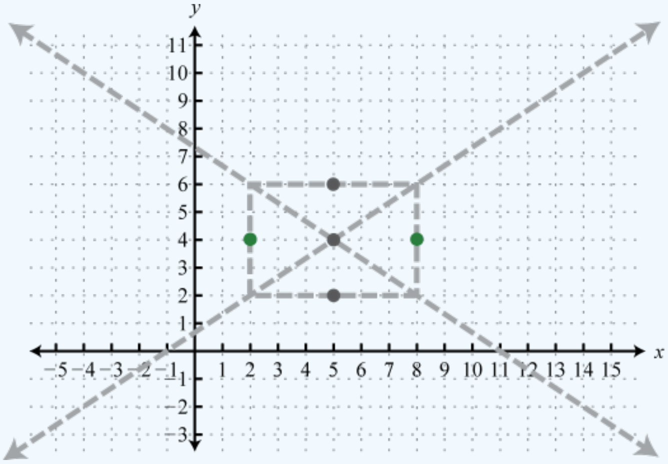

Figure $8.4.7$

Use these dashed lines as a guide to graph the hyperbola opening left and right passing through the vertices.

Answer

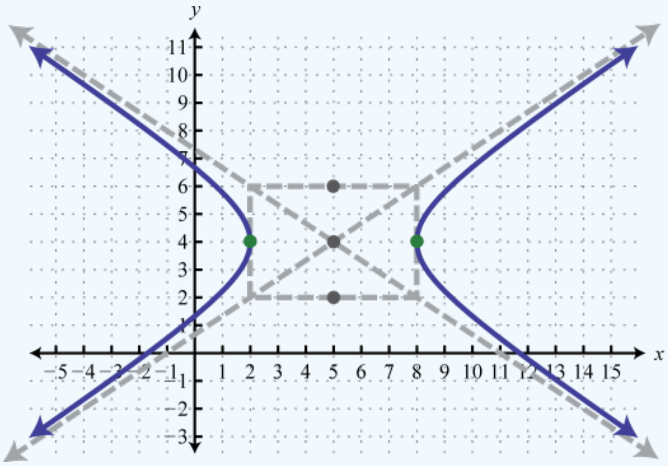

Figure 8.4.8

Example 8.4.2:

Graph: $\frac{(y-2)^2}{4} - \frac{(x+1)^2}{36} = 1$.

Solution

In this case, the expression involving y has a positive leading coefficient; therefore, the hyperbola opens upward and downward. Here $a = \sqrt{36} = 6$ and $b = \sqrt{4} = 2$. From the center $(-1, 2)$mark points 6 units left and right as well as 2 units up and down. Connect these points with a rectangle. The lines through the corners of this rectangle define the asymptotes.

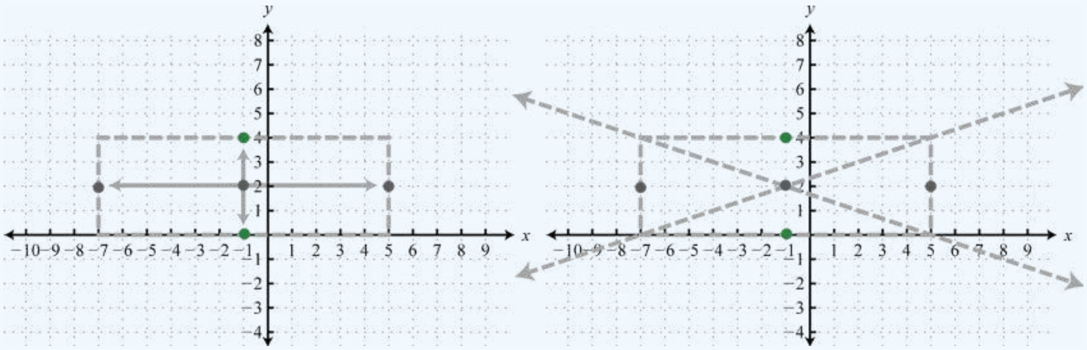

Figure 8.4.9

Use these dashed lines as a guide to graph the hyperbola opening upward and downward passing through the vertices.

Answer:

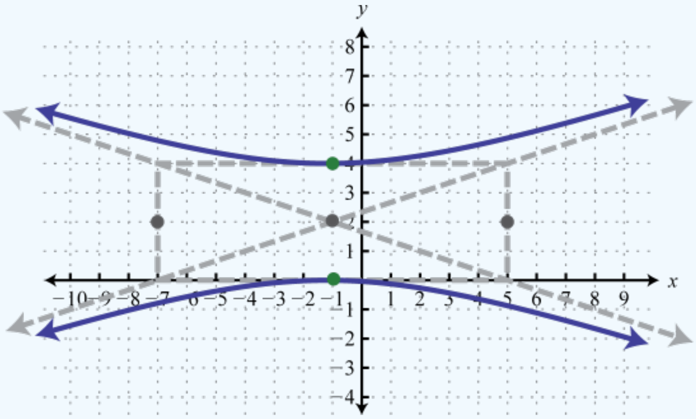

Figure 8.4.10

Note

When given a hyperbola opening upward and downward, as in the previous example, it is a common error to interchange the values for the center, h and k. This is the case because the quantity involving the variable y usually appears first in standard form. Take care to ensure that the y-value of the center comes from the quantity involving the variable y and that the x-value of the center is obtained from the quantity involving the variable x.

As with any graph, we are interested in finding the x- and y-intercepts.

Example $8.4.3$:

Find the intercepts: $\frac{(y-2)^2}{4}-\frac{(x+1)^2}{36}=1$.

Solution

To find the x-intercepts set $y=0$ and solve for x.

$$\begin{aligned}
\frac{(0-2)^2}{4}-\frac{(x+1)^2}{36}&=1\\
1-\frac{(x+1)^2}{36}&=1\\
-\frac{(x+1)^2}{36}&=0\\
(x+1)^2&=0\\
x+1&=0\\
x&=-1
\end{aligned}$$

Therefore there is only one x-intercept, $(-1,0)$. To find the y-intercept set $x=0$ and solve for y.

$$\begin{aligned}
\frac{(y-2)^2}{4}-\frac{(0+1)^2}{36}&=1\\
\frac{(y-2)^2}{4}-\frac{1}{36}&=1\\
\frac{(y-2)^2}{4}&=\frac{37}{36}\\
\frac{(y-2)}{2}&=\frac{\sqrt{37}}{36}\\
y-2&=\pm\frac{\sqrt{37}}{3}\\
y&=2\pm\frac{\sqrt{37}}{3}=\frac{6\pm\sqrt{37}}{3}
\end{aligned}$$

Therefore there are two y-intercepts, $\left(0,\frac{6-\sqrt{37}}{3}\right)\approx(0,-0.03)$ and $\left(0,\frac{6+\sqrt{37}}{3}\right)\approx(0,4.03)$.Take a moment to compare these to the sketch of the graph in the previous example.

Answer:

x-intercept: $(-1,0)$; y-intercepts: $\left(0,\frac{6-\sqrt{37}}{3}\right)$ and $\left(0,\frac{6+\sqrt{37}}{3}\right)$.

Consider the hyperbola centered at the origin,

$$9x^2-5y^2=45$$

Standard form requires one side to be equal to 1. In this case, we can obtain standard form by dividing both sides by 45.

$$\begin{aligned}\frac{9x^2-5y^2}{45}&=\frac{45}{45}\\ \frac{9x^2}{45}-\frac{5y^2}{45}&=\frac{45}{45}\\ \frac{x^2}{5}-\frac{y^2}{9}&=1\end{aligned}$$

This can be written as follows:

$$\frac{(x-0)^2}{5}-\frac{(y-0)^2}{9}=1$$

In this form, it is clear that the center is $(0,0)$, $a=\sqrt{5}$, and $b=3$. The graph follows.

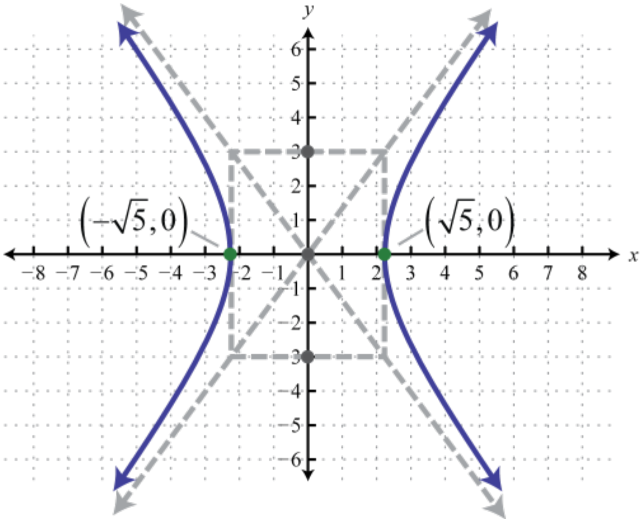

Figure 8.4.11

Exercise 8.4.1:

Graph: $\frac{y^2}{25}-\frac{(x-5)^2}{9}=1$:

Answer:

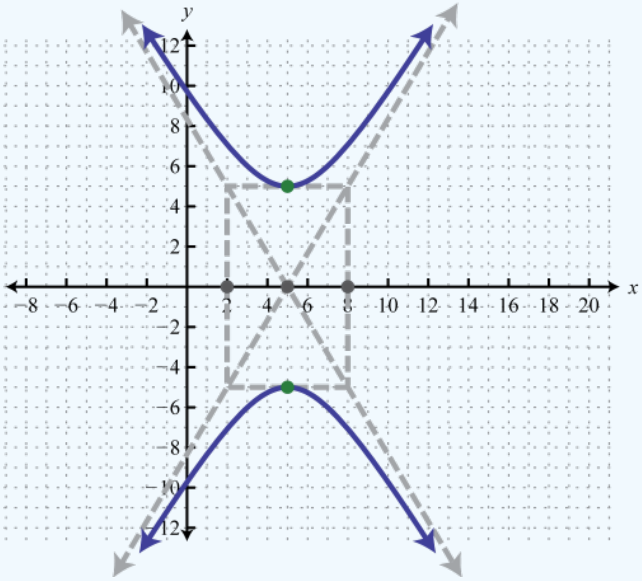

Figure 8.4.12

www.youtube.com/v/Dc2Vw_IiT1Q

The Hyperbola in General Form

We have seen that the graph of a hyperbola is completely determined by its center, vertices, and asymptotes; which can be read from its equation in standard form. However, the equation is not always given in standard form. The equation of a **hyperbola in general form**[31] follows:

$$\begin{aligned} px^2 - qy^2 + cx + dy + e = 0 &\quad \textit{Hyperbola opens left and right.} \\ qy^2 - px^2 + cx + dy + e = 0 &\quad \textit{Hyperbola opens upward and downward.} \end{aligned}$$

where $p, q > 0$. The steps for graphing a hyperbola given its equation in general form are outlined in the following example.

Example $8.4.4$:

Graph: $4x^2 - 9y^2 + 32x - 54y - 53 = 0$.

Solution

Begin by rewriting the equation in standard form.

Step 1: Group the terms with the same variables and move the constant to the right side. Factor so that the leading coefficient of each grouping is 1.

$$\begin{aligned} 4x^2 - 9y^2 + 32x - 54y - 53 &= 0 \\ \left(4x^2 + 32x + \_\_\right) + \left(-9y^2 - 54y + \_\_\right) &= 53 \\ 4\left(x^2 + 8x + \_\_\right) - 9\left(y^2 + 6y + \_\_\right) &= 53 \end{aligned}$$

Step 2: Complete the square for each grouping. In this case, for the terms involving x use $\left(\frac{8}{2}\right)^2 = 4^2 = 16$ and for the terms involving y use $\left(\frac{6}{2}\right)^2 = (3)^2 = 9$. The factor in front of each grouping affects the value used to balance the equation on the right,

$$4(x^2 + 8x + 16) - 9(y^2 + 6y + 9) = 53 + 64 - 81$$

Because of the distributive property, adding 16 inside of the first grouping is equivalent to adding $4 \cdot 16 = 64$. Similarly, adding 9 inside of the second grouping is equivalent to adding $-9 \cdot 9 = -81$. Now factor and then divide to obtain 1 on the right side.

$$\begin{aligned} 4(x+4)^2 - 9(y+3)^2 &= 36 \\ \frac{4(x+4)^2 - 9(y+3)^2}{36} &= \frac{36}{36} \\ \frac{4(x+4)^2}{36} - \frac{9(y+3)^2}{36} &= \frac{36}{36} \\ \frac{(x+4)^2}{9} - \frac{(y+3)^2}{4} &= 1 \end{aligned}$$

Step 3: Determine the center, a, and b, and then use this information to sketch the graph. In this case, the center is $(-4, -3)$, $a = \sqrt{9} = 3$, and $b = \sqrt{4} = 2$. Because the leading coefficient of the expression involving x is positive and the coefficient of the expression involving y is negative, we graph a hyperbola opening left and right.

Answer:

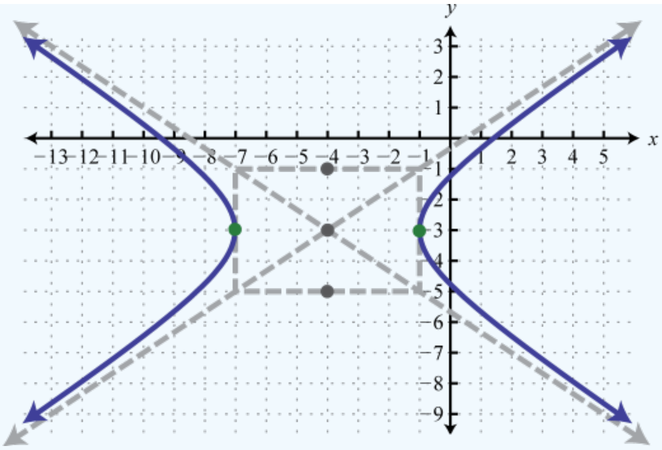

Figure 8.4.13

Exercise 8.4.2:

Graph: $4y^2-x^2-40y-12x+60=0$.

Answer:

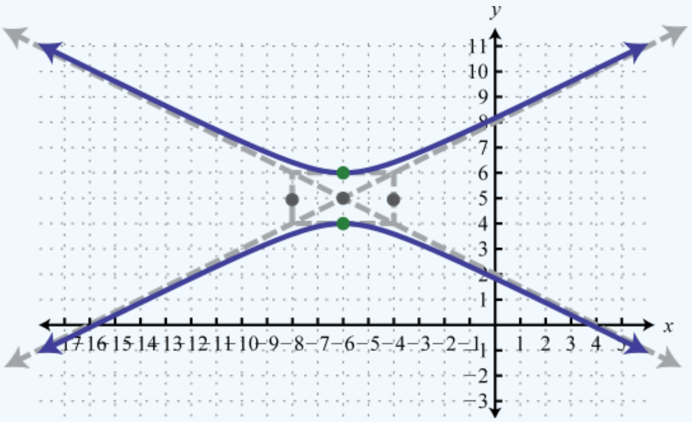

Figure 8.4.14

www.youtube.com/v/s4F6IZLgVhs

Identifying the Conic Sections

In this section, the challenge is to identify a conic section given its equation in general form. To distinguish between the conic sections, use the exponents and coefficients. If the equation is quadratic in only one variable and linear in the other, then its graph will be a parabola.

Table 8.4.2

Parabola: $a>0$	
$y=a(x-h)^2+k$ $y=ax^2+bx+c$	$x=a(y-k)^2+h$ $x=ay^2+by+c$

<table>
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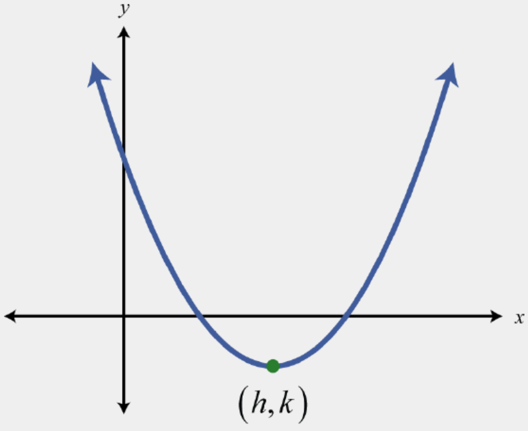
<tr><td>

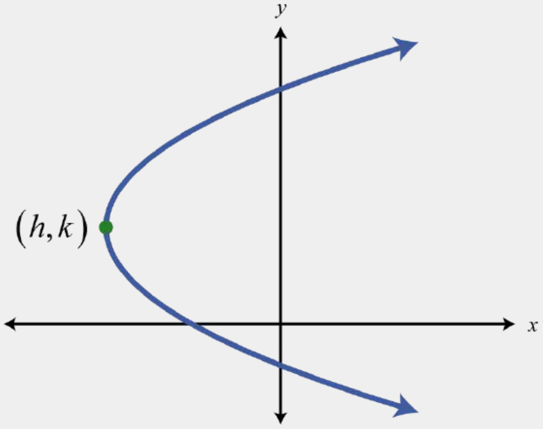
Figure 8.4.15</td><td>

Figure 8.4.16</td></tr>
</table>

Table 8.4.3

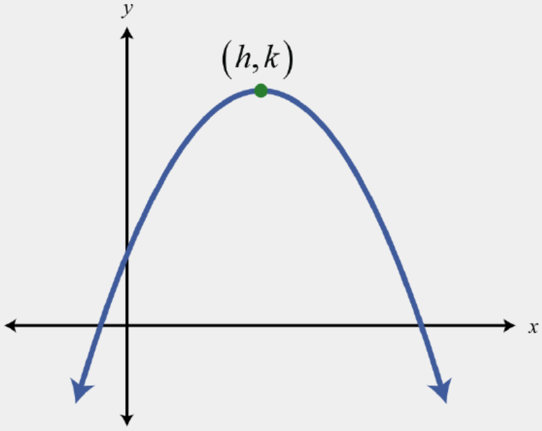
<table>
<tr><th colspan="2">Parabola: $a<0$</th></tr>
<tr><td>$y=a(x-h)^2+k$
$y=ax^2+bx+c$</td><td>$x=a(y-k)^2+h$
$x=ay^2+by+c$</td></tr>
<tr><td>

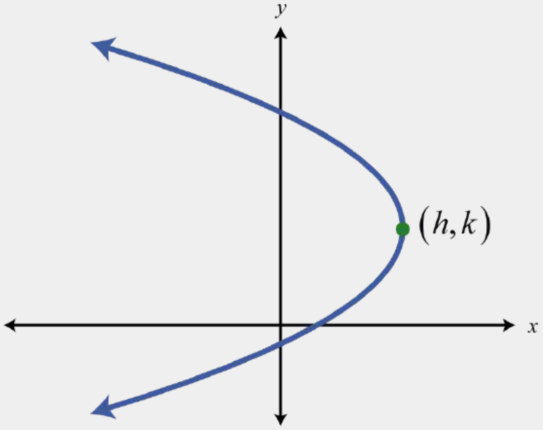
Figure 8.4.17</td><td>

Figure 8.4.18</td></tr>
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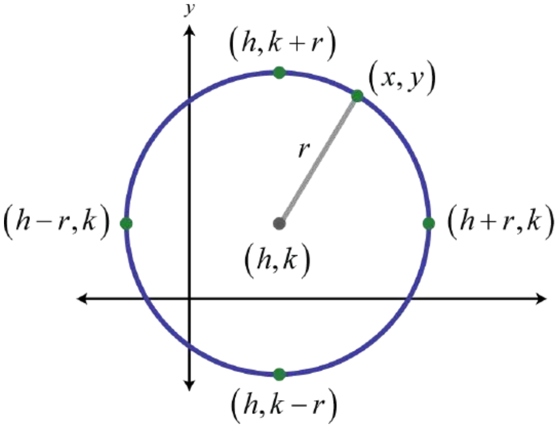
If the equation is quadratic in both variables, where the coefficients of the squared terms are the same, then its graph will be a circle.

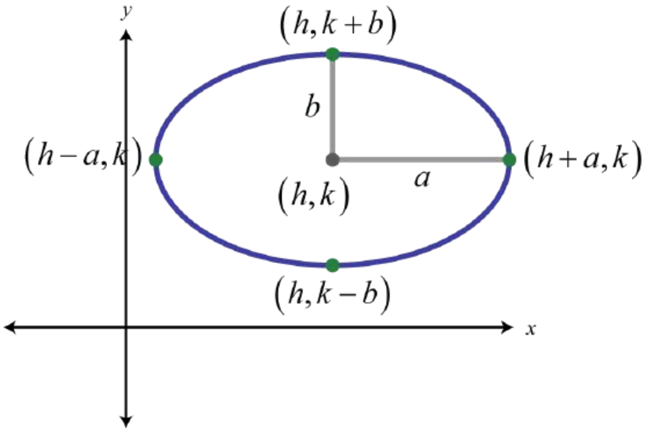
Table 8.4.4

Circle:	

Circle:	
$(x-h)^2+(y-k)^2=r^2$ $x^2+y^2+cx+dy+e=0$	 y x $(h,k+r)$ (x,y) r $(h-r,k)$ $(h+r,k)$ (h,k) $(h,k-r)$ ***Figure*** 8.4.19

If the equation is quadratic in both variables where the coefficients of the squared terms are different but have the same sign, then its graph will be an ellipse.

Table 8.4.5

Ellipse: $a,b>0$ and $p,q>0$	
$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ $px^2+qy^2+cx+dy+e=0$	 y x $(h,k+b)$ b $(h-a,k)$ $(h+a,k)$ a (h,k) $(h,k-b)$ ***Figure*** 8.4.20

If the equation is quadratic in both variables where the coefficients of the squared terms have different signs, then its graph will be a hyperbola.

Table 8.4.6

Hyperbola: $a,b>0$ and $p,q>0$	
$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ $px^2-qy^2+cx+dy+e=0$	$\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$ $qy^2-px^2+cx+dy+e=0$

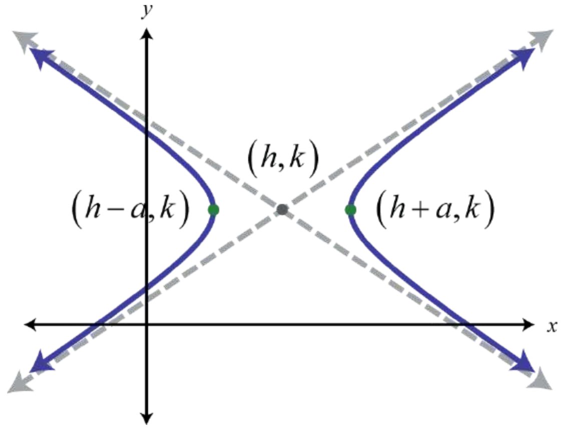


Figure 8.4.21

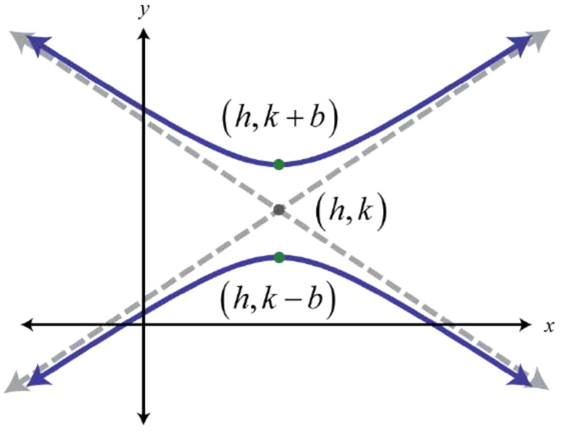


Figure 8.4.22

Example 8.4.5:

Identify the graph of each equation as a parabola, circle, ellipse, or hyperbola.

1. $4x^2+4y^2-1=0$
2. $3x^2-2y^2-12=0$
3. $x-y^2-6y+11=0$

Solution

1. The equation is quadratic in both x and y where the leading coefficients for both variables is the same, 4.

$$4x^2+4y^2-1=0$$
$$4x^2+4y^2=1$$
$$x^2+y^2=\frac{1}{4}$$

This is an equation of a circle centered at the origin with radius $1/2$.

2. The equation is quadratic in both x and y where the leading coefficients for both variables have different signs.

$$3x^2-2y^2-12=0$$
$$\tfrac{3x^2-2y^2}{12}=\tfrac{12}{12}$$
$$\tfrac{x^2}{4}-\tfrac{y^2}{6}=1$$

This is an equation of a hyperbola opening left and right centered at the origin.

3. The equation is quadratic in y only.

$$x-y^2+6y-11=0$$
$$x=y^2-6y+\quad+11$$
$$x=\left(y^2-6y+9\right)+11-9$$
$$x=(y-3)^2+2$$

This is an equation of a parabola opening right with vertex $(2,3)$.

Answer:

1. Circle

2. Hyperbola

3. Parabola

Key Takeaways

- The graph of a hyperbola is completely determined by its center, vertices, and asymptotes.
- The center, vertices, and asymptotes are apparent if the equation of a hyperbola is given in standard form: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ or $\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$.
- To graph a hyperbola, mark points a units left and right from the center and points b units up and down from the center. Use these points to draw the fundamental rectangle; the lines through the corners of this rectangle are the asymptotes. If the coefficient of x^2 is positive, draw the branches of the hyperbola opening left and right through the points determined by a. If the coefficient of y^2 is positive, draw the branches of the hyperbola opening up and down through the points determined by b.
- The orientation of the transverse axis depends the coefficient of x^2 and y^2.
- If the equation of a hyperbola is given in general form $px^2-qy^2+cx+dy+e=0$ or $qy^2-px^2+cx+dy+e=0$ where $p,q>0$, group the terms with the same variables, and complete the square for both groupings to obtain standard form.
- We recognize the equation of a hyperbola if it is quadratic in both x and y where the coefficients of the squared terms are opposite in sign.

Exercise $8.4.3$

Given the equation of a hyperbola in standard form, determine its center, which way the graph opens, and the vertices.

1. $\frac{(x-6)^2}{16}-\frac{(y+4)^2}{9}=1$
2. $\frac{(y-3)^2}{25}-\frac{(x+1)^2}{64}=1$
3. $\frac{(y+9)^2}{5}-x^2=1$
4. $\frac{(x-5)^2}{12}-y^2=1$
5. $4(y+10)^2-25(x+1)^2=100$
6. $9(x-1)^2-5(y+10)^2=45$

Answer

1. Center: $(6,-4)$; $a=4$; $b=3$; opens left and right; vertices: $(2,-4),(10,-4)$

3. Center: $(0,-9)$; $a=1, b=\sqrt{5}$; opens upward and downward; vertices: $(0,-9-\sqrt{5}),(0,-9+\sqrt{5})$

5. Center: $(-1,-10)$; $a=2, b=5$; opens upward and downward; vertices: $(-1,-15),(-1,-5)$

Exercise $8.4.4$

Determine the standard form for the equation of a hyperbola given the following information.

1. Center $(2,7)$, $a=6$, $b=3$, opens left and right.
2. Center $(-9,1)$, $a=7$, $b=2$, opens up and down.
3. Center $(10,-3)$, $a=\sqrt{7}$, $b=5\sqrt{2}$, opens up and down.
4. Center $(-7,-2)$, $a=3\sqrt{3}$, $b=\sqrt{5}$, opens left and right.
5. Center $(0,-8)$, $a=\sqrt{2}b=1$, opens up and down.
6. Center $(0,0)$, $a=2\sqrt{6}$, $b=4$, opens left and right.

Answer

1. $\frac{(x-2)^2}{36}-\frac{(y-7)^2}{9}=1$

3. $\frac{(y+3)^2}{50}-\frac{(x-10)^2}{7}=1$

5. $\frac{(y+8)^2}{1}-\frac{x^2}{2}=1$

Exercise $8.4.5$

Graph.

1. $\frac{(x-3)^2}{9}-\frac{(y+1)^2}{16}=1$
2. $\frac{(x+3)^2}{4}-\frac{(y-1)^2}{25}=1$
3. $\frac{(x-2)^2}{16}-\frac{(y+3)^2}{1}=1$

4. $\frac{(y+2)^2}{9}-\frac{(x+2)^2}{36}=1$
5. $\frac{(y-1)^2}{4}-\frac{(x-2)^2}{16}=1$
6. $(y+2)^2-\frac{(x+3)^2}{9}=1$
7. $4(x+3)^2-9(y-3)^2=36$
8. $16x^2-4(y-1)^2=64$
9. $4(y-1)^2-25x^2=100$
10. $9y^2-16x^2=144$
11. $\frac{(x-2)^2}{12}-\frac{(y-4)^2}{9}=1$
12. $\frac{(x+2)^2}{4}-\frac{(y-1)^2}{8}=1$
13. $\frac{(y+1)^2}{5}-\frac{(x-3)^2}{2}=1$
14. $\frac{(y-4)^2}{3}-\frac{(x+6)^2}{18}=1$
15. $4x^2-3(y-3)^2=12$
16. $7(x+1)^2-2y^2=14$
17. $6y^2-3x^2=18$
18. $10x^2-3y^2=30$

Answer

1.

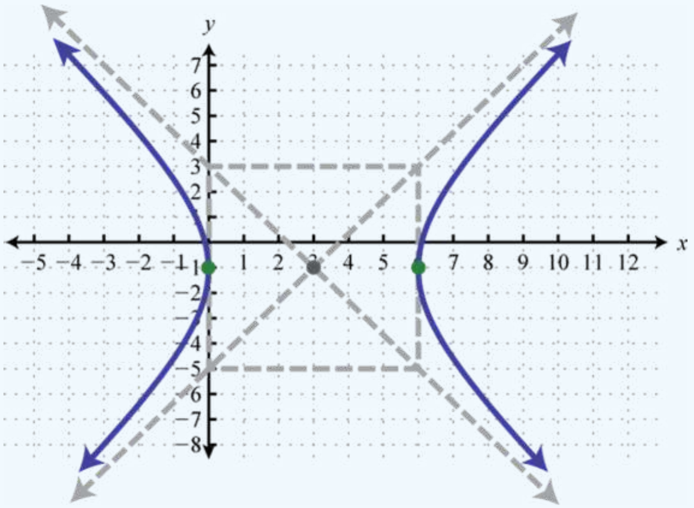


Figure 8.4.23

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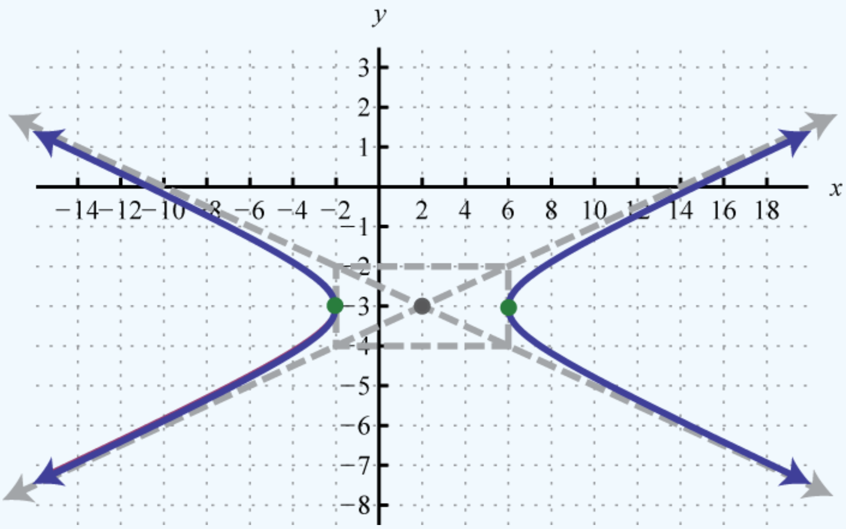

Figure 8.4.24

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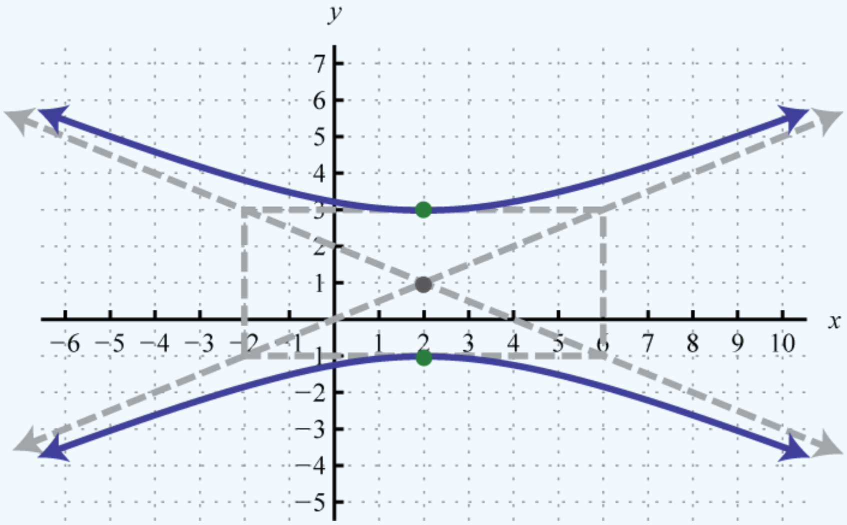

Figure 8.4.25

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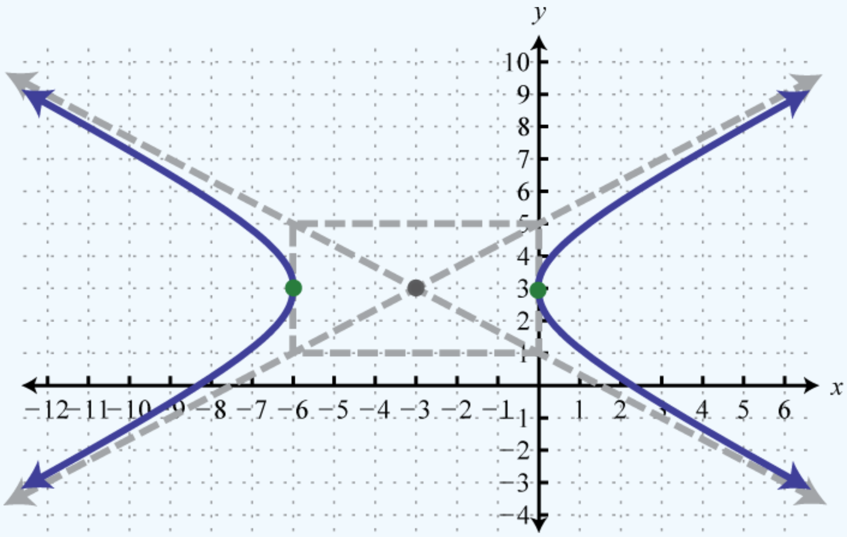

Figure 8.4.26

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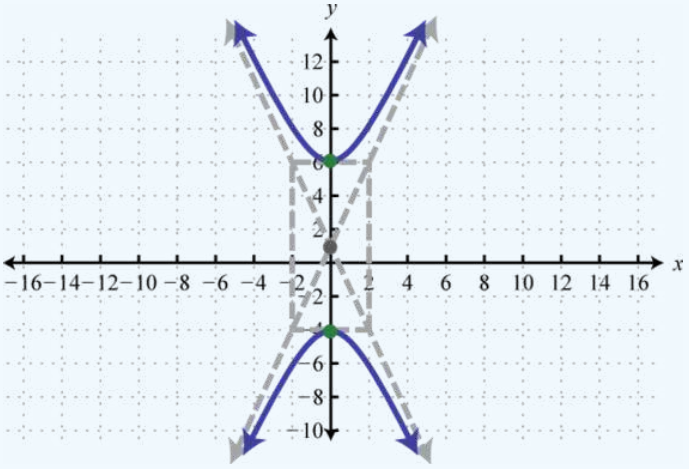

Figure 8.4.27

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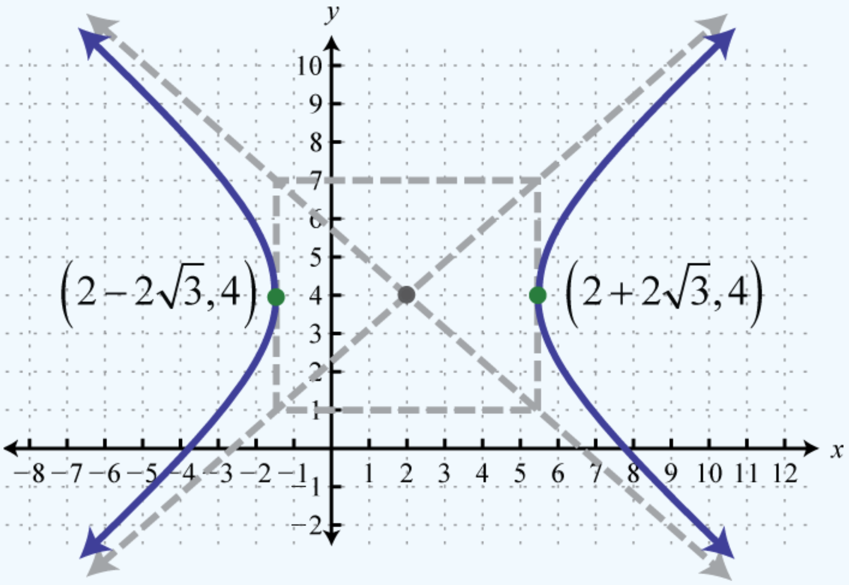

Figure 8.4.28

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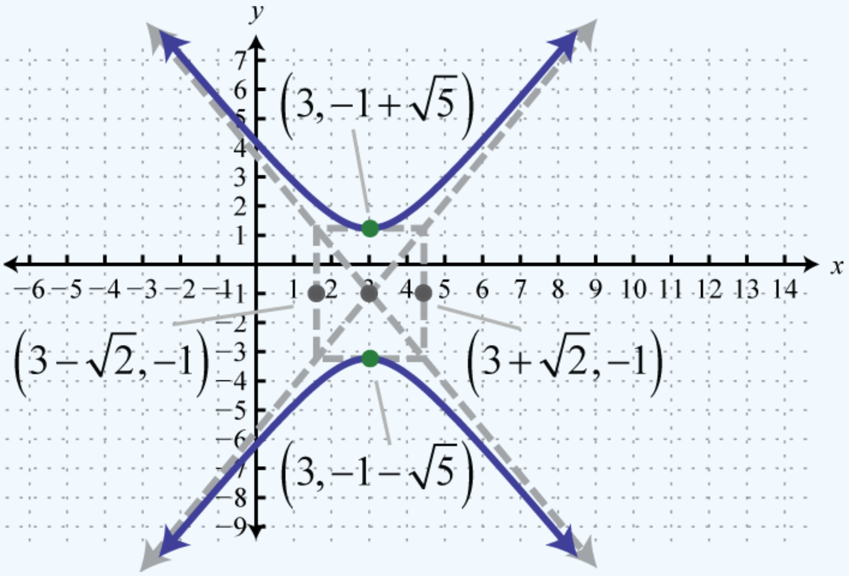

Figure 8.4.29

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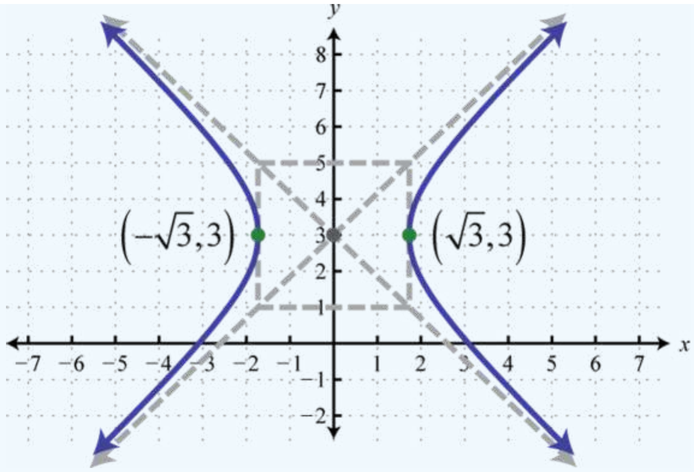

Figure 8.4.30

17.

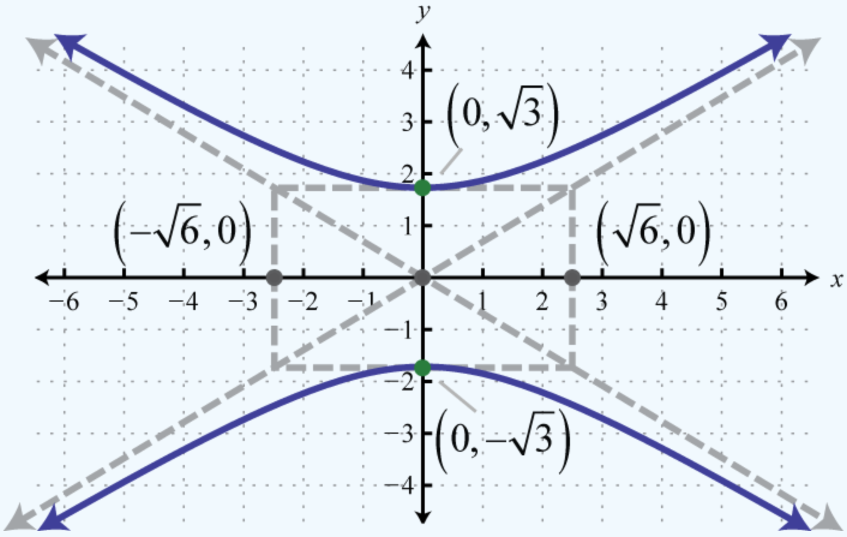

Figure 8.4.31

Exercise 8.4.6

Find the x- and y-intercepts.

1. $\frac{(x-1)^2}{9}-\frac{(y-4)^2}{4}=1$
2. $\frac{(x+4)^2}{16}-\frac{(y-3)^2}{9}=1$
3. $\frac{(y-1)^2}{4}-\frac{(x+1)^2}{36}=1$
4. $\frac{(y+2)^2}{4}-\frac{(x-1)^2}{16}=1$
5. $2x^2-3(y-1)^2=12$
6. $6(x-5)^2-2y^2=12$
7. $36x^2-2y^2=9$
8. $6y^2-4x^2=2$
9. Find the equation of the hyperbola with vertices $(\pm 2,3)$ and a conjugate axis that measures 12 units.
10. Find the equation of the hyperbola with vertices $(4,7)$ and $(4,3)$ and a conjugate axis that measures 6 units.

Answer

1. x-intercepts: $(1\pm 3\sqrt{5},0)$ y-intercepts: none

3. x-intercepts: none y-intercepts: $\left(0,\frac{3\pm\sqrt{37}}{3}\right)$

5. x-intercepts: $\left(\pm\frac{\sqrt{30}}{2}, 0\right)$ y-intercepts: none

7. x-intercepts: $(\pm\frac{1}{2}, 0)$ y-intercepts: none

9. $\frac{x^2}{4} - \frac{(y-3)^2}{36} = 1$

Exercise 8.4.7

Rewrite in standard form and graph.

1. $4x^2 - 9y^2 + 16x + 54y - 101 = 0$
2. $9x^2 - 25y^2 - 18x - 100y - 316 = 0$
3. $4y^2 - 16x^2 - 64x + 8y - 124 = 0$
4. $9y^2 - 4x^2 - 24x - 72y + 72 = 0$
5. $y^2 - 36x^2 - 72x - 12y - 36 = 0$
6. $9y^2 - x^2 + 8x - 36y + 11 = 0$
7. $36x^2 - 4y^2 + 24y - 180 = 0$
8. $x^2 - 25y^2 - 2x - 24 = 0$
9. $25x^2 - 64y^2 + 200x + 640y - 2,800 = 0$
10. $49y^2 - 4x^2 + 40x + 490y + 929 = 0$
11. $3x^2 - 2y^2 + 24x + 8y + 34 = 0$
12. $4x^2 - 8y^2 - 24x + 80y - 196 = 0$
13. $3y^2 - x^2 - 2x - 6y - 16 = 0$
14. $12y^2 - 5x^2 + 40x + 48y - 92 = 0$
15. $4x^2 - 16y^2 + 12x + 16y - 11 = 0$
16. $4x^2 - y^2 - 4x - 2y - 16 = 0$
17. $4y^2 - 36x^2 + 108x - 117 = 0$
18. $4x^2 - 9y^2 + 8x + 6y - 33 = 0$

Answer

1. $\frac{(x+2)^2}{9} - \frac{(y-3)^2}{4} = 1$;

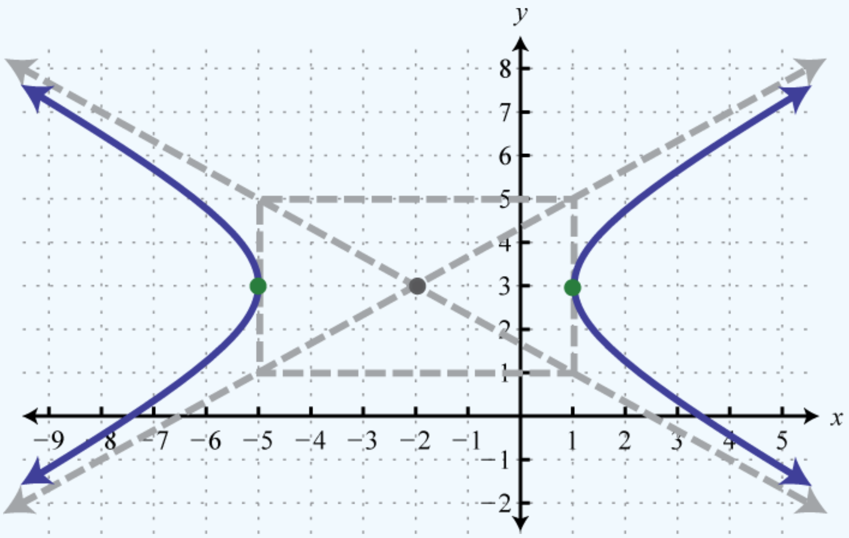

Figure 8.4.32

3. $\frac{(y+1)^2}{16} - \frac{(x+2)^2}{4} = 1$;

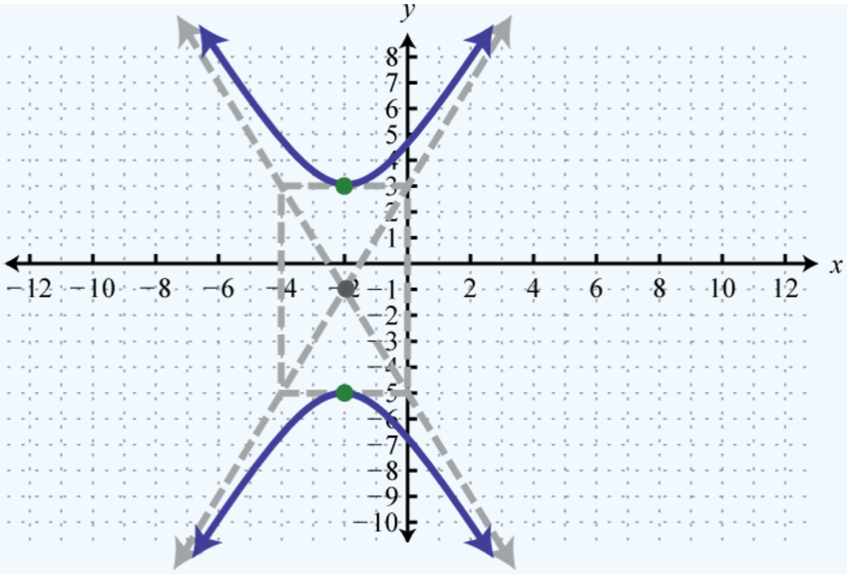

Figure 8.4.33

5. $\frac{(y-6)^2}{36} - (x+1)^2 = 1$;

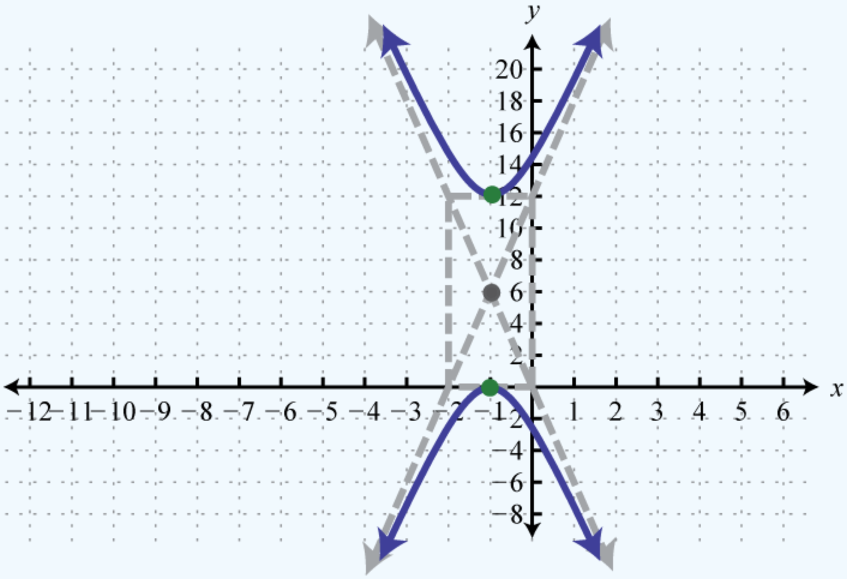

Figure 8.4.34

7. $\frac{x^2}{4} - \frac{(y-3)^2}{36} = 1$;

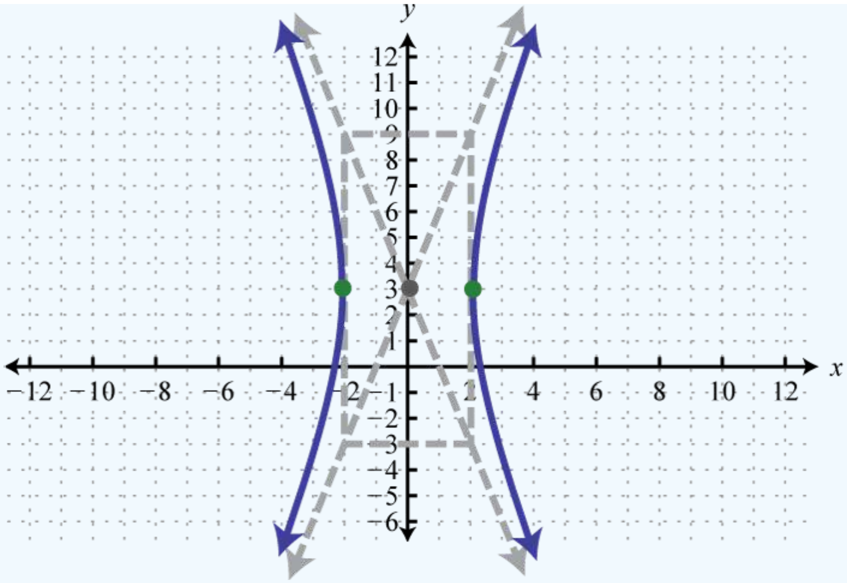

Figure 8.4.35

9. $\frac{(x+4)^2}{64} - \frac{(y-5)^2}{25} = 1$;

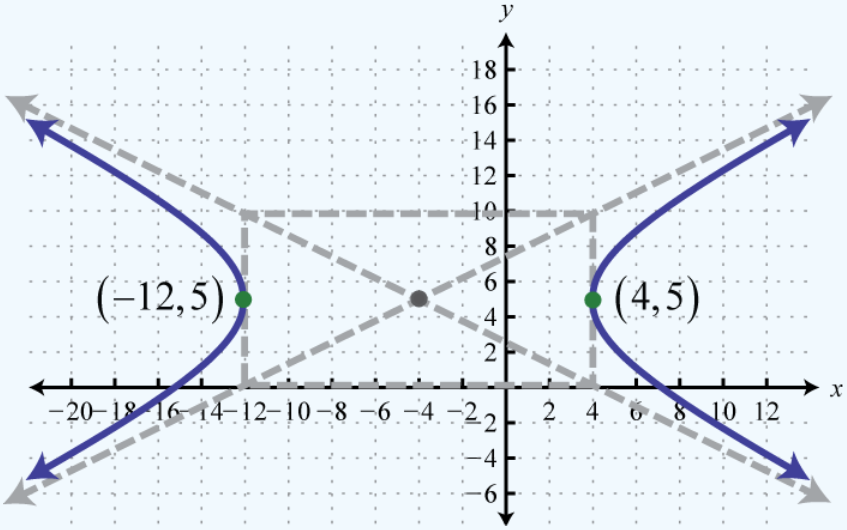

Figure 8.4.36

11. $\frac{(x+4)^2}{2} - \frac{(y-2)^2}{3} = 1$;

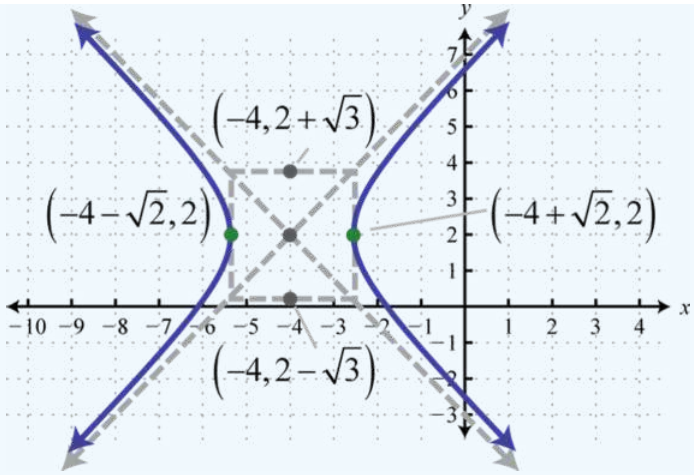

Figure 8.4.37

13. $\frac{(y-1)^2}{6} - \frac{(x+1)^2}{18} = 1$;

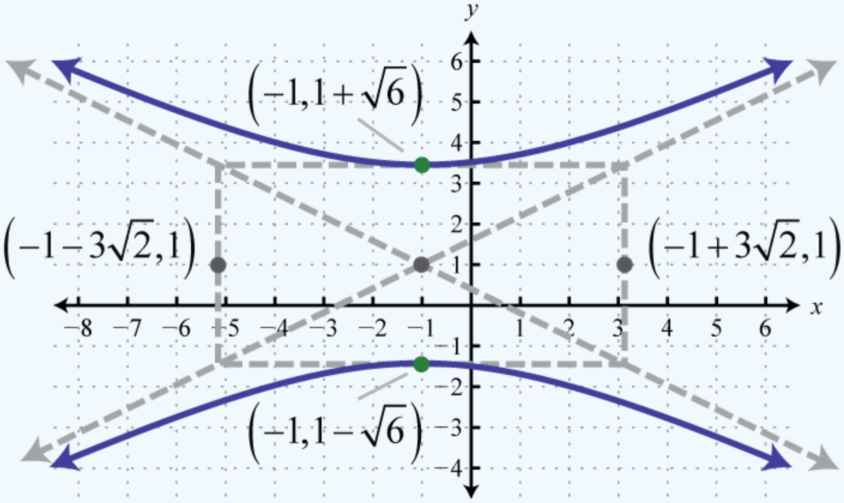

Figure 8.4.38

15. $\frac{(x+\frac{3}{2})^2}{4} - \frac{(y-\frac{1}{2})^2}{1} = 1$

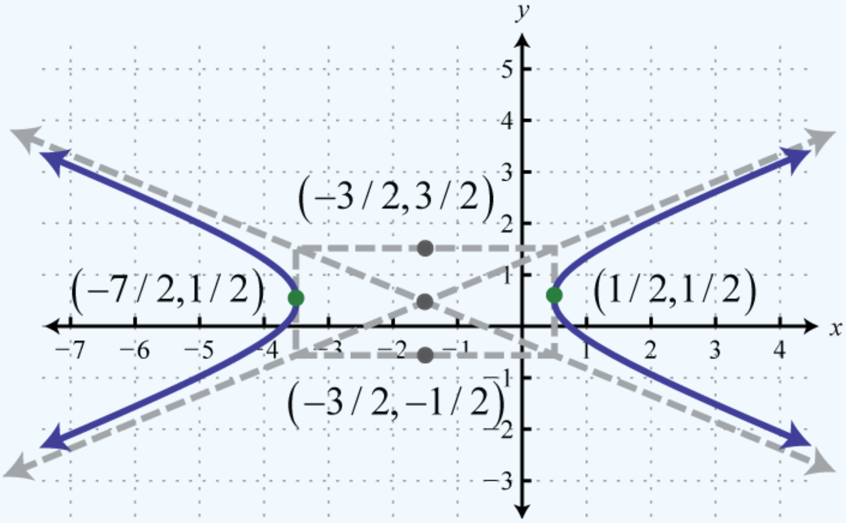

Figure 8.4.39

17. $\frac{y^2}{9} - \left(x-\frac{3}{2}\right)^2 = 1$;

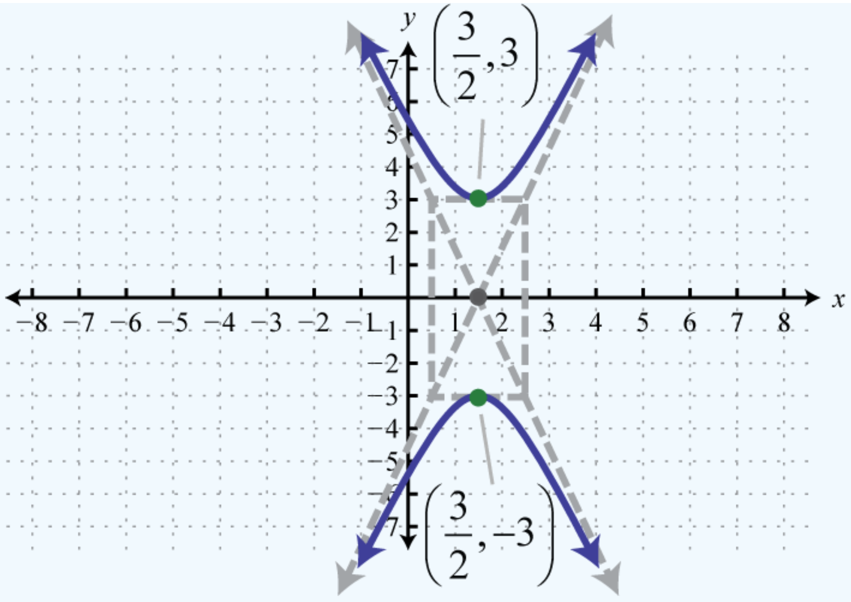

Figure 8.4.40

Exercise 8.4.8

Given the general form, determine the intercepts.

1. $3x^2-y^2-11x-8y-4=0$
2. $4y^2-8x^2+2x+9y-9=0$
3. $x^2-y^2+2x+2y-4=0$
4. $y^2-x^2+6y-8x-16=0$
5. $5x^2-2y^2-4x-3y=0$
6. $2x^2-3y^2-4x-5y+1=0$

Answer

1. x -intercepts: $\left(-\frac{1}{3},0\right),(4,0)$; y -intercepts: $(0,-4\pm 2\sqrt{3})$

3. x -intercepts: $(-1\pm\sqrt{5},0)$; y -intercepts: none

5. x-intercepts: $(0,0),\left(\frac{4}{5},0\right)$; y-intercepts: $(0,0),\left(0,-\frac{3}{2}\right)$

Exercise 8.4.9

Find the equations of the asymptotes to the given hyperbola.

1. $\frac{(y-5)^2}{9}-\frac{(x+8)^2}{16}=1$
2. $\frac{(x+9)^2}{36}-\frac{(y-4)^2}{4}=1$
3. $16x^2-4y^2-24y-96x+44=0$
4. $4y^2-x^2-8y-4x-4=0$

Answer

1. $y=-\frac{3}{4}x-1, y=\frac{3}{4}x+11$

3. $y=-2x+3, y=2x-9$

Exercise 8.4.10

Given the graph of a hyperbola, determine its equation in general form.

1.

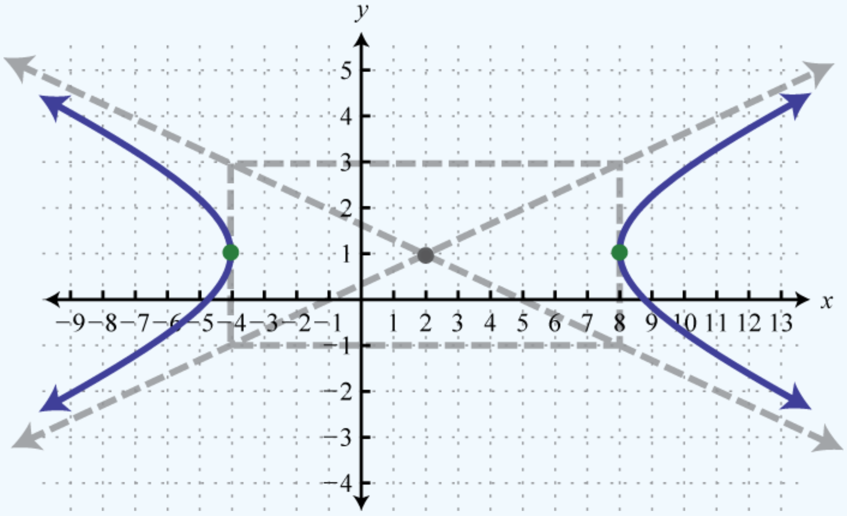

Figure 8.4.41

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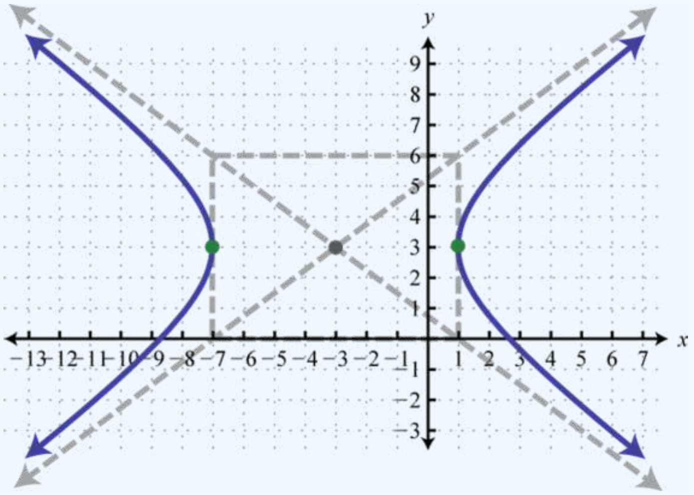

Figure 8.4.42

3.

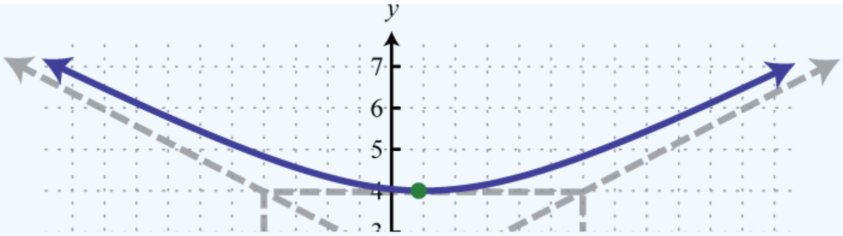

Figure 8.4.43

4.

Figure 8.4.44

Answer

1. $x^2 - 9y^2 - 4x + 18y - 41 = 0$

3. $25y^2 - 4x^2 - 100y + 8x - 4 = 0$

Exercise 8.4.11

Identify the following as the equation of a line, parabola, circle, ellipse, or hyperbola.

1. $x^2 + y^2 + 10x - 2y + 23 = 0$
2. $x^2 + y + 2x - 3 = 0$
3. $2x^2 + y^2 - 12x + 14 = 0$
4. $3x - 2y = 24$
5. $x^2 - y^2 + 36 = 0$
6. $4x^2 + 4y^2 - 32 = 0$
7. $x^2 - y^2 - 2x + 2y - 1 = 0$
8. $x - y^2 + 2y + 1 = 0$
9. $3x + 3y + 5 = 0$
10. $8x^2 + 4y^2 - 144x - 12y + 641 = 0$

Answer

1. Circle

3. Ellipse

5. Hyperbola

7. Hyperbola

9. Line

Exercise 8.4.12

Identify the conic sections and rewrite in standard form.

1. $x^2 - y - 6x + 11 = 0$
2. $x^2 + y^2 - 12x - 6y + 44 = 0$

3. $x^2-2y^2-4x-12y-18=0$
4. $25y^2-2x^2+36x-50y-187=0$
5. $7x^2+4y^2-84x+16y+240=0$
6. $4x^2+4y^2-80x+399=0$
7. $4x^2+4y^2+4x-32y+29=0$
8. $16x^2-4y^2-32x+20y-25=0$
9. $9x-18y^2+12y+7=0$
10. $16x^2+12y^2-24x-48y+9=0$

Answer

1. Parabola; $y=(x-3)^2+2$

3. Hyperbola; $\frac{(x-2)^2}{4}-\frac{(y+3)^2}{2}=1$

5. Ellipse; $\frac{(x-6)^2}{4}+\frac{(y+2)^2}{7}=1$

7. Circle; $\left(x+\frac{1}{2}\right)^2+(y-4)^2=9$

9. Parabola; $x=2\left(y-\frac{1}{3}\right)^2-1$

Exercise 8.4.13

1. Develop a formula for the equations of the asymptotes of a hyperbola. Share it along with an example on the discussion board.
2. Make up your own equation of a hyperbola, write it in general form, and graph it.
3. Do all hyperbolas have intercepts? What are the possible numbers of intercepts for a hyperbola? Explain.
4. Research and discuss real-world examples of hyperbolas.

Answer

1. Answer may vary

3. Answer may vary

Footnotes

[23] The set of points in a plane whose distances from two fixed points, called foci, has an absolute difference that is equal to a positive constant.

[24] The two separate curves of a hyperbola.

[25] Points on the separate branches of a hyperbola where the distance is a minimum.

[26] The equation of a hyperbola written in the form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$.The center is (h,k), a defines the transverse axis, and b defines the conjugate axis.

[27] The equation of a hyperbola written in the form $\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$. The center is (h,k), b defines the transverse axis, and a defines the conjugate axis.

[28] The line segment formed by the vertices of a hyperbola.

[29] A line segment through the center of a hyperbola that is perpendicular to the transverse axis.

[30] The rectangle formed using the endpoints of a hyperbolas, transverse and conjugate axes.

[31] The equation of a hyperbola written in the form $px^2-qy^2+cx+dy+e=0$ or $qy^2-px^2-cx+dy+e=0$ where $p,q>0$.

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8.5: Solving Nonlinear Systems

Learning Objectives

- Identify nonlinear systems.
- Solve nonlinear systems using the substitution method.

Nonlinear Systems

A system of equations where at least one equation is not linear is called a **nonlinear system**[32]. In this section we will use the substitution method to solve nonlinear systems. Recall that solutions to a system with two variables are ordered pairs (x, y) that satisfy both equations.

Example $8.5.1$:

Solve: $\left\{\begin{array}{l} x+2y=0 \\ x^2+y^2=5 \end{array}\right.$.

Solution

In this case we begin by solving for x in the first equation.

$$\left\{\begin{array}{l} x+2y=0 \\ x^2+y^2=5 \end{array}\right. \Longrightarrow x=-2y$$

Substitute $x=-2y$ into the second equation and then solve for y.

$$\begin{aligned} (-2y)^2+y^2 &= 5 \\ 4y^2+y^2 &= 5 \\ 5y^2 &= 5 \\ y^2 &= 1 \\ y &= \pm 1 \end{aligned}$$

Here there are two answers for y; use $x=-2y$ to find the corresponding x-values.

Table $8.5.1$

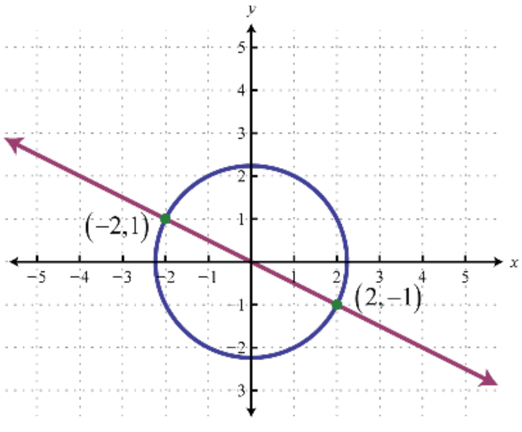
Using $y=-1$	Using $y=1$
$\begin{aligned} x &= -2y \\ &= -2(-1) \\ &= 2 \end{aligned}$	$\begin{aligned} x &= -2y \\ &= -2(1) \\ &= -2 \end{aligned}$

This gives us two ordered pair solutions, $(2,-1)$ and $(-2,1)$.

Answer:

$(2,-1), (-2,1)$

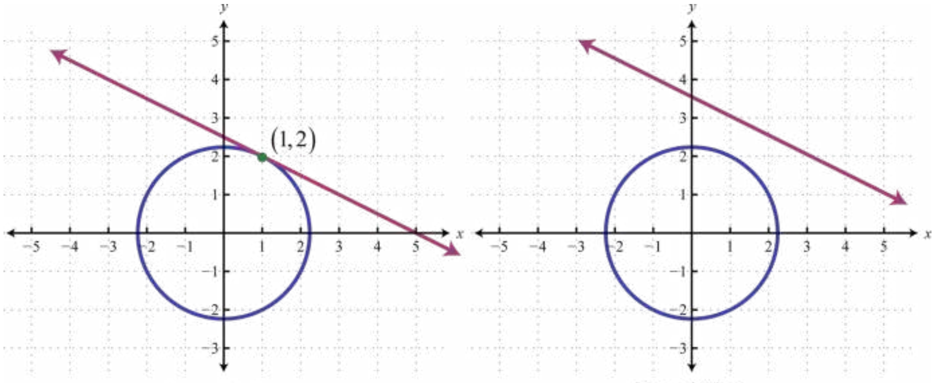
In the previous example, the given system consisted of a line and a circle. Graphing these equations on the same set of axes, we can see that the two ordered pair solutions correspond to the two points of intersection.



Two solutions.

Figure 8.5.1

If we are given a system consisting of a circle and a line, then there are 3 possibilities for real solutions—two solutions as pictured above, one solution, or no solution.



One solution. No solution.

Figure 8.5.2

Example 8.5.2

Solve: $\left\{\begin{array}{c}x+y=3\\x^{2}+y^{2}=2\end{array}\right.$.

Solution

Solve for y in the first equation.

$$\left\{\begin{array}{c}x+y\\x^{2}+y^{2}\end{array}\right.$$

Next, substitute $y=3-x$ into the second equation and then solve for x.

$$\begin{aligned}x^{2}+(3-x)^{2}&=2\\x^{2}+9-6x+x^{2}&=2\\2x^{2}-6x+9&=2\\2x^{2}-6x+7&=0\end{aligned}$$

The resulting equation does not factor. Furthermore, using $a=2, b=-6$, and $c=7$ we can see that the discriminant is negative:

$$\begin{aligned}b^{2}-4ac&=(-6)^{2}-4(2)(7)\\&=36-56\\&=-20\end{aligned}$$

We conclude that there are no real solutions to this equation and thus no solution to the system.

Answer:

$\emptyset$

Exercise 8.5.1

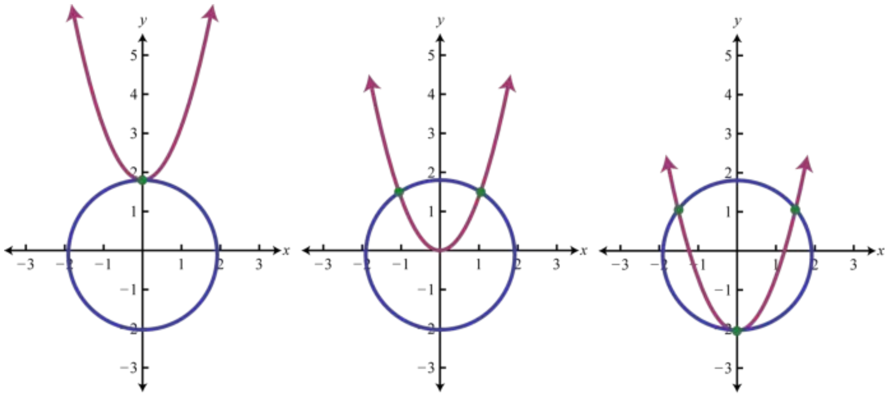
Solve: $\begin{cases} x-y=5 \\ x^2+(y+1)^2=8 \end{cases}$

Answer

$(2,-3)$

www.youtube.com/v/ToIrjw-8SNA

If given a circle and a parabola, then there are 5 possibilities for solutions.

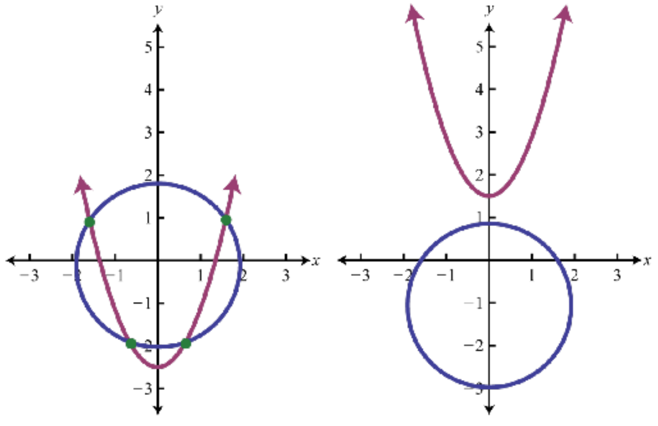


one solution

two solutions

three solutions

Figure 8.5.3



four solutions

no solution

Figure 8.5.4

When using the substitution method, we can perform the substitution step using entire algebraic expressions. The goal is to produce a single equation in one variable that can be solved using the techniques learned up to this point in our study of algebra.

Example 8.5.3:

Solve: $\begin{cases} x^2+y^2=2 \\ y-x^2=-2 \end{cases}$.

Solution

We can solve for x^2 in the second equation.

$$\begin{cases} x^2+y^2=2 \\ y-x^2=-2 \quad \Rightarrow \quad y+2=x^2 \end{cases}$$

Substitute $x^2=y+2$ into the first equation and then solve for y.

$$\begin{aligned} y+2+y^2&=2 \\ y^2+y&=0 \\ y(y+1)&=0 \\ y=0 \quad &\text{or} \quad y=-1 \end{aligned}$$

Back substitute into $x^2=y+2$ to find the corresponding x-values.

Table $8.5.2$

Using $y=-1$	Using $y=0$
$x^2=y+2$ $x^2=-1+2$ $x^2=1$ $x=\pm 1$	$x^2=y+2$ $x^2=0+2$ $x^2=2$ $x=\pm\sqrt{2}$

This leads us to four solutions, $(\pm 1,-1)$ and $(\pm\sqrt{2},0)$.

Answer:

$(\pm 1,-1),(\pm\sqrt{2},0)$

Example $8.5.4$

Solve: $\left\{\begin{aligned} (x-1)^2-2y^2&=4 \\ x^2+y^2&=9 \end{aligned}\right.$

Solution

We can solve for y^2 in the second equation,

$$\left\{\begin{aligned} (x-1)^2-2y^2&=4 \\ x^2+y^2&=9 \end{aligned}\right. \Longrightarrow y^2=9-x^2$$

Substitute $y^2=9-x^2$ into the first equation and then solve for x.

$$\begin{aligned} (x-1)^2-2\left(9-x^2\right)&=4 \\ x^2-2x+1-18+2x^2&=0 \\ 3x^2-2x-21&=0 \\ (3x+7)(x-3)&=0 \\ 3x+7=0 &\text{ or } x-3=0 \\ x=-\frac{7}{3} &\quad x=3 \end{aligned}$$

Back substitute into $y^2=9-x^2$ to find the corresponding y-values.

Table $8.5.3$

Using $x=-\frac{7}{3}$	Using $x=3$
$y^2=9-\left(-\frac{7}{3}\right)^2$ $y^2=\frac{9}{1}-\frac{49}{9}$ $y^2=\frac{32}{9}$ $y=\pm\frac{\sqrt{32}}{3}=\pm\frac{4\sqrt{2}}{3}$	$y^2=9-(3)^2$ $y^2=0$ $y=0$

This leads to three solutions, $\left(-\frac{7}{3},\pm\frac{4\sqrt{2}}{3}\right)$ and $(3,0)$.

Answer:

$(3,0),\left(-\frac{7}{3},\pm\frac{4\sqrt{2}}{3}\right)$

Example 8.5.5

Solve: $\begin{cases} x^2+y^2=2 \\ xy=1 \end{cases}$.

Solution

Solve for y in the second equation.

$$\begin{cases} x^2+y^2=2 \\ xy=1 \end{cases} \Longrightarrow y=\frac{1}{x}$$

Substitute $y=\frac{1}{x}$ into the first equation and then solve for x.

$$\begin{aligned} x^2+\left(\tfrac{1}{x}\right)^2&=2 \\ x^2+\tfrac{1}{x^2}&=2 \end{aligned}$$

This leaves us with a rational equation. Make a note that $x \neq 0$ and multiply both sides by x^2.

$$\begin{aligned} x^2\left(x^2+\frac{1}{x^2}\right)&=2\cdot x^2 \\ x^4+1&=2x^2 \\ x^4-2x^2+1&=0 \\ \left(x^2-1\right)\left(x^2-1\right)&=0 \end{aligned}$$

At this point we can see that both factors are the same. Apply the zero product property.

$$\begin{aligned} x^2-1&=0 \\ x^2&=1 \\ x&=\pm 1 \end{aligned}$$

Back substitute into $y=\frac{1}{x}$ to find the corresponding y-values.

Using $x=-1$	Using $x=1$
$\begin{aligned} y&=\frac{1}{x} \\ &=\frac{1}{-1} \\ &=-1 \end{aligned}$	$\begin{aligned} y&=\frac{1}{x} \\ &=\frac{1}{1} \\ &=1 \end{aligned}$

This leads to two solutions.

Answer:

$(1,1),(-1,-1)$

Exercise 8.5.2

Solve: $\begin{cases} \frac{1}{x}+\frac{1}{y}=4 \\ \frac{1}{x^2}+\frac{1}{y^2}=40 \end{cases}$

Answer

$\left(-\frac{1}{2}, \frac{1}{6}\right)\left(\frac{1}{6},-\frac{1}{2}\right)$

www.youtube.com/v/n8JJ_ybegkM

Key Takeaways

- Use the substitution method to solve nonlinear systems.
- Streamline the solving process by using entire algebraic expressions in the substitution step to obtain a single equation with one variable.
- Understanding the geometric interpretation of the system can help in finding real solutions.

Exercise 8.5.3

Solve.

1. $\begin{cases} x^2+y^2=10 \\ x+y=4 \end{cases}$
2. $\begin{cases} x^2+y^2=5 \\ x-y=-3 \end{cases}$
3. $\begin{cases} x^2+y^2=30 \\ x-3y=0 \end{cases}$
4. $\begin{cases} x^2+y^2=10 \\ 2x-y=0 \end{cases}$
5. $\begin{cases} x^2+y^2=18 \\ 2x-2y=-12 \end{cases}$
6. $\begin{cases} (x-4)^2+y^2=25 \\ 4x-3y=16 \end{cases}$
7. $\begin{cases} 3x^2+2y^2=21 \\ 3x-y=0 \end{cases}$
8. $\begin{cases} x^2+5y^2=36 \\ x-2y=0 \end{cases}$
9. $\begin{cases} 4x^2+9y^2=36 \\ 2x+3y=6 \end{cases}$
10. $\begin{cases} 4x^2+y^2=4 \\ 2x+y=-2 \end{cases}$
11. $\begin{cases} 2x^2+y^2=1 \\ x+y=1 \end{cases}$
12. $\begin{cases} 4x^2+3y^2=12 \\ 2x-y=2 \end{cases}$
13. $\begin{cases} x^2-2y^2=35 \\ x-3y=0 \end{cases}$
14. $\begin{cases} 5x^2-7y^2=39 \\ 2x+4y=0 \end{cases}$
15. $\begin{cases} 9x^2-4y^2=36 \\ 3x+2y=0 \end{cases}$
16. $\begin{cases} x^2+y^2=25 \\ x-2y=-12 \end{cases}$
17. $\begin{cases} 2x^2+3y=9 \\ 8x-4y=12 \end{cases}$
18. $\begin{cases} 2x-4y^2=3 \\ 3x-12y=6 \end{cases}$
19. $\begin{cases} 4x^2+3y^2=12 \\ x-\frac{3}{2}=0 \end{cases}$
20. $\begin{cases} 5x^2+4y^2=40 \\ y-3=0 \end{cases}$
21. The sum of the squares of two positive integers is 10. If the first integer is added to twice the second integer, the sum is 7. Find the integers.
22. The diagonal of a rectangle measures $\sqrt{5}$ units and has a perimeter equal to 6 units. Find the dimensions of the rectangle.
23. For what values of b will the following system have real solutions? $\begin{cases} x^2+y^2=1 \\ y=x+b \end{cases}$
24. For what values of m will be the following system have real solutions? $\begin{cases} x^2-y^2=1 \\ y=mx \end{cases}$

Answer

1. $(1,3),(3,1)$

3. $(-3\sqrt{3},-\sqrt{3}),(3\sqrt{3},\sqrt{3})$

5. $(-3,3)$

7. $(-1,-3),(1,3)$

9. $(0,2),(3,0)$

11. $(0,1),\left(\frac{2}{3},\frac{1}{3}\right)$

13. $(-3\sqrt{5},-\sqrt{5}),(3\sqrt{5},\sqrt{5})$

15. $\emptyset$

17. $\left(\frac{-3+3\sqrt{5}}{2},-6+3\sqrt{5}\right),\left(\frac{-3-3\sqrt{5}}{2},-6-3\sqrt{5}\right)$

19. $\left(\frac{3}{2},-1\right),\left(\frac{3}{2},1\right)$

21. $1,3$

23. $b\in[-\sqrt{2},\sqrt{2}]$

Exercise 8.5.4

Solve.

1. $\begin{cases} x^2+y^2=4 \\ y-x^2=2 \end{cases}$
2. $\begin{cases} x^2+y^2=4 \\ y-x^2=-2 \end{cases}$
3. $\begin{cases} x^2+y^2=4 \\ y-x^2=3 \end{cases}$
4. $\begin{cases} x^2+y^2=4 \\ 4y-x^2=-4 \end{cases}$
5. $\begin{cases} x^2+3y^2=9 \\ y^2-x=3 \end{cases}$
6. $\begin{cases} x^2+3y^2=9 \\ x+y^2=-4 \end{cases}$
7. $\begin{cases} 4x^2-3y^2=12 \\ x^2+y^2=1 \end{cases}$
8. $\begin{cases} x^2+y^2=1 \\ x^2-y^2=1 \end{cases}$
9. $\begin{cases} x^2+y^2=1 \\ 4y^2-x^2-4y=0 \end{cases}$
10. $\begin{cases} x^2+y^2=4 \\ 2x^2-y^2+4x=0 \end{cases}$
11. $\begin{cases} 2(x-2)^2+y^2=6 \\ (x-3)^2+y^2=4 \end{cases}$
12. $\begin{cases} x^2+y^2-6y=0 \\ 4x^2+5y^2+20y=0 \end{cases}$
13. $\begin{cases} x^2+4y^2=25 \\ 4x^2+y^2=40 \end{cases}$
14. $\begin{cases} x^2-2y^2=-10 \\ 4x^2+y^2=10 \end{cases}$
15. $\begin{cases} 2x^2+y^2=14 \\ x^2-(y-1)^2=6 \end{cases}$
16. $\begin{cases} 3x^2-(y-2)^2=12 \\ x^2+(y-2)^2=1 \end{cases}$
17. The difference of the squares of two positive integers is 12. The sum of the larger integer and the square of the smaller is equal to 8. Find the integers.

18. The difference between the length and width of a rectangle is 4 units and the diagonal measures 8 units. Find the dimensions of the rectangle. Round off to the nearest tenth.
19. The diagonal of a rectangle measures p units and has a perimeter equal to $2q$ units. Find the dimensions of the rectangle in terms of p and q.
20. The area of a rectangle is p square units and its perimeter is $2q$ units. Find the dimensions of the rectangle in terms of p and q.

Answer

1. $(0,2)$
3. $\emptyset$
5. $(-3,0),(0,-\sqrt{3}),(0,\sqrt{3})$
7. $\emptyset$
9. $(0,1),\left(-\frac{2\sqrt{5}}{5},-\frac{1}{5}\right),\left(\frac{2\sqrt{5}}{5},-\frac{1}{5}\right)$
11. $(3,-2),(3,2)$
13. $(-3,-2),(-3,2),(3,-2),(3,2)$
15. $(-\sqrt{7},0),(\sqrt{7},0),\left(-\frac{\sqrt{55}}{3},\frac{4}{3}\right),\left(\frac{\sqrt{55}}{3},\frac{4}{3}\right)$
17. $2,4$
19. $\frac{q+\sqrt{2p^2-q^2}}{2}$ units by $\frac{q-\sqrt{2p^2-q^2}}{2}$ units

Exercise $8.5.5$

Solve.

1. $\begin{cases} x^2+y^2=26 \\ xy=5 \end{cases}$
2. $\begin{cases} x^2+y^2=10 \\ xy=3 \end{cases}$
3. $\begin{cases} 2x^2-3y^2=5 \\ xy=1 \end{cases}$
4. $\begin{cases} 3x^2-4y^2=-11 \\ xy=1 \end{cases}$
5. $\begin{cases} x^2+y^2=2 \\ xy-2=0 \end{cases}$
6. $\begin{cases} x^2+y^2=1 \\ 2xy-1=0 \end{cases}$
7. $\begin{cases} 4x-y^2=0 \\ xy=2 \end{cases}$
8. $\begin{cases} 3y-x^2=0 \\ xy-9=0 \end{cases}$
9. $\begin{cases} 2y-x^2=0 \\ xy-1=0 \end{cases}$
10. $\begin{cases} x-y^2=0 \\ xy=3 \end{cases}$
11. The diagonal of a rectangle measures $2\sqrt{10}$ units. If the area of the rectangle is 12 square units, find its dimensions.
12. The area of a rectangle is 48 square meters and the perimeter measures 32 meters. Find the dimensions of the rectangle.
13. The product of two positive integers is 72 and their sum is 18. Find the integers.
14. The sum of the squares of two positive integers is 52 and their product is 24. Find the integers.

Answer

1. $(-5,-1),(5,1),(-1,-5),(1,5)$
3. $\left(-\sqrt{3},-\frac{\sqrt{3}}{3}\right),\left(\sqrt{3},\frac{\sqrt{3}}{3}\right)$

5. $\emptyset$

7. $(1,2)$

9. $\left(\sqrt[3]{2}, \frac{\sqrt[3]{4}}{2}\right)$

11. 2 units by 6 units

13. $6, 12$

Exercise 8.5.6

Solve.

1. $\begin{cases} \frac{1}{x}+\frac{1}{y}=4 \\ \frac{1}{x}-\frac{1}{y}=2 \end{cases}$
2. $\begin{cases} \frac{2}{x}-\frac{1}{y}=5 \\ \frac{1}{x}+\frac{1}{y}=2 \end{cases}$
3. $\begin{cases} \frac{1}{x}+\frac{2}{y}=1 \\ \frac{3}{x}-\frac{1}{y}=2 \end{cases}$
4. $\begin{cases} \frac{1}{x}+\frac{1}{y}=6 \\ \frac{1}{x^2}+\frac{1}{y^2}=20 \end{cases}$
5. $\begin{cases} \frac{1}{x}+\frac{1}{y}=2 \\ \frac{1}{x^2}+\frac{1}{y^2}=34 \end{cases}$
6. $\begin{cases} xy-16=0 \\ 2x^2-y=0 \end{cases}$
7. $\begin{cases} x+y^2=4 \\ y=\sqrt{x} \end{cases}$
8. $\begin{cases} y^2-(x-1)^2=1 \\ \qquad y=\sqrt{x} \end{cases}$
9. $\begin{cases} y=2^x \\ y=2^{2x}-56 \end{cases}$
10. $\begin{cases} y=3^{2x}-72 \\ y-3^x=0 \end{cases}$
11. $\begin{cases} y=e^{4x} \\ y=e^{2x}+6 \end{cases}$
12. $\begin{cases} y-e^{2x}=0 \\ y-e^x=0 \end{cases}$

Answer

1. $\left(\frac{1}{3}, 1\right)$

3. $\left(\frac{7}{5}, 7\right)$

5. $\left(-\frac{1}{3}, \frac{1}{5}\right), \left(\frac{1}{5}, -\frac{1}{3}\right)$

7. $(2, \sqrt{2})$

9. $(3,8)$

11. $\left(\frac{\ln 3}{2}, 9\right)$

Exercise 8.5.7

1. How many real solutions can be obtained from a system that consists of a circle and a hyperbola? Explain.
2. Make up your own nonlinear system, solve it, and provide the answer. Also, provide a graph and discuss the geometric interpretation of the solutions.

Answer

1. Answer may vary

Footnotes

[32] A system of equations where at least one equation is not linear.

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8.E: Conic Sections (Exercises)

Exercise $8.E.1$

Calculate the distance and midpoint between the given two points.

1. $(0,2)$ and $(-4,-1)$
2. $(6,0)$ and $(-2,-6)$
3. $(-2,4)$ and $(-6,-8)$
4. $(\frac{1}{2},-1)$ and $(\frac{5}{2},-\frac{1}{2})$
5. $(0,-3\sqrt{2})$ and $(\sqrt{5},-4\sqrt{2})$
6. $(-5\sqrt{3},\sqrt{6})$ and $(-3\sqrt{3},\sqrt{6})$

Answer

1. Distance: 5 units; midpoint: $(-2,\frac{1}{2})$

3. Distance: $4\sqrt{10}$ units; midpoint: $(-4,-2)$

5. Distance: $\sqrt{7}$ units; midpoint: $\left(\frac{\sqrt{5}}{2},-\frac{7\sqrt{2}}{2}\right)$

Exercise $8.E.2$

Determine the area of a circle whose diameter is defined by the given two points.

1. $(-3,3)$ and $(3,-3)$
2. $(-2,-9)$ and $(-10,-15)$
3. $(\frac{2}{3},-\frac{1}{2})$ and $(-\frac{1}{3},\frac{3}{2})$
4. $(2\sqrt{5},-2\sqrt{2})$ and $(0,-4\sqrt{2})$

Answer

1. 18π square units

3. $\frac{5\pi}{4}$ square units

Exercise $8.E.3$

Rewrite in standard form and give the vertex.

1. $y=x^2-10x+33$
2. $y=2x^2-4x-1$
3. $y=x^2-3x-1$
4. $y=-x^2-x-2$
5. $x=y^2+10y+10$
6. $x=3y^2+12y+7$
7. $x=-y^2+8y-3$
8. $x=5y^2-5y+2$

Answer

1. $y=(x-5)^2+8$; vertex: $(5,8)$

3. $y=\left(x-\frac{3}{2}\right)^2-\frac{13}{4}$; vertex: $(\frac{3}{2},-\frac{13}{4})$

5. $x=(y+5)^2-15$; vertex: $(-15,-5)$

7. $x=-(y-4)^2+13$; vertex: $(13,4)$

Exercise $8.E.4$

Rewrite in standard form and graph. Be sure to find the vertex and all intercepts.

1. $y=x^2-20x+75$
2. $y=-x^2-10x+75$
3. $y=-2x^2-12x-24$
4. $y=4x^2+4x+6$
5. $x=y^2-10y+16$
6. $x=-y^2+4y+12$
7. $x=-4y^2+12y$
8. $x=9y^2+18y+12$
9. $x=-4y^2+4y+2$
10. $x=-y^2-5y+2$

Answer

1. $y=(x-10)^2-25$;

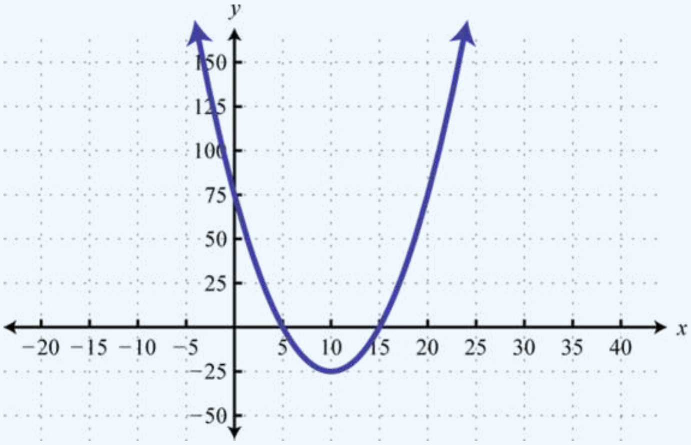

Figure 8.E.1

3. $y=-2(x+3)^2-6$;

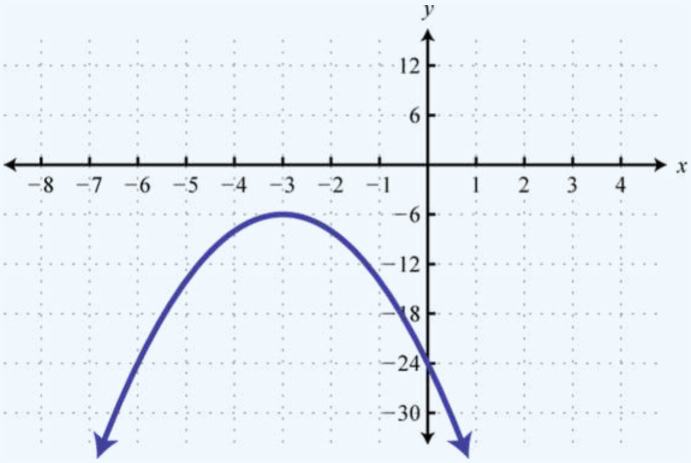

Figure 8.E.2

5. $x=(y-5)^2-9$;

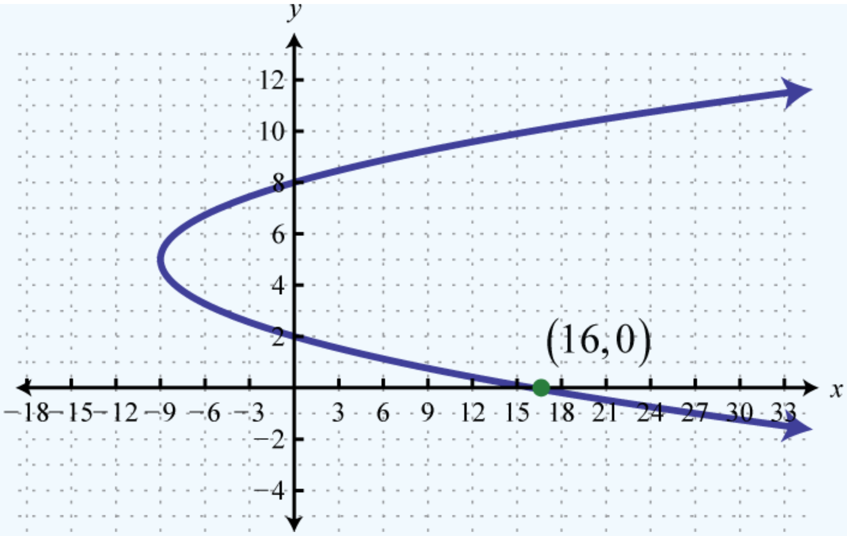

Figure 8.E.3

7. $x=-4\left(y-\frac{3}{2}\right)^{2}+9$;

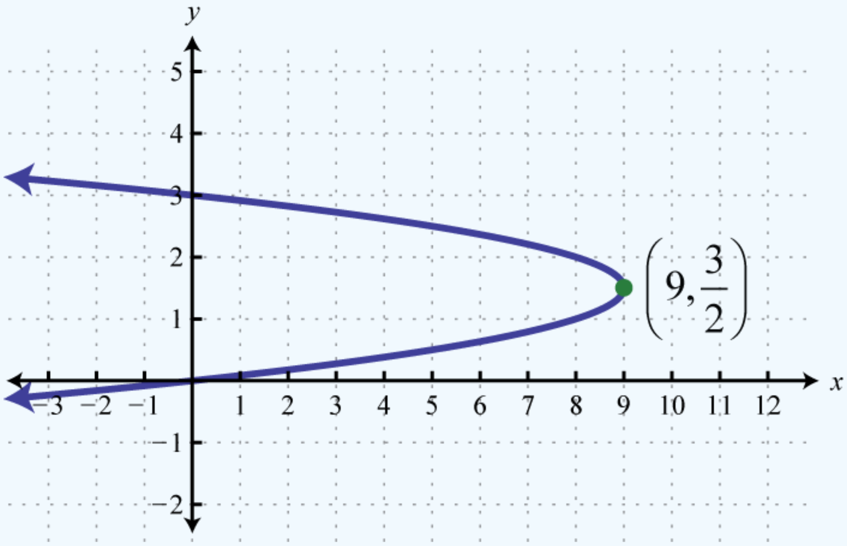

Figure 8.E.4

9. $x=-4\left(y-\frac{1}{2}\right)^{2}+3$;

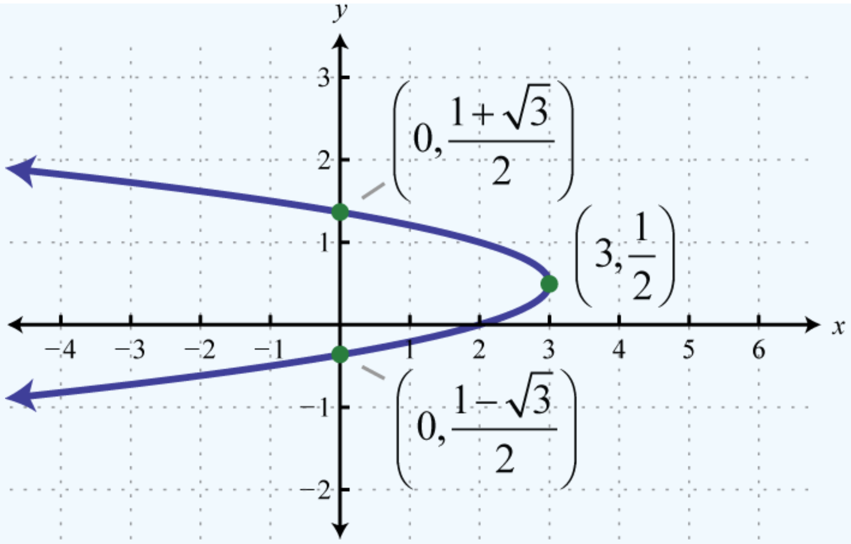

Figure 8.E.5

Exercise $8.E.5$

Determine the center and radius given the equation of a circle in standard form.

1. $(x-6)^2+y^2=9$
2. $(x+8)^2+(y-10)^2=1$
3. $x^2+y^2=5$
4. $\left(x-\frac{3}{8}\right)^2+\left(y+\frac{5}{2}\right)^2=\frac{1}{2}$

Answer

1. Center: $(6,0)$; radius: $r=3$

3. Center: $(0,0)$; radius: $r=\sqrt{5}$

Exercise $8.E.6$

Determine standard form for the equation of the circle:

1. Center $(-7,2)$ with radius $r=10$
2. Center $\left(\frac{1}{3},-1\right)$ with radius $r=\frac{2}{3}$
3. Center $(0,-5)$ with radius $r=2\sqrt{7}$
4. Center $(1,0)$ with radius $r=\frac{5\sqrt{3}}{2}$
5. Circle whose diameter is defined by $(-4,10)$ and $(-2,8)$
6. Circle whose diameter is defined by $(3,-6)$ and $(0,-4)$

Answer

1. $(x+7)^2+(y-2)^2=100$

3. $x^2+(y+5)^2=28$

5. $(x+3)^2+(y-9)^2=2$

Exercise $8.E.7$

Find the x- and y-intercepts.

1. $(x-3)^2+(y+5)^2=16$
2. $(x+5)^2+(y-1)^2=4$

3. $x^2+(y-2)^2=20$
4. $(x-3)^2+(y+3)^2=8$
5. $x^2+y^2-12y+27=0$
6. $x^2+y^2-4x+2y+1=0$

Answer

1. x-intercepts: none; y-intercepts: $(0,-5\pm\sqrt{7})$

3. x-intercepts: $(\pm 4,0)$; y-intercepts: $(0,2\pm 2\sqrt{5})$

5. x-intercepts: none; y-intercepts: $(0,3),(0,9)$

Exercise $8.E.8$

Graph.

1. $(x+8)^2+(y-6)^2=4$
2. $(x-20)^2+\left(y+\frac{15}{2}\right)^2=\frac{225}{4}$
3. $x^2+y^2=24$
4. $(x-1)^2+y^2=\frac{1}{4}$
5. $x^2+(y-7)^2=27$
6. $(x+1)^2+(y-1)^2=2$

Answer

1.

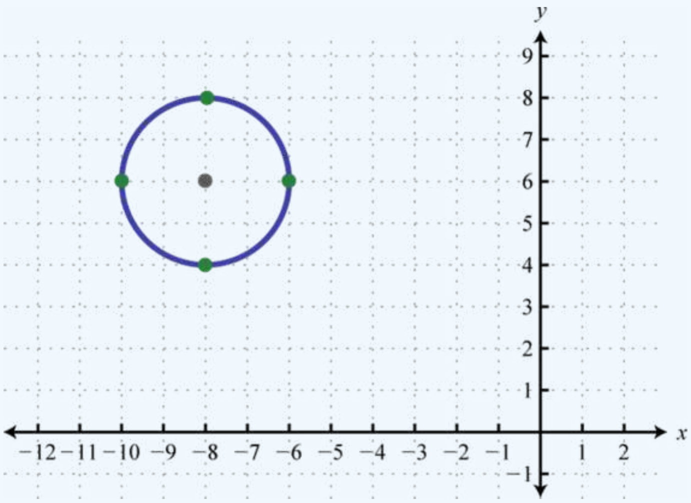

Figure 8.E.6

3.

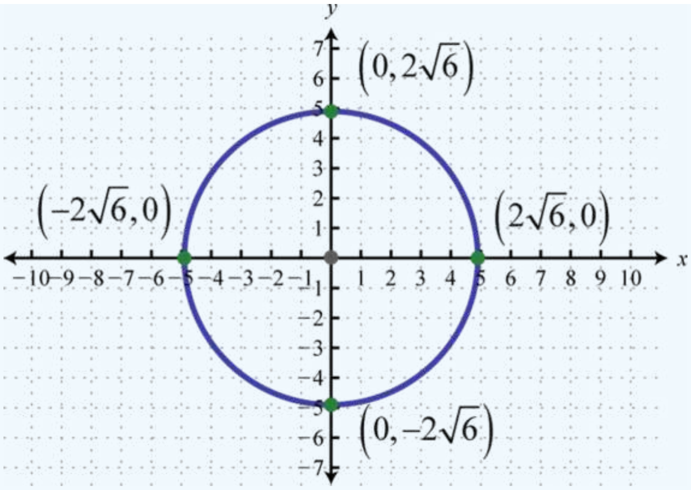

Figure 8.E.7

5.

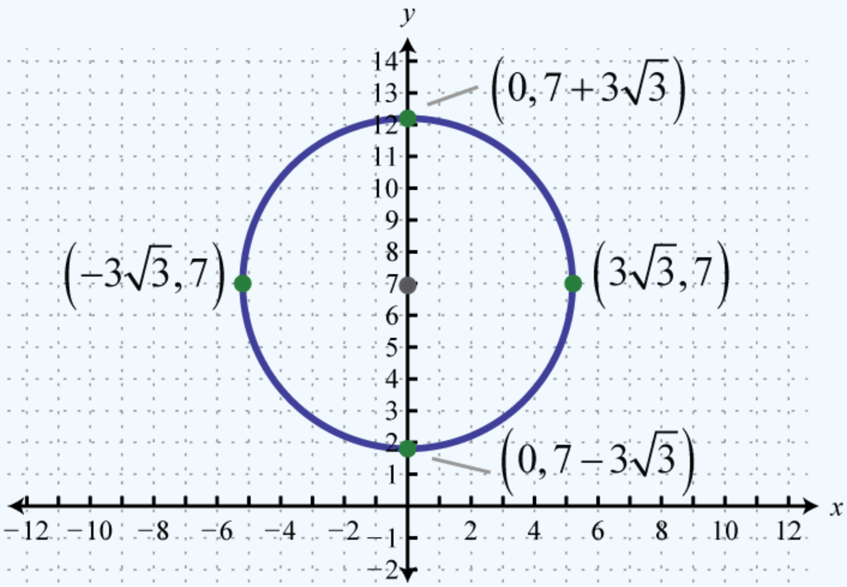

Figure 8.E.8

Exercise $8.E.9$

Rewrite in standard form and graph.

1. $x^2+y^2-6x+4y-3=0$
2. $x^2+y^2+8x-10y+16=0$
3. $2x^2+2y^2-2x-6y-3=0$
4. $4x^2+4y^2+8y+1=0$
5. $x^2+y^2-5x+y-\frac{1}{2}=0$
6. $x^2+y^2+12x-8y=0$

Answer

1. $(x-3)^2+(y+2)^2=16$;

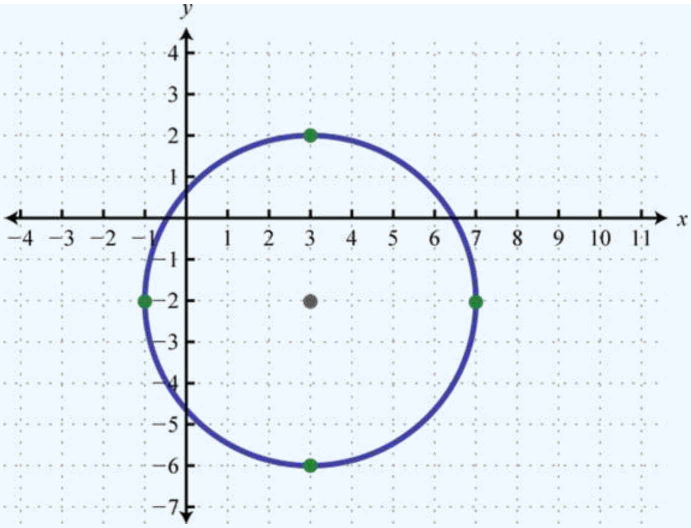

Figure 8.E.9

3. $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{3}{2}\right)^2=4$;

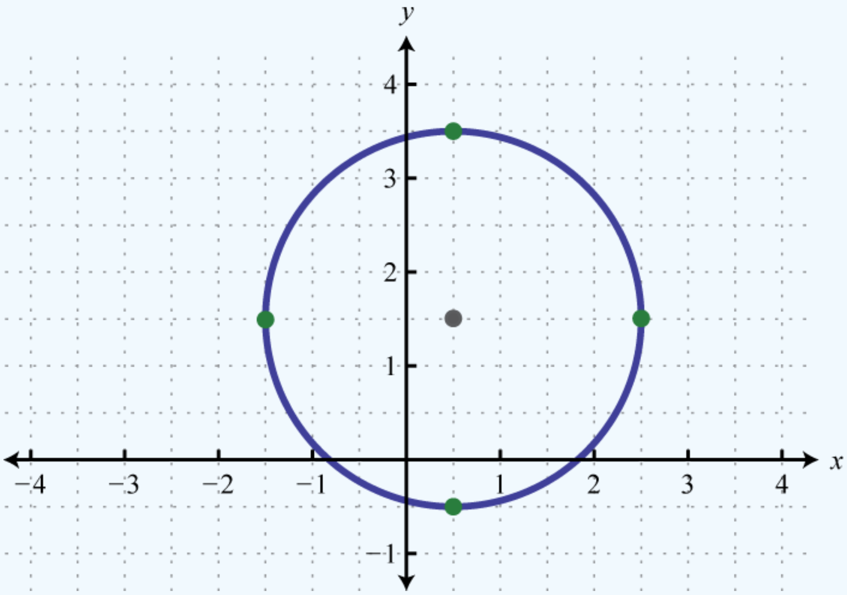

Figure 8.E.10

5. $\left(x-\frac{5}{2}\right)^2+\left(y+\frac{1}{2}\right)^2=7$;

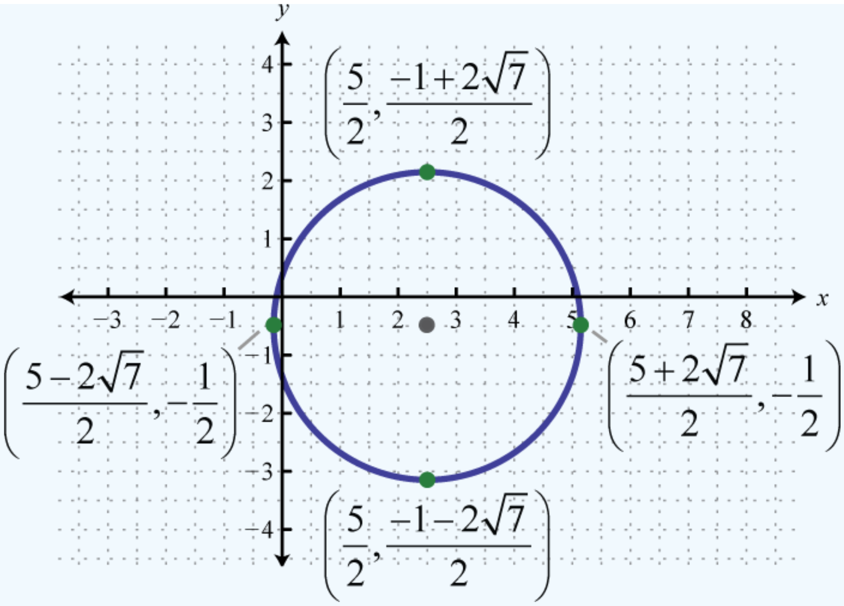

Figure 8.E.11

Exercise $8.E.10$

Given the equation of an ellipse in standard form, determine its center, orientation, major radius, and minor radius.

1. $\frac{(x+12)^2}{16} + \frac{(y-10)^2}{4} = 1$
2. $\frac{(x+3)^2}{3} + \frac{y^2}{25} = 1$
3. $x^2 + \frac{(y-5)^2}{12} = 1$
4. $\frac{(x-8)^2}{5} + \frac{(y+8)}{18} = 1$

Answer

1. Center: $(-12, 10)$; orientation: horizontal; major radius: 4 units; minor radius: 2 units

3. Center: $(0, 5)$; orientation: vertical; major radius: $2\sqrt{3}$ units; minor radius: 1 unit

Exercise $8.E.11$

Determine the standard form for the equation of the ellipse given the following information.

1. Center $(0, -4)$ with $a = 3$ and $b = 4$
2. Center $(3, 8)$ with $a = 1$ and $b = \sqrt{7}$
3. Center $(0, 0)$ with $a = 5$ and $b = \sqrt{2}$
4. Center $(-10, -30)$ with $a = 10$ and $b = 1$

Answer

1. $\frac{x^2}{9} + \frac{(y+4)^2}{16} = 1$

3. $\frac{x^2}{25} + \frac{y^2}{2} = 1$

Exercise $8.E.12$

Find the x- and y-intercepts.

1. $\frac{(x+2)^2}{4} + \frac{y^2}{9} = 1$
2. $\frac{(x-1)^2}{2} + \frac{(y+1)^2}{3} = 1$
3. $5x^2 + 2y^2 = 20$
4. $5(x-3)^2 + 6y^2 = 120$

Answer

1. x -intercepts: $(-4,0),(0,0)$; y -intercepts: $(0,0)$
3. x -intercepts: $(\pm 2,0)$; y -intercepts: $(0,\pm\sqrt{10})$

Exercise $8.E.13$

Graph.

1. $\frac{(x-10)^2}{25}+\frac{(y+5)^2}{4}=1$
2. $\frac{(x+6)^2}{9}+\frac{(y-8)^2}{36}=1$
3. $\frac{\left(x-\frac{3}{2}\right)^2}{4}+\left(y-\frac{7}{2}\right)^2=1$
4. $\left(x-\frac{2}{3}\right)^2+\frac{y^2}{4}=1$
5. $\frac{x^2}{2}+\frac{y^2}{5}=1$
6. $\frac{(x+2)^2}{8}+\frac{(y-3)^2}{12}=1$

Answer

1.

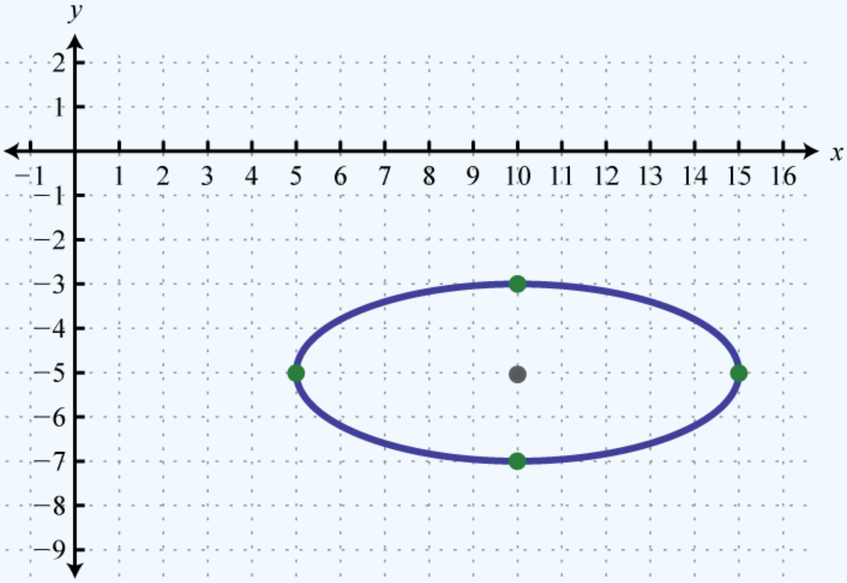

Figure 8.E.12

3.

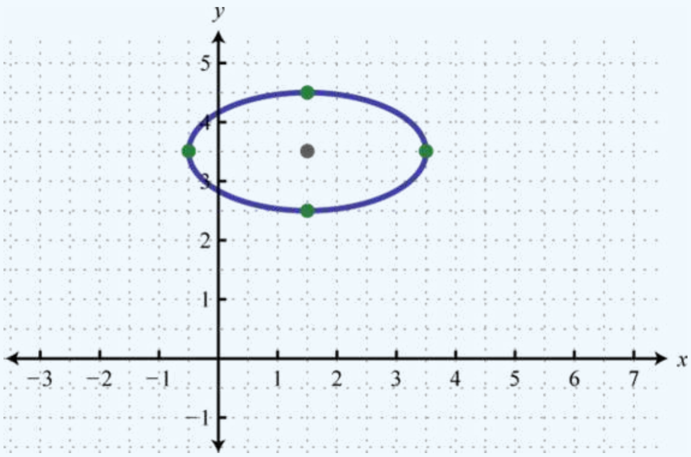

Figure 8.E.13

5.

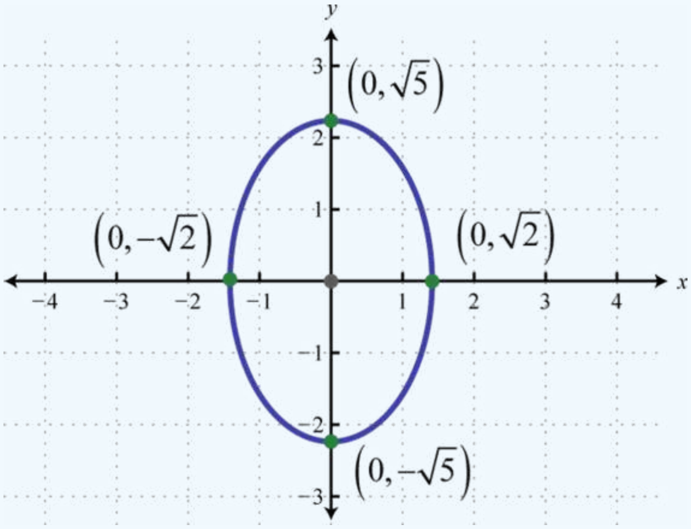

Figure 8.E.14

Exercise $8.E.14$

Rewrite in standard form and graph.

1. $4x^2+9y^2-8x+90y+193=0$
2. $9x^2+4y^2+108x-80y+580=0$
3. $x^2+9y^2+6x+108y+324=0$
4. $25x^2+y^2-350x-8y+1,216=0$
5. $8x^2+12y^2-16x-36y-13=0$
6. $10x^2+2y^2-50x+14y+7=0$

Answer

1. $\frac{(x-1)^2}{9}+\frac{(y+5)^2}{4}=1$;

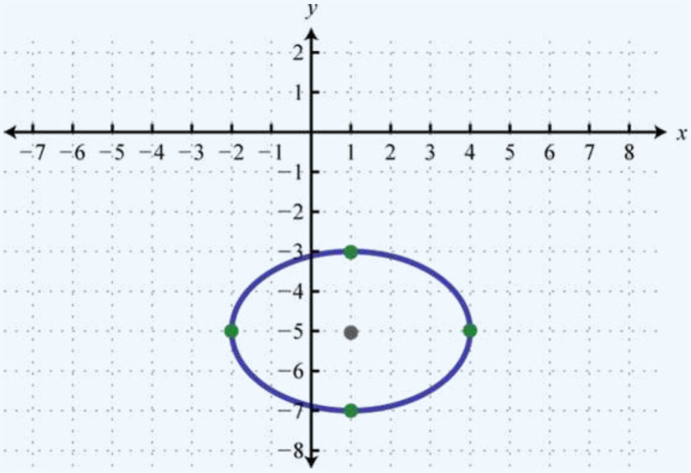

Figure 8.E.15

3. $\frac{(x+3)^2}{9}+(y+6)^2=1$;

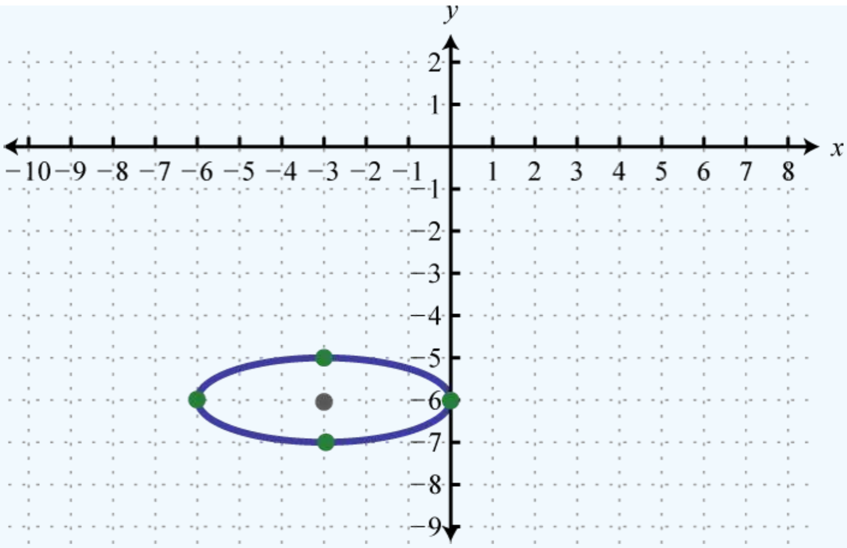

Figure 8.E.16

5. $\frac{(x-1)^2}{6}+\frac{\left(y-\frac{3}{2}\right)^2}{4}=1$;

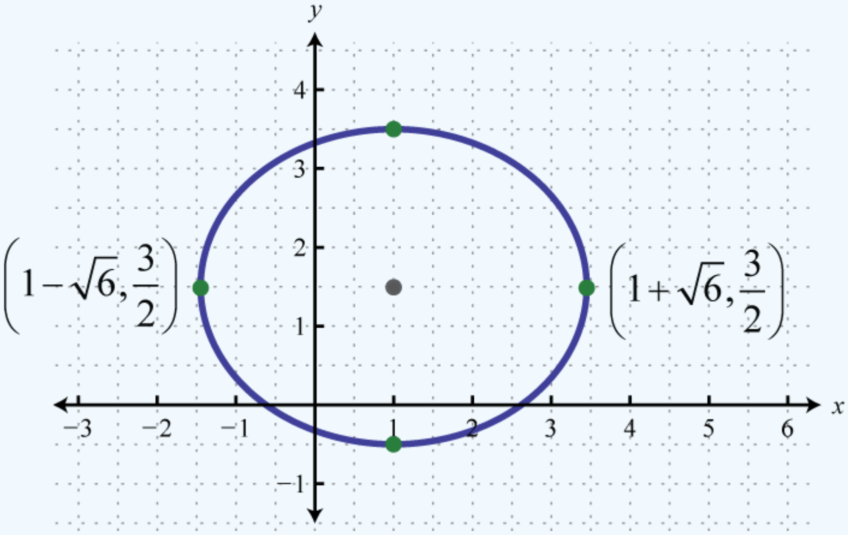

Figure 8.E.17

Exercise $8.E.15$

Given the equation of a hyperbola in standard form, determine its center, which way the graph opens, and the vertices.

1. $\frac{(x-10)^2}{4}-\frac{(y+5)^2}{16}=1$
2. $\frac{(x+7)^2}{2}-\frac{(y-8)^2}{8}=1$
3. $\frac{(y-20)^2}{3}-(x-15)^2=1$
4. $3y^2-12(x-1)^2=36$

Answer

1. Center: $(10,-5)$; opens left and right; vertices: $(8,-5),(12,-5)$

3. Center: $(15,20)$; opens upward and downward; vertices: $(15,20-\sqrt{3}),(15,20+\sqrt{3})$

Exercise $8.E.16$

Determine the standard form for the equation of the hyperbola.

1. Center $(-25, 10), a=3, b=\sqrt{5}$, opens up and down.
2. Center $(9, -12), a=5\sqrt{3}, b=7$, opens left and right.
3. Center $(-4, 0), a=1, b=6$, opens left and right.
4. Center $(-2, -3), a=10\sqrt{2}, b=2\sqrt{3}$, opens up and down.

Answer

1. $\frac{(y-10)^2}{5}-\frac{(x+25)^2}{9}=1$

3. $(x+4)^2-\frac{y^2}{36}=1$

Exercise $8.E.17$

Find the x- and y-intercepts.

1. $\frac{(x-1)^2}{4}-\frac{(y+3)^2}{9}=1$
2. $\frac{(x+4)^2}{8}-\frac{(y-2)^2}{12}=1$
3. $4(y-2)^2-x^2=16$
4. $6(y+1)^2-3(x-1)^2=18$

Answer

1. x -intercepts: $(1\pm 2\sqrt{2}, 0)$; y -intercepts: none

3. x -intercepts: $(0,0)$; y -intercepts: $(0,0), (0,4)$

Exercise $8.E.18$

Graph.

1. $\frac{(x-10)^2}{25}-\frac{(y+5)^2}{100}=1$
2. $\frac{(x-4)^2}{4}-\frac{(y-8)^2}{16}=1$
3. $\frac{(y-3)^2}{9}-\frac{(x-6)^2}{81}=1$
4. $\frac{(y+1)^2}{4}-\frac{(x+1)^2}{25}=1$
5. $\frac{y^2}{27}-\frac{(x-3)^2}{9}=1$
6. $\frac{x^2}{2}-\frac{y^2}{3}=1$

Answer

1.

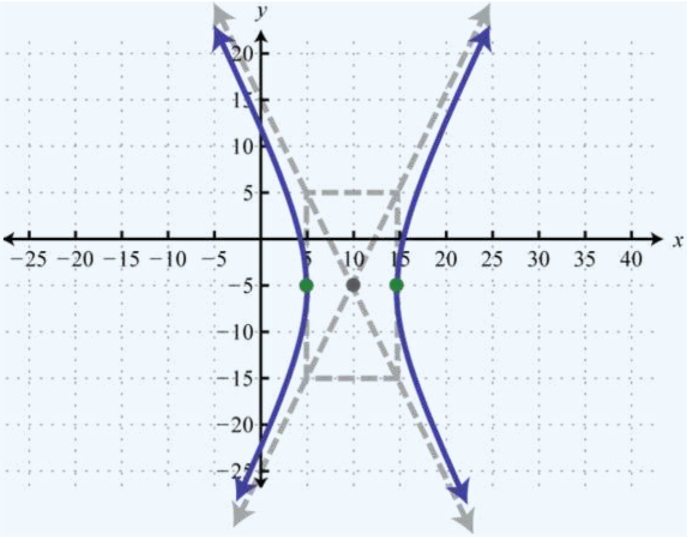

Figure 8.E.18

3.

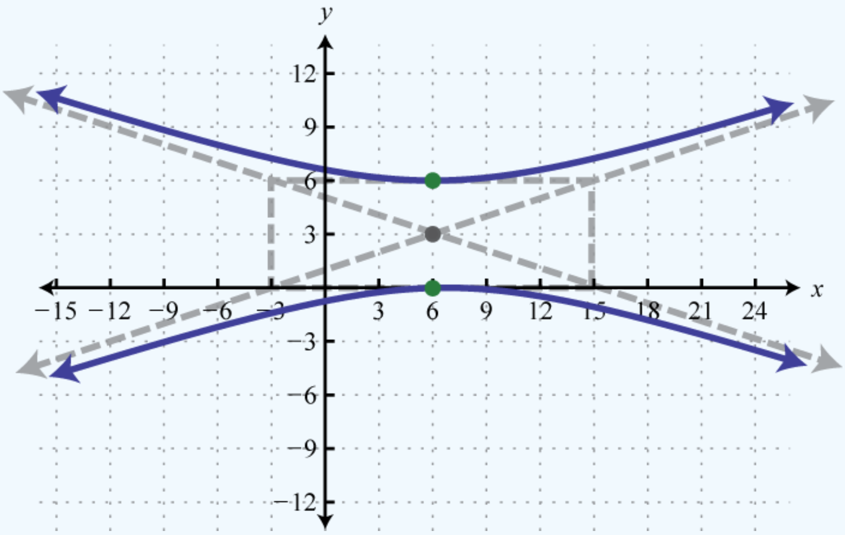

Figure 8.E.19

5.

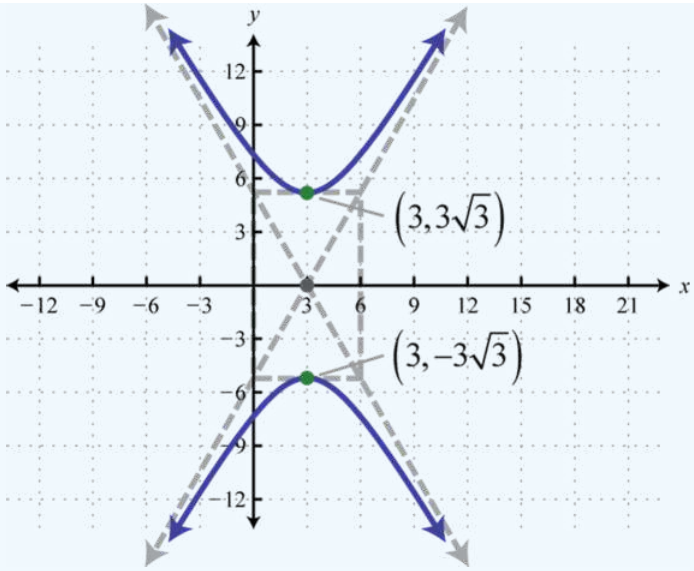

Figure 8.E.20

Exercise 8.E.19

Rewrite in standard form and graph.

1. $4x^2 - 9y^2 - 8x - 90y - 257 = 0$
2. $9x^2 - y^2 - 108x + 16y + 224 = 0$
3. $25y^2 - 2x^2 - 100y + 50 = 0$
4. $3y^2 - x^2 - 2x - 10 = 0$
5. $8y^2 - 12x^2 + 24y - 12x - 33 = 0$
6. $4y^2 - 4x^2 - 16y - 28x - 37 = 0$

Answer

1. $\frac{(x-1)^2}{9} - \frac{(y+5)^2}{4} = 1$;

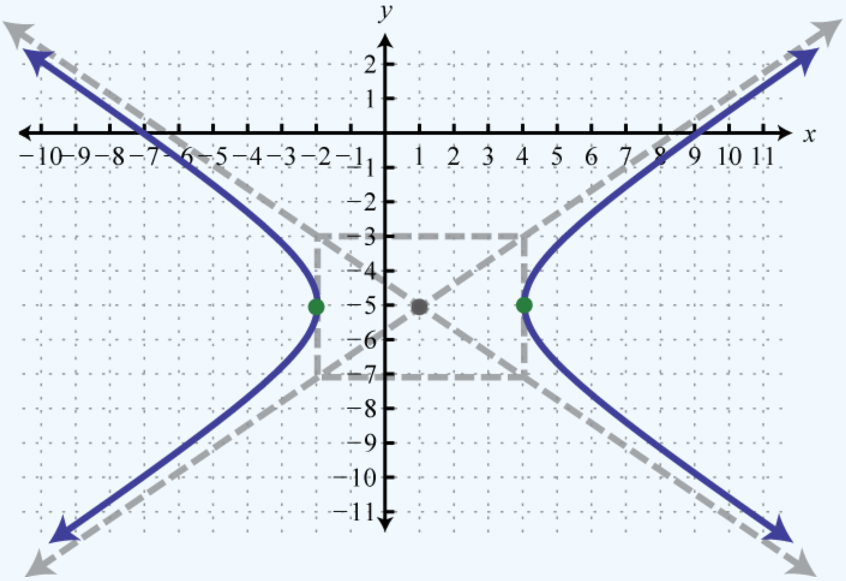

Figure 8.E.21

3. $\frac{(y-2)^2}{2} - \frac{x^2}{25} = 1$;

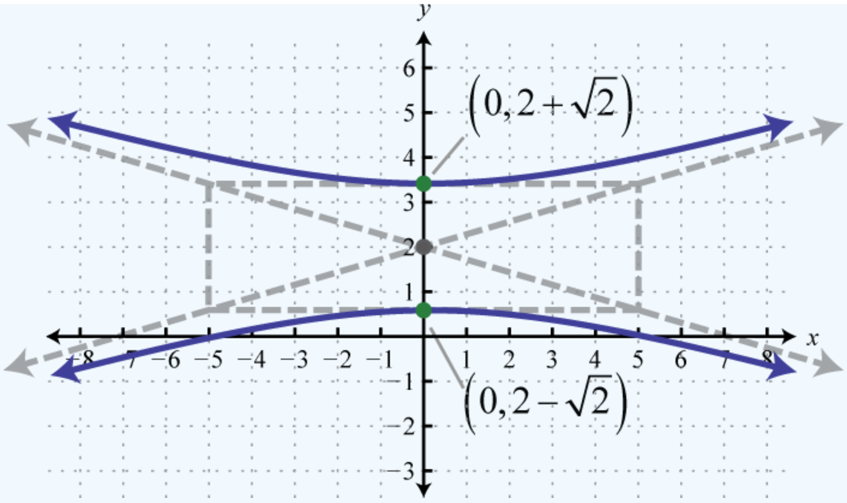

Figure 8.E.22

5. $\frac{(y+\frac{3}{2})^2}{6} - \frac{(x+\frac{1}{2})^2}{4} = 1$

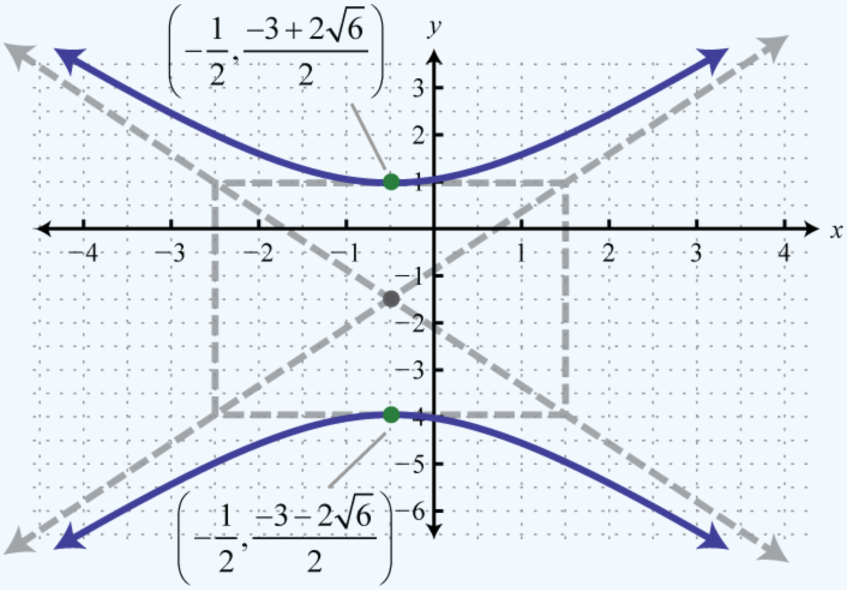

Figure 8.E.23

Exercise $8.E.20$

Identify the conic sections and rewrite in standard form.

1. $x^2 + y^2 - 2x - 8y + 16 = 0$
2. $x^2 + 2y^2 + 4x - 24y + 74 = 0$
3. $x^2 - y^2 - 6x - 4y + 3 = 0$
4. $x^2 + y - 10x + 22 = 0$
5. $x^2 + 12y^2 - 12x + 24 = 0$
6. $x^2 + y^2 + 10y + 22 = 0$
7. $4y^2 - 20x^2 + 16y + 20x - 9 = 0$
8. $16x - 16y^2 + 24y - 25 = 0$
9. $9x^2 - 9y^2 - 6x - 18y - 17 = 0$
10. $4x^2 + 4y^2 + 4x - 8y + 1 = 0$

Answer

1. Circle;$(x-1)^2+(y-4)^2=1$
3. Hyperbola; $\frac{(x-3)^2}{2}-\frac{(y+2)^2}{2}=1$
5. Ellipse; $\frac{(x-6)^2}{12}+y^2=1$
7. Hyperbola; $\frac{(y+2)^2}{5}-\left(x-\frac{1}{2}\right)^2=1$
9. Hyperbola; $\left(x-\frac{1}{3}\right)^2-(y+1)^2=1$

Exercise $8.E.21$

Given the graph, write the equation in general form.

1.

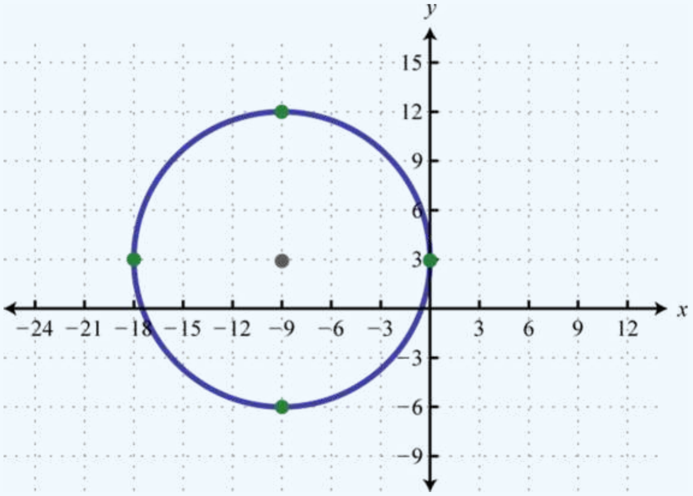

Figure 8.E.24

2.

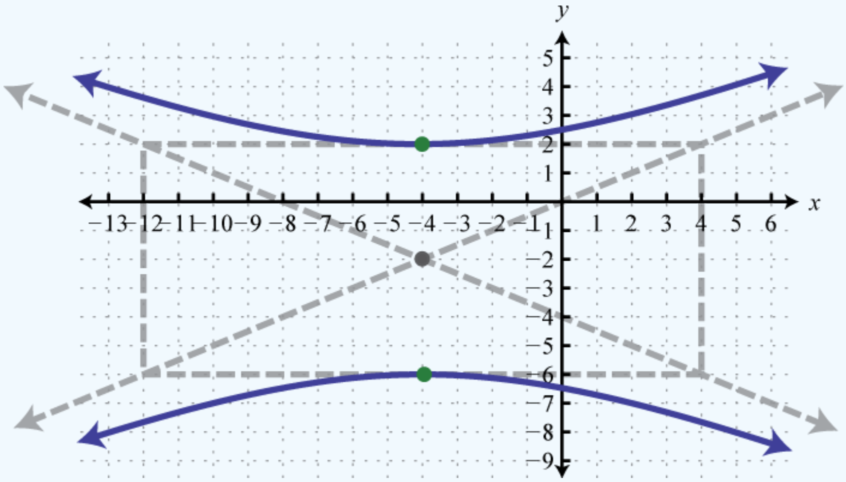

Figure 8.E.25

3.

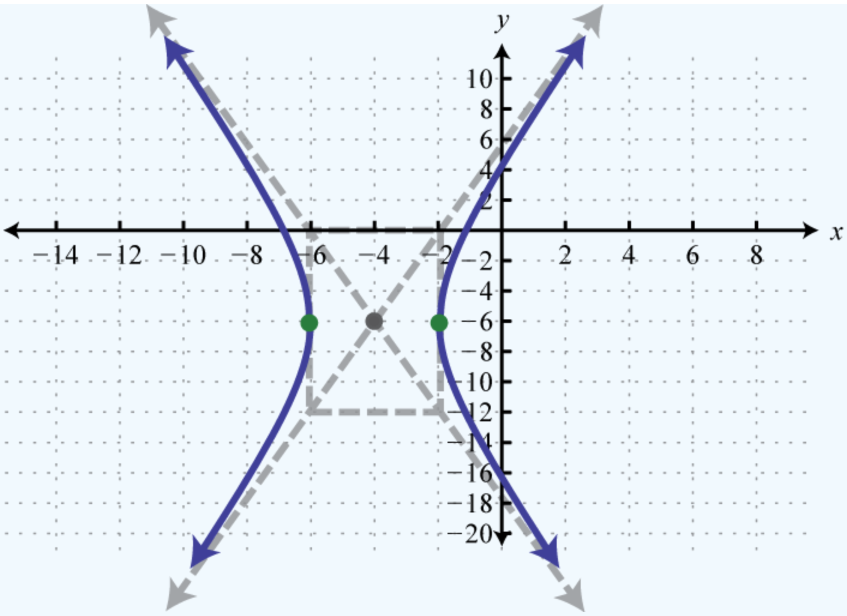

Figure 8.E.26

4.

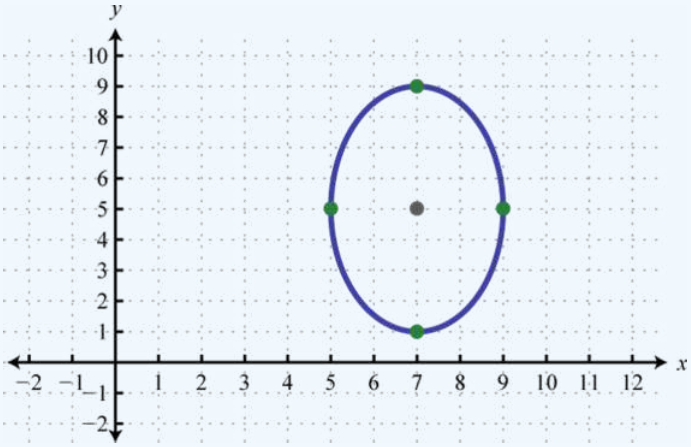

Figure 8.E.27

5.

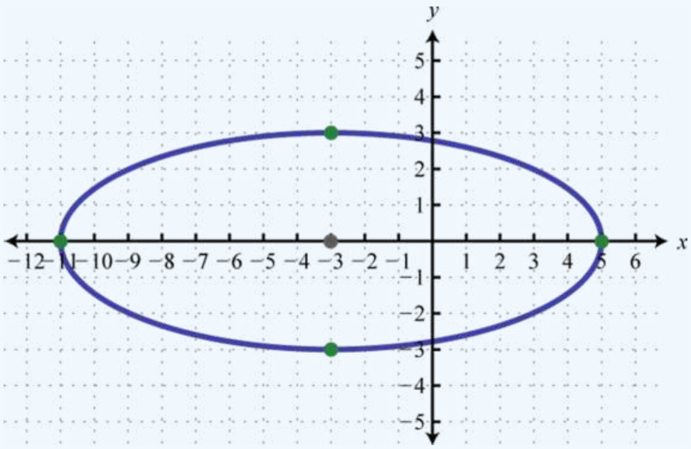

Figure 8.E.28

6.

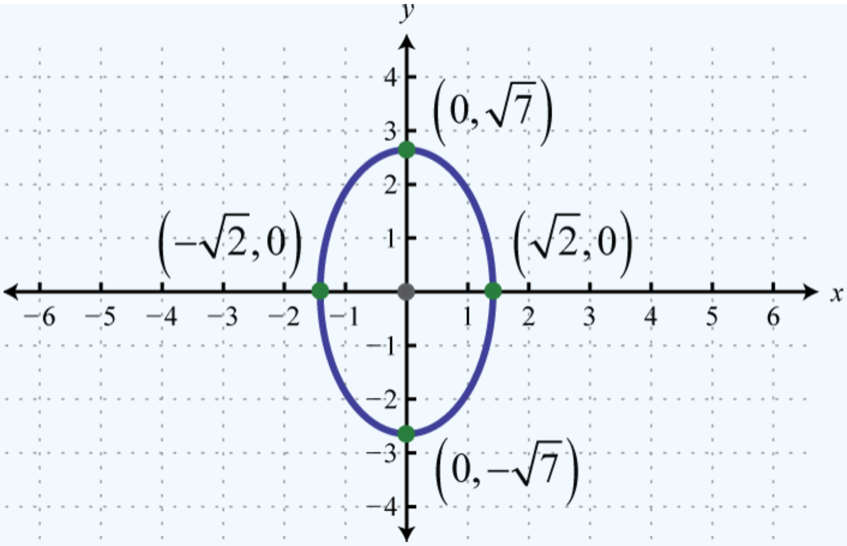

Figure 8.E.29

Answer

1. $x^2+y^2+18x-6y+9=0$

3. $9x^2-y^2+72x-12y+72=0$

5. $9x^2+64y^2+54x-495=0$

Exercise $8.E.22$

Solve.

1. $\begin{cases} x^2+y^2=8 \\ x-y=4 \end{cases}$
2. $\begin{cases} x^2+y^2=1 \\ x+2y=1 \end{cases}$
3. $\begin{cases} x^2+3y^2=4 \\ 2x-y=1 \end{cases}$
4. $\begin{cases} 2x^2+y^2=5 \\ x+y=3 \end{cases}$
5. $\begin{cases} 3x^2-2y^2=1 \\ x-y=2 \end{cases}$
6. $\begin{cases} x^2-3y^2=10 \\ x-2y=1 \end{cases}$
7. $\begin{cases} 2x^2+y^2=11 \\ 4x+y^2=5 \end{cases}$
8. $\begin{cases} x^2+4y^2=1 \\ 2x^2+4y=5 \end{cases}$
9. $\begin{cases} 5x^2-y^2=10 \\ x^2+y=2 \end{cases}$
10. $\begin{cases} 2x^2+y^2=1 \\ 2x-4y^2=-3 \end{cases}$
11. $\begin{cases} x^2+4y^2=10 \\ xy=2 \end{cases}$
12. $\begin{cases} y+x^2=0 \\ xy-8=0 \end{cases}$

13. $\begin{cases} \frac{1}{x} + \frac{1}{y} = 10 \\ \frac{1}{x} - \frac{1}{y} = 6 \end{cases}$
14. $\begin{cases} \frac{1}{x} + \frac{1}{y} = 1 \\ y - x = 2 \end{cases}$
15. $\begin{cases} x - 2y^2 = 3 \\ y = \sqrt{x - 4} \end{cases}$
16. $\begin{cases} (x-1)^2 + y^2 = 1 \\ \quad y - \sqrt{x} = 0 \end{cases}$

Answer

1. $(2, -2)$
3. $\left(-\frac{1}{13}, -\frac{15}{13}\right), (1, 1)$
5. $(-9, -11), (1, -1)$
7. $(-1, -3), (-1, 3)$
9. $(-\sqrt{2}, 0), (\sqrt{2}, 0), (-\sqrt{7}, -5), (\sqrt{7}, -5)$
11. $(\sqrt{2}, \sqrt{2}) \cdot (-\sqrt{2}, -\sqrt{2}) \cdot \left(2\sqrt{2}, \frac{\sqrt{2}}{2}\right) \cdot \left(-2\sqrt{2}, -\frac{\sqrt{2}}{2}\right)$
13. $\left(\frac{1}{8}, \frac{1}{2}\right)$
15. $(5, 1)$

Sample Exam

Exercise $8.E.23$

1. Given two points $(-4, -6)$ and $(2, -8)$:
    1. Calculate the distance between them.
    2. Find the midpoint between them.
2. Determine the area of a circle whose diameter is defined by the points $(4, -3)$ and $(-1, 2)$.

Answer

1. (1) $2\sqrt{10}$ units; (2) $(-1, -7)$

Exercise $8.E.24$

Rewrite in standard form and graph. Find the vertex and all intercepts if any.

1. $y = -x^2 + 6x - 5$
2. $x = 2y^2 + 4y - 6$
3. $x = -3y^2 + 3y + 1$
4. Find the equation of a circle in standard form with center $(-6, 3)$ and radius $2\sqrt{5}$ units.

Answer

1. $y = -(x-3)^2 + 4$;

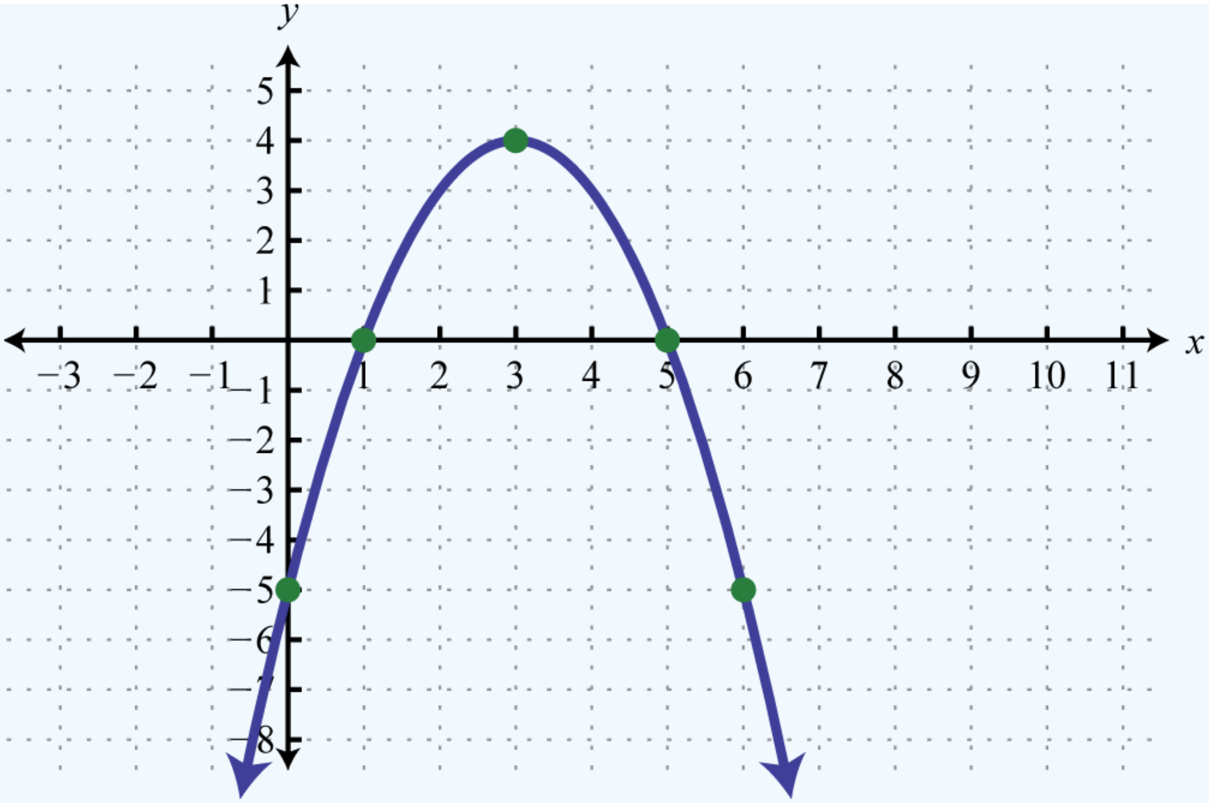

Figure 8.E.30

3. $x=-3\left(y-\frac{1}{2}\right)^2+\frac{7}{4}$;

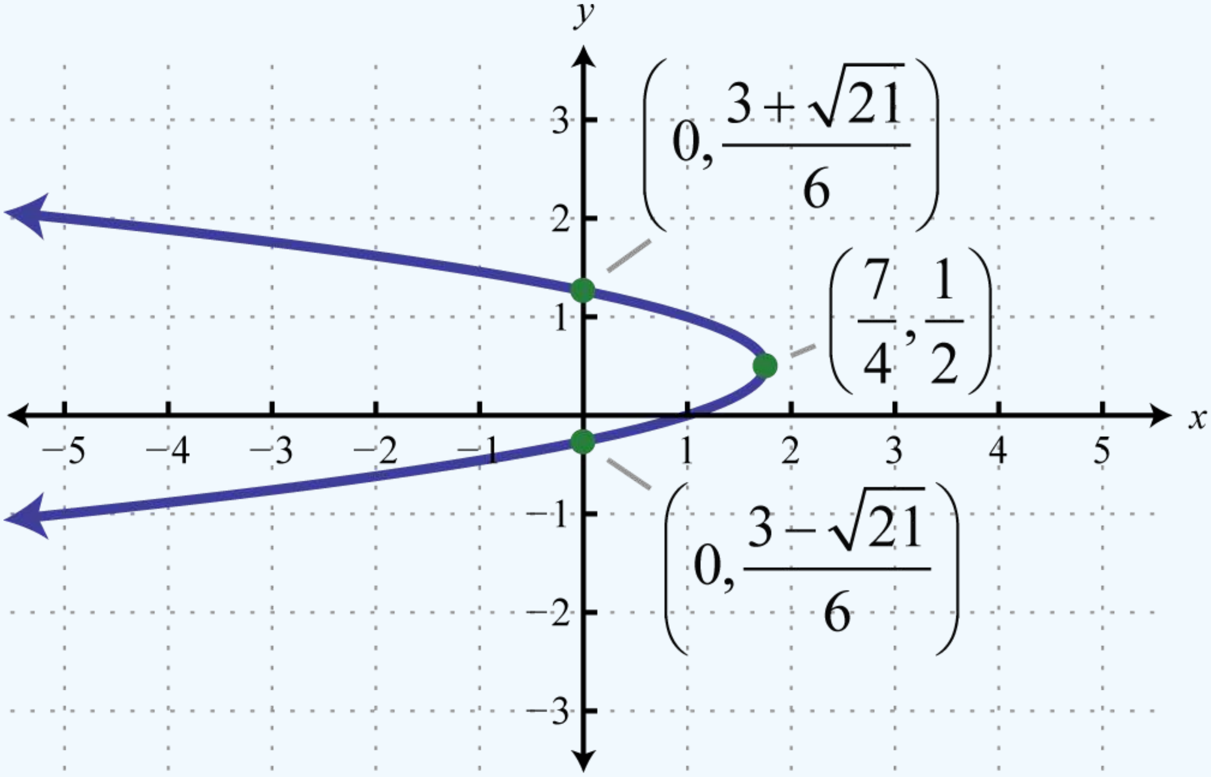

Figure 8.E.31

Exercise $8.E.25$

Sketch the graph of the conic section given its equation in standard form.

1. $(x-4)^2+(y+1)^2=45$
2. $\frac{(x+3)^2}{4}+\frac{y^2}{9}=1$
3. $\frac{y^2}{3}-\frac{x^2}{9}=1$
4. $\frac{x^2}{16}-(y-2)^2=1$

Answer

1.

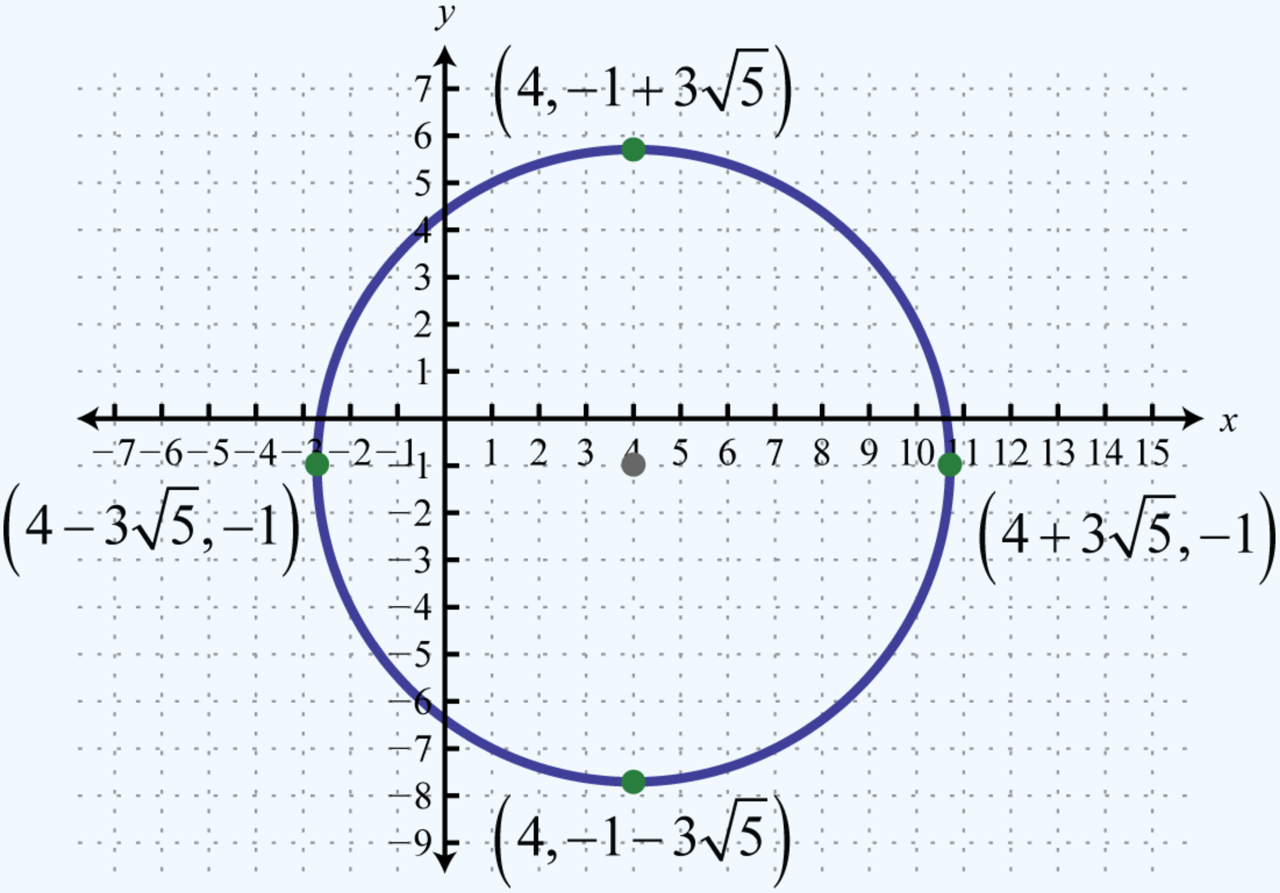

Figure 8.E.32

3.

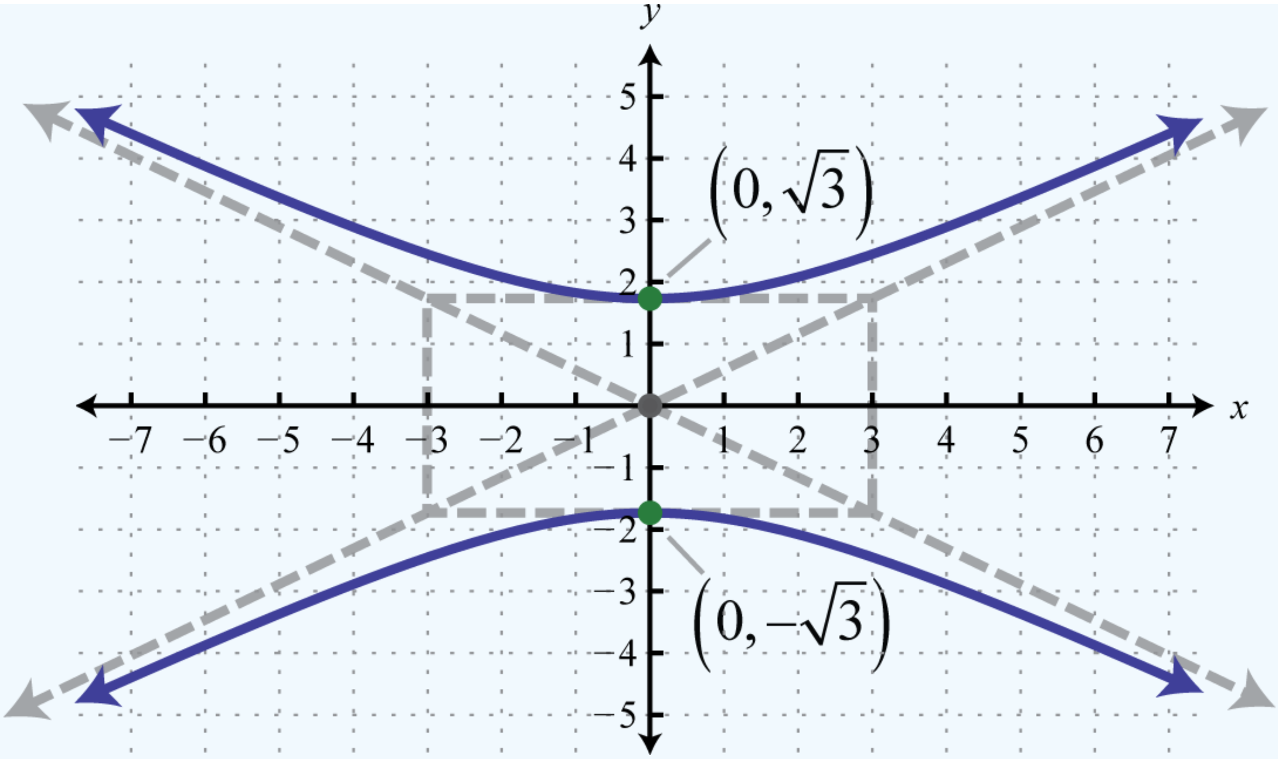

Figure 8.E.33

Exercise $8.E.26$

Rewrite in standard form and graph.

1. $9x^2+4y^2-144x+16y+556=0$
2. $x-y^2+6y+7=0$
3. $x^2+y^2+20x-20y+100=0$
4. $4y^2-x^2+40y-30x-225=0$

Answer

1. $\frac{(x-8)^2}{4}+\frac{(y+2)^2}{9}=1$;

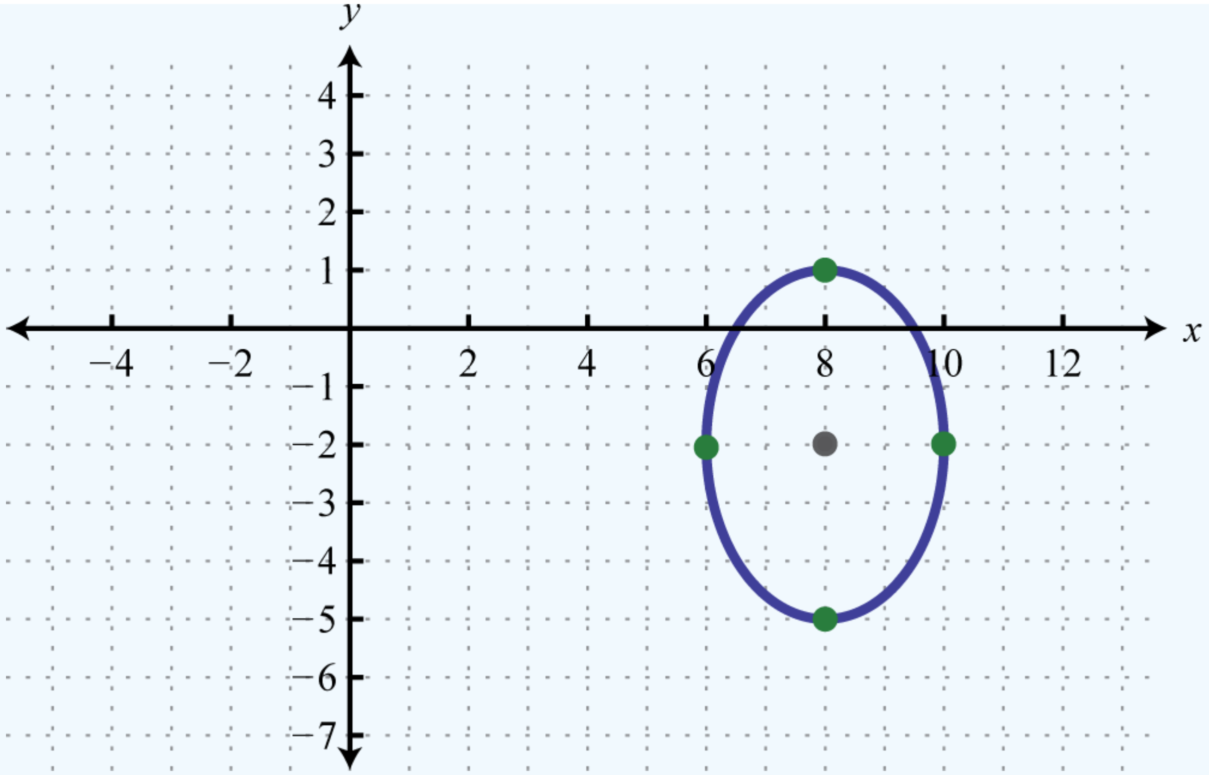

Figure 8.E.34

3. $(x+10)^2+(y-10)^2=100$;

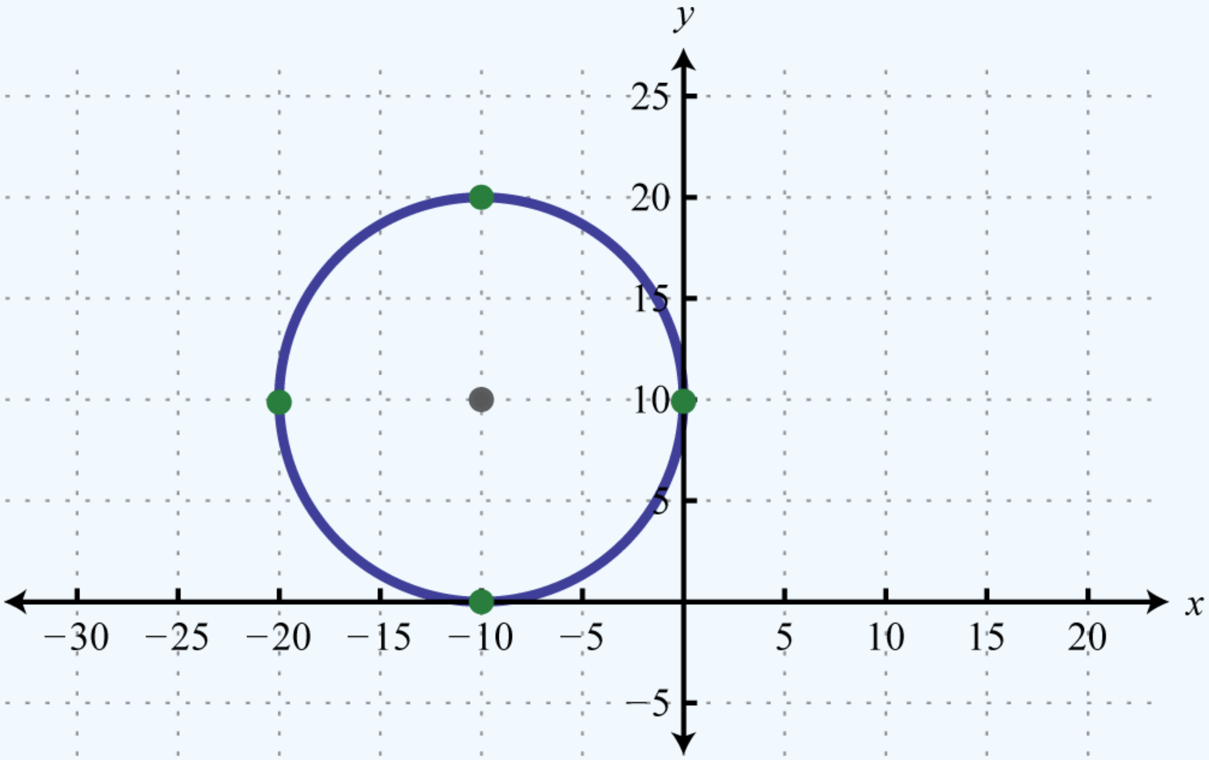

Figure 8.E.35

Exercise $8.E.27$

Find the x- and y-intercepts.

1. $x=-2(y-4)^2+9$
2. $\frac{(y-1)^2}{12}-(x+1)^2=1$

Answer

1. x -intercept: $(-23,0)$; y -intercepts: $\left(0,\frac{8\pm3\sqrt{2}}{2}\right)$

Exercise $8.E.28$

Solve.

1. $\begin{cases} x+y=2 \\ y=-x^2+4 \end{cases}$
2. $\begin{cases} y-x^2=-3 \\ x^2+y^2=9 \end{cases}$
3. $\begin{cases} 2x-y=1 \\ (x+1)^2+2y^2=1 \end{cases}$
4. $\begin{cases} x^2+y^2=6 \\ xy=3 \end{cases}$

Answer

1. $(-1,3),(2,0)$

3. $\emptyset$

Exercise $8.E.29$

1. Find the equation of an ellipse in standard form with vertices $(-3,-5)$ and $(5,-5)$ and a minor radius 2 units in length.
2. Find the equation of a hyperbola in standard form opening left and right with vertices $(\pm\sqrt{5},0)$ and a conjugate axis that measures 10 units.
3. Given the graph of the ellipse, determine its equation in general form.

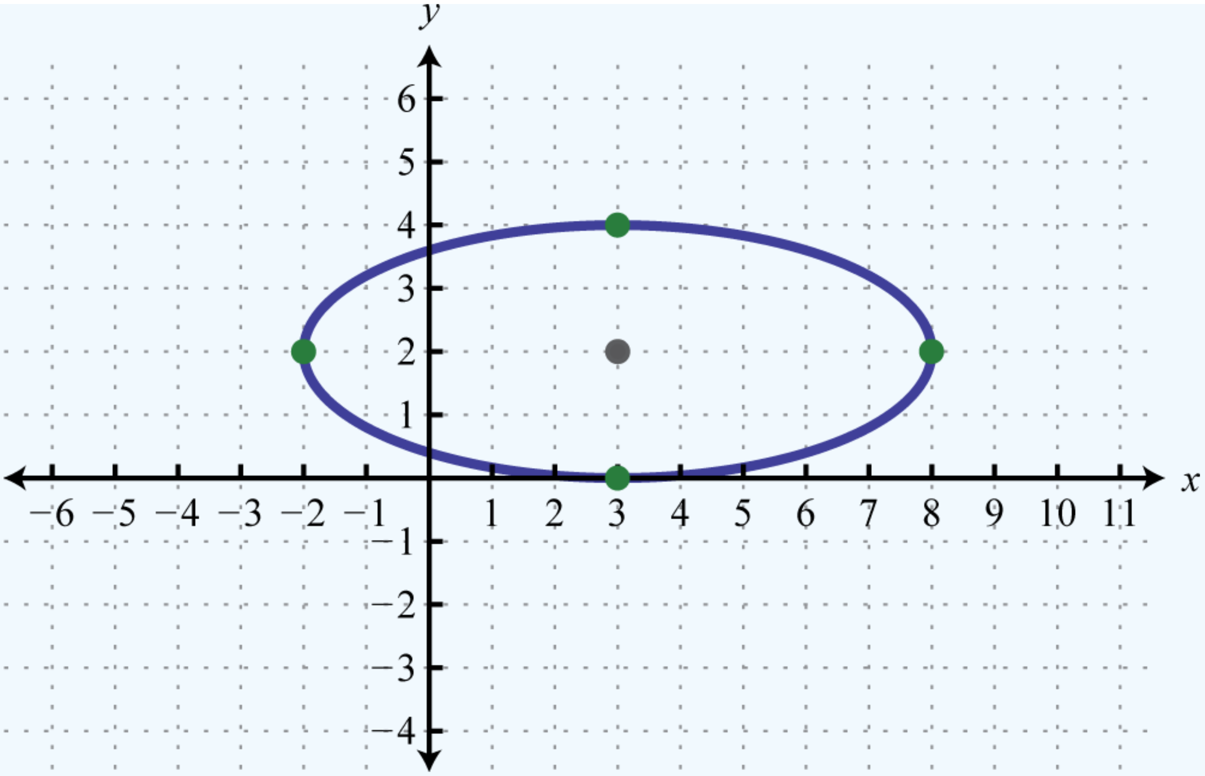

Figure 8.E.36

4. A rectangular deck has an area of 80 square feet and a perimeter that measures 36 feet. Find the dimensions of the deck.

5. The diagonal of a rectangle measures $2\sqrt{13}$ centimeters and the perimeter measures 20 centimeters. Find the dimensions of the rectangle.

Answer

1. $\frac{(x-1)^2}{16}+\frac{(y+5)^2}{4}=1$

3. $4x^2+25y^2-24x-100y+36=0$

5. 6 centimeters by 4 centimeters

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CHAPTER OVERVIEW

9: Sequences, Series, and the Binomial Theorem

9.1: Introduction to Sequences and Series
9.2: Arithmetic Sequences and Series
9.3: Geometric Sequences and Series
9.4: Binomial Theorem
9.E: Sequences, Series, and the Binomial Theorem (Exercises)

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9.1: Introduction to Sequences and Series

Skills to Develop

- Find any element of a sequence given a formula for its general term.
- Use sigma notation and expand corresponding series.
- Distinguish between a sequence and a series.
- Calculate the nth partial sum of sequence.

Sequences

A **sequence**[1] is a function whose domain is a set of consecutive natural numbers beginning with 1. For example, the following equation with domain $\{1, 2, 3, \ldots\}$ defines an **infinite sequence**[2]:

$$a(n) = 5n - 3 \text{ or } a_n = 5n - 3$$

The elements in the range of this function are called terms of the sequence. It is common to define the nth term, or the **general term of a sequence**[3], using the subscripted notation a_n, which reads "a sub n." Terms can be found using substitution as follows:

$$\begin{aligned} General\ term: &\quad a_n = 5n - 3 \\ First\ term (n=1): &\quad a_1 = 5(1) - 3 = 2 \\ Second\ term (n=2): &\quad a_2 = 5(2) - 3 = 7 \\ Third\ term (n=3): &\quad a_3 = 5(3) - 3 = 12 \\ Fourth\ term (n=4): &\quad a_4 = 5(4) - 3 = 17 \\ Fifth\ term (n=5): &\quad a_5 = 5(5) - 3 = 22 \\ &\quad \vdots \end{aligned}$$

This produces an ordered list,

$$2, 7, 12, 17, 22, \ldots$$

The ellipsis $(\ldots)$ indicates that this sequence continues forever. Unlike a set, order matters. If the domain of a sequence consists of natural numbers that end, such as $\{1, 2, 3, \ldots, k\}$, then it is called a **finite sequence**[4].

Example 9.1.1:

Given the general term of a sequence, find the first 5 terms as well as the 100^{th} term: $a_n = \frac{n(n-1)}{2}$.

Solution:

To find the first 5 terms, substitute $1, 2, 3, 4$, and 5 for n and then simplify.

$$\begin{aligned} a_1 &= \tfrac{1(1-1)}{2} = \tfrac{1(0)}{2} = \tfrac{0}{2} = 0 \\ a_2 &= \tfrac{2(2-1)}{2} = \tfrac{2(1)}{2} = \tfrac{2}{2} = 1 \\ a_3 &= \tfrac{3(3-1)}{2} = \tfrac{3(2)}{2} = \tfrac{6}{2} = 3 \\ a_4 &= \tfrac{4(4-1)}{2} = \tfrac{4(3)}{2} = \tfrac{12}{2} = 6 \\ a_5 &= \tfrac{5(5-1)}{2} = \tfrac{5(4)}{2} = \tfrac{20}{2} = 10 \end{aligned}$$

Use $n = 100$ to determine the 100^{th} term in the sequence.

$$a_{100} = \frac{100(100-1)}{2} = \frac{100(99)}{2} = \frac{9{,}900}{2} = 4,950$$

Answer:

First five terms: $0, 1, 3, 6, 10$; $a_{100} = 4,950$

Sometimes the general term of a sequence will alternate in sign and have a variable other than n.

Example 9.1.2:

Find the first 5 terms of the sequence: $a_n = (-1)^n x^{n+1}$.

Solution:

Here we take care to replace n with the first 5 natural numbers and not x.

$$\begin{aligned} a_1 &= (-1)^1 x^{1+1} = -x^2 \\ a_2 &= (-1)^2 x^{2+1} = x^3 \\ a_3 &= (-1)^3 x^{3+1} = -x^4 \\ a_4 &= (-1)^4 x^{4+1} = x^5 \\ a_5 &= (-1)^5 x^{5+1} = -x^6 \end{aligned}$$

Answer:

$-x^2, x^3, -x^4, x^5, -x^6$

Exercise 9.1.1

Find the first 5 terms of the sequence: $a_n = (-1)^{n+1} 2^n$.

Answer

$2, -4, 8, -16, 32.$

http://www.youtube.com/v/uuQ3jYL-g_I

One interesting example is the Fibonacci sequence. The first two numbers in the Fibonacci sequence are 1, and each successive term is the sum of the previous two. Therefore, the general term is expressed in terms of the previous two as follows:

$$F_n = F_{n-2} + F_{n-1}$$

Here $F_1 = 1, F_2 = 1$, and $n > 2$. A formula that describes a sequence in terms of its previous terms is called a **recurrence relation**[5].

Example 9.1.3:

Find the first 7 Fibonacci numbers.

Solution:

Given that $F_1 = 1$ and $F_2 = 1$, use the recurrence relation $F_n = F_{n-2} + F_{n-1}$ where n is an integer starting with $n = 3$ to find the next 5 terms:

$$\begin{aligned} F_3 &= F_{3-2} + F_{3-1} = F_1 + F_2 = 1 + 1 = 2 \\ F_4 &= F_{4-2} + F_{4-1} = F_2 + F_3 = 1 + 2 = 3 \\ F_5 &= F_{5-2} + F_{5-1} = F_3 + F_4 = 2 + 3 = 5 \\ F_6 &= F_{6-2} + F_{6-1} = F_4 + F_5 = 3 + 5 = 8 \\ F_7 &= F_{7-2} + F_{7-1} = F_5 + F_6 = 5 + 8 = 13 \end{aligned}$$

Answer:

$1, 1, 2, 3, 5, 8, 13$

***Figure 9.1.1**: Leonardo Fibonacci (1170–1250)*

Fibonacci numbers appear in applications ranging from art to computer science and biology. The beauty of this sequence can be visualized by constructing a Fibonacci spiral. Consider a tiling of squares where each side has a length that matches each Fibonacci number:

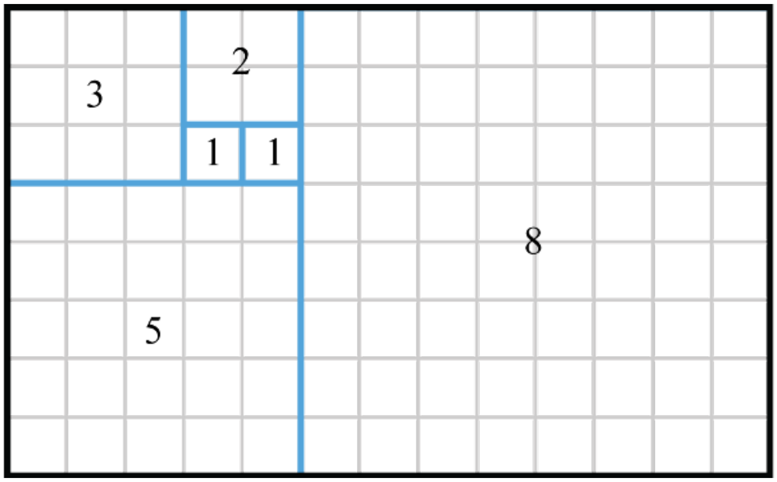

Figure 9.1.2

Connecting the opposite corners of the squares with an arc produces a special spiral shape.

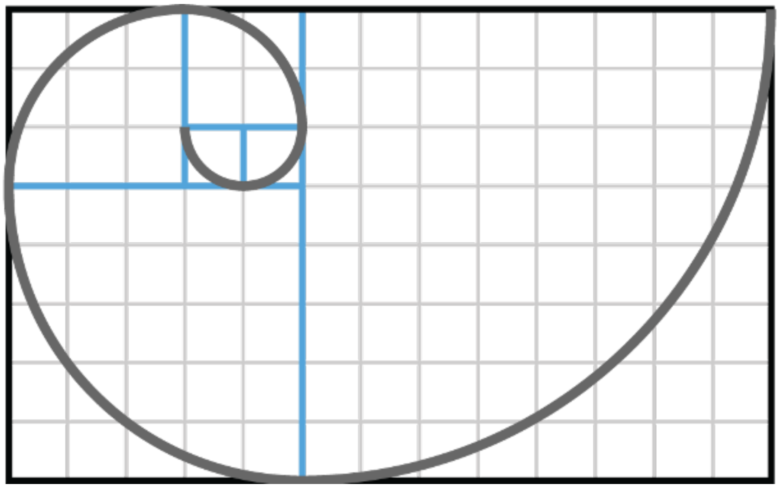

Figure 9.1.3

This shape is called the Fibonacci spiral and approximates many spiral shapes found in nature.

Series

A **series**[6] is the sum of the terms of a sequence. The sum of the terms of an infinite sequence results in an **infinite series**[7], denoted S_∞. The sum of the first n terms in a sequence is called a **partial sum**[8], denoted S_n. For example, given the sequence of positive odd integers $1, 3, 5, \ldots$ we can write:

$$\begin{array}{ll} S_\infty = 1+3+5+7+9+\ldots & \textit{Infinite series} \\ S_5 = 1+3+5+7+9=25 & \textit{5th partial sum} \end{array}$$

Example 9.1.4:

Determine the 3^{rd} and 5^{th} partial sums of the sequence: $3, -6, 12, -24, 48, \ldots$

Solution:

$$S_3 = 3 + (-6) + 12 = 9$$
$$S_5 = 3 + (-6) + 12 + (-24) + 48 = 33$$

Answer:

$S_3 = 9; S_5 = 33$

If the general term is known, then we can express a series using **sigma**[9] (or **summation**[10]) **notation**:

$$\begin{aligned} S_\infty &= \sum_{n=1}^{\infty} n^2 = 1^2 + 2^2 + 3^2 + \ldots && \textit{Infinite series} \\ S_3 &= \sum_{n=1}^{3} n^2 = 1^2 + 2^2 + 3^2 && \textit{3rd partial sum} \end{aligned}$$

The symbol Σ (upper case Greek letter sigma) is used to indicate a series. The expressions above and below indicate the range of the **index of summation**[11], in this case represented by n. The lower number indicates the starting integer and the upper value indicates the ending integer. The nth partial sum S_n can be expressed using sigma notation as follows:

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

This is read, "the sum of a_k as k goes from 1 to n." Replace n with ∞ to indicate an infinite sum.

Example 9.1.5:

Evaluate: $\sum_{k=1}^{5} (-3)^{n-1}$.

$$\begin{aligned} \sum_{k=1}^{5} (-3)^{k-1} &= (-3)^{1-1} + (-3)^{2-1} + (-3)^{3-1} + (-3)^{4-1} + (-3)^{5-1} \\ &= (-3)^0 + (-3)^1 + (-3)^2 + (-3)^3 + (-3)^4 \\ &= 1 - 3 + 9 - 27 + 81 \\ &= 61 \end{aligned}$$

Answer:

61

When working with sigma notation, the index does not always start at 1.

Example 9.1.1:

Evaluate: $\sum_{k=2}^{5} (-1)^k (3k)$

Solution:

Here the index is expressed using the variable k, which ranges from 2 to 5.

Exercise 9.1.2

Evaluate: $\sum_{n=1}^{5} (15 - 9n)$.

Answer

-60

http://www.youtube.com/v/aZ3sPd8N1TU

Infinity is used as the upper bound of a sum to indicate an infinite series.

Example 9.1.7:

Write in expanded form: $\sum_{i=0}^{\infty} \frac{n}{n+1}$.

Solution:

In this case we begin with $n = 0$ and add three dots to indicate that this series continues forever.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{n}{n+1} &= \frac{0}{0+1}+\frac{1}{1+1}+\frac{2}{2+1}+\frac{3}{3+1}+\cdots\\ &= \frac{0}{1}+\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots\\ &= 0+\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots\end{aligned}$$

Answer:

$0+\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots$

When expanding a series, take care to replace only the variable indicated by the index.

Example 9.1.8:

Write in expanded form: $\sum_{i=1}^{\infty}(-1)^{i-1}x^{2i}$.

Solution:

$$\begin{aligned}\sum_{i=1}^{\infty}(-1)^{i-1}x^{2i} &= (-1)^{1-1}x^{2(1)}+(-1)^{2-1}x^{2(2)}+(-1)^{3-1}x^{2(3)}+\cdots\\ &= (-1)^{0}x^{2(1)}+(-1)^{1}x^{2(2)}+(-1)^{2}x^{2(3)}+\cdots\\ &= x^2-x^4+x^6-\cdots\end{aligned}$$

Answer:

$x^2-x^4+x^6-\cdots$

Key Takeaways

- A sequence is a function whose domain consists of a set of natural numbers beginning with 1. In addition, a sequence can be thought of as an ordered list.
- Formulas are often used to describe the nth term, or general term, of a sequence using the subscripted notation a_n.
- A series is the sum of the terms in a sequence. The sum of the first n terms is called the nth partial sum and is denoted S_n.
- Use sigma notation to denote summations in a compact manner. The nth partial sum, using sigma notation, can be written $S_n = \sum_{k=1}^{n}a_k$. The symbol Σ denotes a summation where the expression below indicates that the index k starts at 1 and iterates through the natural numbers ending with the value n above.

Exercise 9.1.3

Find the first 5 terms of the sequence as well as the 30^{th} term.

1. $a_n=2n$
2. $a_n=2n+1$
3. $a_n=\frac{n^2-1}{2}$
4. $a_n=\frac{n}{2n-1}$
5. $a_n=(-1)^n(n+1)^2$
6. $a_n=(-1)^{n+1}n^2$
7. $a_n=3^{n-1}$
8. $a_n=2^{n-2}$
9. $a_n=\left(\frac{1}{2}\right)^n$
10. $a_n=\left(-\frac{1}{3}\right)^n$
11. $a_n=\frac{(-1)^{n-1}}{3n-1}$
12. $a_n=\frac{2(-1)^n}{n+5}$
13. $a_n=1+\frac{1}{n}$
14. $a_n=\frac{n^2+1}{n}$

Answer

1. $2,4,6,8,10; a_{30}=60$

3. $0,\frac{3}{2},4,\frac{15}{2},12; a_{30}=\frac{899}{2}$

5. $-4,9,-16,25,-36; a_{30}=961$

7. $1, 3, 9, 27, 81; a_{30} = 3^{29}$

9. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}; a_{30} = \frac{1}{2^{30}}$

11. $\frac{1}{2}, -\frac{1}{5}, \frac{1}{8}, -\frac{1}{11}, \frac{1}{14}; a_{30} = -\frac{1}{89}$

13. $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}; a_{30} = \frac{31}{30}$

Exercise 9.1.4

Find the first 5 terms of the sequence.

1. $a_n = 2x^{2n-1}$
2. $a_n = (2x)^{n-1}$
3. $a_n = \frac{x^n}{n+4}$
4. $a_n = \frac{x^{2n}}{x-2}$
5. $a_n = \frac{nx^{2n}}{n+1}$
6. $a_n = \frac{(n+1)x^n}{n^2}$
7. $a_n = (-1)^n x^{3n}$
8. $a_n = (-1)^{n-1} x^{n+1}$

Answer

1. $2x, 2x^3, 2x^5, 2x^7, 2x^9$

3. $\frac{x}{5}, \frac{x^2}{6}, \frac{x^3}{7}, \frac{x^4}{8}, \frac{x^5}{9}$

5. $\frac{x^2}{2}, \frac{2x^4}{3}, \frac{3x^6}{4}, \frac{4x^8}{5}, \frac{5x^{10}}{6}$

7. $-x^3, x^6, -x^9, x^{12}, -x^{15}$

Exercise 9.1.5

Find the first 5 terms of the sequence defined by the given recurrence relation.

1. $a_n = a_{n-1} + 5$ where $a_1 = 3$
2. $a_n = a_{n-1} - 3$ where $a_1 = 4$
3. $a_n = 3a_{n-1}$ where $a_1 = -2$
4. $a_n = -2a_{n-1}$ where $a_1 = -1$
5. $a_n = na_{n-1}$ where $a_1 = 1$
6. $a_n = (n-1)a_{n-1}$ where $a_1 = 1$
7. $a_n = 2a_{n-1} - 1$ where $a_1 = 0$
8. $a_n = 3a_{n-1} + 1$ where $a_1 = -1$
9. $a_n = a_{n-2} + 2a_{n-1}$ where $a_1 = -1$ and $a_2 = 0$
10. $a_n = 3a_{n-1} - a_{n-2}$ where $a_1 = 0$ and $a_2 = 2$
11. $a_n = a_{n-1} - a_{n-2}$ where $a_1 = 1$ and $a_2 = 3$
12. $a_n = a_{n-2} + a_{n-1} + 2$ where $a_1 = -4$ and $a_2 = -1$

Answer

1. $3, 8, 13, 18, 23$

3. $-2, -6, -18, -54, -162$

5. $1, 2, 6, 24, 120$

7. $0, -1, -3, -7, -15$

9. $-1, 0, -1, -2, -5$

11. $1, 3, 2, -1, -3$

Exercise 9.1.6

Find the indicated term.

1. $a_n = 2 - 7n; a_{12}$
2. $a_n = 3n - 8; a_{20}$
3. $a_n = -4(5)^{n-4}; a_7$
4. $a_n = 6\left(\frac{1}{3}\right)^{n-6}; a_9$
5. $a_n = 1 + \frac{1}{n}; a_{10}$
6. $a_n = (n+1)5^{n-3}; a_5$
7. $a_n = (-1)^n 2^{2n-3}; a_4$
8. $a_n = n(n-1)(n-2); a_6$
9. An investment of $\$4,500$ is made in an account earning 2% interest compounded quarterly. The balance in the account after n quarters is given by $a_n = 4500\left(1 + \frac{0.02}{4}\right)^n$. Find the amount in the account after each quarter for the first two years. Round to the nearest cent.
10. The value of a new car after n years is given by the formula $a_n = 18,000\left(\frac{3}{4}\right)^n$. Find and interpret a_7. Round to the nearest whole dollar.
11. The number of comparisons a computer algorithm makes to sort n names in a list is given by the formula $a_n = n \log_2 n$. Determine the number of comparisons it takes this algorithm to sort 2×10^6 (2 million) names.
12. The number of comparisons a computer algorithm makes to search n names in a list is given by the formula $a_n = n^2$ Determine the number of comparisons it takes this algorithm to search 2×10^6 (2 million) names.

Answer

1. -82

3. -500

5. $\frac{11}{10}$

7. 32

9. Year 1: QI: $\$4,522.50$; QII: $\$4,545.11$; QIII: $\$4,567.84$; QIV: $\$4,590.68$; Year 2: QI: $\$4,613.63$; QII: $\$4,636.70$; QIII: $\$4,659.88$; QIV: $\$4,683.18$

11. Approximately 4×10^7 comparisons

Exercise 9.1.7

Find the indicated partial sum.

1. $3, 5, 9, 17, 33, \ldots; S_4$
2. $-5, 7, -29, 79, -245, \ldots; S_4$
3. $4, 1, -4, -11, -20, \ldots; S_5$
4. $0, 2, 6, 12, 20, \ldots; S_3$
5. $a_n = 2 - 7n; S_5$
6. $a_n = 3n - 8; S_5$
7. $a_n = -4(5)^{n-4}; S_3$
8. $a_n = 6\left(\frac{1}{3}\right)^{n-6}; S_3$
9. $a_n = 1 + \frac{1}{n}; S_4$
10. $a_n = (n+1)5^{n-3}; S_3$
11. $a_n = (-1)^n 2^{2n-3}; S_5$
12. $a_n = n(n-1)(n-2); S_4$

Answer

1. 34

3. -30

5. -95

7. $-\frac{124}{125}$

9. $\frac{73}{12}$

11. $-\frac{205}{2}$

Exercise 9.1.8

Evaluate.

1. $\sum_{k=1}^{5} 3k$
2. $\sum_{k=1}^{6} 2k$
3. $\sum_{i=2}^{6} i^2$
4. $\sum_{i=0}^{4} (i+1)^2$
5. $\sum_{n=1}^{5} (-1)^{n+1}2^n$
6. $\sum_{n=5}^{10} (-1)^n n^2$
7. $\sum_{k=-2}^{2} \left(\frac{1}{2}\right)^k$
8. $\sum_{k=-4}^{0} \left(\frac{1}{3}\right)^k$
9. $\sum_{k=0}^{4} (-2)^{k+1}$
10. $\sum_{k=-1}^{3} (-3)^{k-1}$
11. $\sum_{n=1}^{5} 3$
12. $\sum_{n=1}^{7} -5$
13. $\sum_{k=-2}^{3} k(k+1)$
14. $\sum_{k=-2}^{2}(k-2)(k+2)$

Answer

1. 45

3. 90

5. 22

7. $\frac{31}{4}$

9. -22

11. 15

13. 22

Exercise 9.1.9

Write in expanded form.

1. $\sum_{n=1}^{\infty} \frac{n-1}{n}$
2. $\sum_{n=1}^{\infty} \frac{n}{2n-1}$
3. $\sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^{n-1}$
4. $\sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^{n+1}$
5. $\sum_{n=1}^{\infty} 3\left(\frac{1}{5}\right)^n$
6. $\sum_{n=0}^{\infty} 2\left(\frac{1}{3}\right)^n$
7. $\sum_{k=0}^{\infty} (-1)^k x^{k+1}$
8. $\sum_{k=1}^{\infty} (-1)^{k+1} x^{k-1}$
9. $\sum_{i=0}^{\infty} (-2)^{i+1} x^i$
10. $\sum_{i=1}^{\infty} (-3)^{i-1} x^{3i}$
11. $\sum_{k=1}^{\infty}(2k-1)x^{2k}$
12. $\sum_{k=1}^{\infty} \frac{kx^{k-1}}{k+1}$

Answer

1. $0+\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots$

3. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots$

5. $\frac{3}{5}+\frac{3}{25}+\frac{3}{125}+\frac{3}{625}+\cdots$
7. $x-x^2+x^3-x^4+\cdots$
9. $-2+4x-8x^2+16x^3-\ldots$
11. $x^2+3x^4+5x^6+7x^8+\cdots$

Exercise 9.1.10

Express the following series using sigma notation.

1. $x+2x^2+3x^3+4x^4+5x^5$
2. $\frac{1}{2}x^2+\frac{2}{3}x^3+\frac{3}{4}x^4+\frac{4}{5}x^5+\frac{5}{6}x^6$
3. $2+2^2x+2^3x^2+2^4x^3+2^5x^4$
4. $3x+3^2x^2+3^3x^3+3^4x^4+3^5x^5$
5. $2x+4x^2+8x^3+\cdots+2^nx^n$
6. $x+3x^2+9x^3+\cdots+3^nx^{n+1}$
7. $5+(5+d)+(5+2d)+\cdots+(5+nd)$
8. $2+2r^1+2r^2+\cdots+2r^{n-1}$
9. $\frac{3}{4}+\frac{3}{8}+\frac{3}{16}+\cdots+3\left(\frac{1}{2}\right)^n$
10. $\frac{8}{3}+\frac{16}{4}+\frac{32}{5}+\cdots+\frac{2^n}{n}$
11. A structured settlement yields an amount in dollars each year, represented by n, according to the formula $p_n=10,000(0.70)^{n-1}$. What is the total amount gained from the settlement after 5 years?
12. The first row of seating in a small theater consists of 14 seats. Each row thereafter consists of 2 more seats than the previous row. If there are 7 rows, how many total seats are in the theater?

Answer

1. $\sum_{k=1}^{5}kx^k$
3. $\sum_{k=1}^{5}2^kx^{k-1}$
5. $\sum_{k=1}^{n}2^kx^k$
7. $\sum_{k=0}^{n}(5+kd)$
9. $\sum_{k=2}^{n}3\left(\frac{1}{2}\right)^k$
11. $\$27,731$

Exercise 9.1.11

1. Research and discuss Fibonacci numbers as they are found in nature.
2. Research and discuss the life and contributions of Leonardo Fibonacci.
3. Explain the difference between a sequence and a series. Provide an example of each.

Answer

1. Answer may vary
3. Answer may vary

Footnotes

[1] A function whose domain is a set of consecutive natural numbers starting with 1.

[2] A sequence whose domain is the set of natural numbers $\{1,2,3,\ldots\}$.

[3] An equation that defines the nth term of a sequence commonly denoted using subscripts a_n.

[4] A sequence whose domain is $\{1,2,3,\ldots,k\}$ where k is a natural number.

[5] A formula that uses previous terms of a sequence to describe subsequent terms.

[6] The sum of the terms of a sequence.

[7] The sum of the terms of an infinite sequence denoted S_∞.

[8] The sum of the first n terms in a sequence denoted S_n.

[9] A sum denoted using the symbol Σ (upper case Greek letter sigma).

[10] Used when referring to sigma notation.

[11] The variable used in sigma notation to indicate the lower and upper bounds of the summation.

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9.2: Arithmetic Sequences and Series

Learning Objectives

- Identify the common difference of an arithmetic sequence.
- Find a formula for the general term of an arithmetic sequence.
- Calculate the nth partial sum of an arithmetic sequence.

Arithmetic Sequences

An **arithmetic sequence**[12], or **arithmetic progression**[13], is a sequence of numbers where each successive number is the sum of the previous number and some constant d.

$$a_n = a_{n-1} + d \quad \textit{Arithmetic sequence}$$

And because $a_n - a_{n-1} = d$, the constant d is called the **common difference**[14]. For example, the sequence of positive odd integers is an arithmetic sequence,

$$1, 3, 5, 7, 9, \ldots$$

Here $a_1 = 1$ and the difference between any two successive terms is 2. We can construct the general term $a_n = a_{n-1} + 2$ where,

$$\begin{gathered} a_1 = 1 \\ a_2 = a_1 + 2 = 1 + 2 = 3 \\ a_3 = a_2 + 2 = 3 + 2 = 5 \\ a_4 = a_3 + 2 = 5 + 2 = 7 \\ a_5 = a_4 + 2 = 7 + 2 = 9 \\ \vdots \end{gathered}$$

In general, given the first term a_1 of an arithmetic sequence and its common difference d, we can write the following:

$$\begin{aligned} &a_2 = a_1 + d \\ &a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2d \\ &a_4 = a_3 + d = (a_1 + 2d) + d = a_1 + 3d \\ &a_5 = a_4 + d = (a_1 + 3d) + d = a_1 + 4d \\ &\quad\vdots \end{aligned}$$

From this we see that any arithmetic sequence can be written in terms of its first element, common difference, and index as follows:

$$a_n = a_1 + (n-1)d \quad \textit{Arithmetic Sequence}$$

In fact, any general term that is linear in n defines an arithmetic sequence.

Example 9.2.1:

Find an equation for the general term of the given arithmetic sequence and use it to calculate its 100^{th} term: $7, 10, 13, 16, 19, \ldots$

Solution

Begin by finding the common difference,

$$d = 10 - 7 = 3$$

Note that the difference between any two successive terms is 3. The sequence is indeed an arithmetic progression where $a_1 = 7$ and $d = 3$.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= 7 + (n-1) \cdot 3 \\ &= 7 + 3n - 3 \\ &= 3n + 4 \end{aligned}$$

Therefore, we can write the general term $a_n = 3n + 4$. Take a minute to verify that this equation describes the given sequence. Use this equation to find the 100^{th} term:

$$a_{100} = 3(100) + 4 = 304$$

Answer:

$a_n = 3n + 4; a_{100} = 304$

The common difference of an arithmetic sequence may be negative.

Example $9.2.2$:

Find an equation for the general term of the given arithmetic sequence and use it to calculate its 75^{th} term: $6, 4, 2, 0, -2, \ldots$

Solution

Begin by finding the common difference,

$$d = 4 - 6 = -2$$

Next find the formula for the general term, here $a_1 = 6$ and $d = -2$.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= 6 + (n-1) \cdot (-2) \\ &= 6 - 2n + 2 \\ &= 8 - 2n \end{aligned}$$

Therefore, $a_n = 8 - 2n$ and the 75^{th} term can be calculated as follows:

$$\begin{aligned} a_{75} &= 8 - 2(75) \\ &= 8 - 150 \\ &= -142 \end{aligned}$$

Answer:

$a_n = 8 - 2n; a_{100} = -142$

The terms between given terms of an arithmetic sequence are called **arithmetic means**[15].

Example $9.2.3$:

Find all terms in between $a_1 = -8$ and $a_7 = 10$ of an arithmetic sequence. In other words, find all arithmetic means between the 1^{st} and 7^{th} terms.

Solution

Begin by finding the common difference d. In this case, we are given the first and seventh term:

$$\begin{aligned} a_n &= a_1 + (n-1)d \quad \color{#29abe2}{Use\ n = 7.} \\ a_7 &= a_1 + (7-1)d \\ a_7 &= a_1 + 6d \end{aligned}$$

Substitute $a_1 = -8$ and $a_7 = 10$ into the above equation and then solve for the common difference d.

$$\begin{aligned} 10 &= -8 + 6d \\ 18 &= 6d \\ 3 &= d \end{aligned}$$

Next, use the first term $a_1 = -8$ and the common difference $d = 3$ to find an equation for the nth term of the sequence.

$$\begin{aligned} a_n &= -8 + (n-1) \cdot 3 \\ &= -8 + 3n - 3 \\ &= -11 + 3n \end{aligned}$$

With $a_n = 3n - 11$, where n is a positive integer, find the missing terms.

$$\left.\begin{aligned} a_1 &= 3(1) - 11 = 3 - 11 = -8 \\ a_2 &= 3(2) - 11 = 6 - 11 = -5 \\ a_3 &= 3(3) - 11 = 9 - 11 = -2 \\ a_4 &= 3(4) - 11 = 12 - 11 = 1 \\ a_5 &= 3(5) - 11 = 15 - 11 = 4 \\ a_6 &= 3(6) - 11 = 18 - 11 = 7 \\ a_7 &= 3(7) - 11 = 21 - 111 = 10 \end{aligned}\right\} \color{#29abe2}{arithmetic\ means}$$

Answer:

$-5, -2, 1, 4, 7$

In some cases, the first term of an arithmetic sequence may not be given.

Example 9.2.4:

Find the general term of an arithmetic sequence where $a_3 = -1$ and $a_{10} = 48$.

Solution

To determine a formula for the general term we need a_1 and d. A linear system with these as variables can be formed using the given information and $a_n = a_1 + (n-1)d$:

$$\left\{\begin{array}{l} a_3 = a_1 + (3-1)d \\ a_{10} = a_1 + (10-1)d \end{array}\right. \Rightarrow \left\{\begin{array}{ll} -1 = a_1 + 2d & \color{#1ea7e1}{Use\, a_3 = -1.} \\ 48 = a_1 + 9d & \color{#1ea7e1}{Use\, a_{10} = 48.} \end{array}\right.$$

Eliminate a_1 by multiplying the first equation by -1 and add the result to the second equation.

$$\left\{\begin{array}{l} -1 = a_1 + 2d \\ 48 = a_1 + 9d \end{array}\right. \quad \overset{\color{#1ea7e1}{x(-1)}}{\Longrightarrow} + \left\{\begin{array}{l} 1 = -a_1 - 2d \\ 48 = a_1 + 9d \end{array}\right.$$

$$\begin{aligned} 49 &= 7d \\ 7 &= d \end{aligned}$$

Substitute $d = 7$ into $-1 = a_1 + 2d$ to find a_1.

$$\begin{aligned} -1 &= a_1 + 2(7) \\ -1 &= a_1 + 14 \\ -15 &= a_1 \end{aligned}$$

Next, use the first term $a_1 = -15$ and the common difference $d = 7$ to find a formula for the general term.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= -15 + (n-1) \cdot 7 \\ &= -15 + 7n - 7 \\ &= -22 + 7n \end{aligned}$$

Answer:

$a_n = 7n - 22$

Exercise 9.2.1

Find an equation for the general term of the given arithmetic sequence and use it to calculate its 100^{th} term: $\frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, \ldots$

Answer

$a_n = \frac{1}{2}n + 1; a_{100} = 51$

www.youtube.com/v/_ovjvVKtKpQ

Arithmetic Series

An **arithmetic series**[16] is the sum of the terms of an arithmetic sequence. For example, the sum of the first 5 terms of the sequence defined by $a_n = 2n - 1$ follows:

$$\begin{aligned} S_5 &= \sum_{n=1}^{5}(2n-1) \\ &= [2(\color{#1ea7e1}{1}\color{black}) - 1] + [2(\color{#1ea7e1}{2}\color{black}) - 1] + [2(\color{#1ea7e1}{3}\color{black}) - 1] + [2(\color{#1ea7e1}{4}\color{black}) - 1] + [2(\color{#1ea7e1}{5}\color{black}) - 1] \\ &= 1 + 3 + 5 + 7 + 9 \\ &= 25 \end{aligned}$$

Adding 5 positive odd integers, as we have done above, is managable. However, consider adding the first 100 positive odd integers. This would be very tedious. Therefore, we next develop a formula that can be used to calculate the sum of the first n terms, denoted S_n, of any arithmetic sequence. In general,

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + a_n$$

Writing this series in reverse we have,

$$S_n = a_n + (a_n - d) + (a_n - 2d) + \ldots + a_1$$

And adding these two equations together, the terms involving d add to zero and we obtain n factors of $a_1 + a_n$:

$$\begin{gathered} 2S_n = (a_1 + a_n) + (a_1 + a_n) + \ldots + (a_n + a_1) \\ 2S_n = n(a_1 + a_n) \end{gathered}$$

Dividing both sides by 2 leads us the formula for the nth partial sum of an **arithmetic sequence**[17]:

$$S_n = \frac{n(a_1 + a_n)}{2}$$

Use this formula to calculate the sum of the first 100 terms of the sequence defined by $a_n = 2n - 1$. Here $a_1 = 1$ and $a_{100} = 199$.

$$\begin{aligned} S_{100} &= \frac{100(a_1 + a_{100})}{2} \\ &= \frac{100(1 + 199)}{2} \\ &= 10,000 \end{aligned}$$

Example $9.2.5$:

Find the sum of the first 50 terms of the given sequence: $4, 9, 14, 19, 24, \ldots$

Solution

Determine whether or not there is a common difference between the given terms.

$$d = 9 - 4 = 5$$

Note that the difference between any two successive terms is 5. The sequence is indeed an arithmetic progression and we can write

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= 4 + (n-1)\cdot 5 \\ &= 4 + 5n - 5 \\ &= 5n - 1 \end{aligned}$$

Therefore, the general term is $a_n = 5n - 1$. To calculate the 50^{th} partial sum of this sequence we need the 1^{st} and the 50^{th} terms:

$$\begin{aligned} a_1 &= 4 \\ a_{50} &= 5(50) - 1 = 249 \end{aligned}$$

Next use the formula to determine the 50^{th} partial sum of the given arithmetic sequence.

$$\begin{aligned} S_n &= \frac{n(a_1 + a_n)}{2} \\ S_{50} &= \frac{50 \cdot (a_1 + a_{50})}{2} \\ &= \frac{50(4 + 249)}{2} \\ &= 25(253) \\ &= 6,325 \end{aligned}$$

Answer:

$S_{50} = 6,325$

Example $9.2.6$:

Evaluate: $\sum_{n=1}^{35}(10 - 4n)$.

Solution

In this case, we are asked to find the sum of the first 35 terms of an arithmetic sequence with general term $a_n = 10 - 4n$. Use this to determine the 1^{st} and the 35^{th} term.

$$\begin{aligned} a_1 &= 10 - 4(1) = 6 \\ a_{35} &= 10 - 4(35) = -130 \end{aligned}$$

Next use the formula to determine the 35^{th} partial sum.

$$\begin{aligned} S_n &= \frac{n\left(a_1 + a_n\right)}{2} \\ S_{35} &= \frac{35 \cdot \left(a_1 + a_{35}\right)}{2} \\ &= \frac{35[6 + (-130)]}{2} \\ &= \frac{35(-124)}{2} \\ &= -2,170 \end{aligned}$$

Answer:

$-2,170$

Example $9.2.7$:

The first row of seating in an outdoor amphitheater contains 26 seats, the second row contains 28 seats, the third row contains 30 seats, and so on. If there are 18 rows, what is the total seating capacity of the theater?

Figure $9.2.1$: Roman Theater

Solution

Begin by finding a formula that gives the number of seats in any row. Here the number of seats in each row forms a sequence:

$$26, 28, 30, \ldots$$

Note that the difference between any two successive terms is 2. The sequence is an arithmetic progression where $a_1 = 26$ and $d = 2$.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= 26 + (n-1) \cdot 2 \\ &= 26 + 2n - 2 \\ &= 2n + 24 \end{aligned}$$

Therefore, the number of seats in each row is given by $a_n = 2n + 24$. To calculate the total seating capacity of the 18 rows we need to calculate the 18^{th} partial sum. To do this we need the 1^{st} and the 18^{th} terms:

$$\begin{aligned} a_1 &= 26 \\ a_{18} &= 2(18) + 24 = 60 \end{aligned}$$

Use this to calculate the 18^{th} partial sum as follows:

$$\begin{aligned} S_n &= \frac{n\left(a_1 + a_n\right)}{2} \\ S_{18} &= \frac{18 \cdot \left(a_1 + a_{18}\right)}{2} \\ &= \frac{18(26 + 60)}{2} \\ &= 9(86) \\ &= 774 \end{aligned}$$

Answer:

There are 774 seats total.

Exercise $9.2.2$

Find the sum of the first 60 terms of the given sequence: $5, 0, -5, -10, -15, \ldots$

Answer

$S_{60} = -8,550$

www.youtube.com/v/baYq2a_kBKo

Key Takeaways

- An arithmetic sequence is a sequence where the difference d between successive terms is constant.
- The general term of an arithmetic sequence can be written in terms of its first term a_1, common difference d, and index n as follows: $a_n = a_1 + (n-1)d$.
- An arithmetic series is the sum of the terms of an arithmetic sequence.
- The nth partial sum of an arithmetic sequence can be calculated using the first and last terms as follows: $S_n = \frac{n(a_1 + a_n)}{2}$.

Exercise $9.2.3$

Write the first 5 terms of the arithmetic sequence given its first term and common difference. Find a formula for its general term.

1. $a_1 = 5; d = 3$
2. $a_1 = 12; d = 2$
3. $a_1 = 15; d = -5$
4. $a_1 = 7; d = -4$
5. $a_1 = \frac{1}{2}; d = 1$
6. $a_1 = \frac{2}{3}; d = \frac{1}{3}$
7. $a_1 = 1; d = -\frac{1}{2}$
8. $a_1 = -\frac{5}{4}; d = \frac{1}{4}$
9. $a_1 = 1.8; d = 0.6$
10. $a_1 = -4.3; d = 2.1$

Answer

1. $5, 8, 11, 14, 17; a_n = 3n + 2$

3. $15, 10, 5, 0, -5; a_n = 20 - 5n$

5. $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}; a_n = n - \frac{1}{2}$

7. $1, \frac{1}{2}, 0, -\frac{1}{2}, -1; a_n = \frac{3}{2} - \frac{1}{2}n$

9. $1.8, 2.4, 3, 3.6, 4.2; a_n = 0.6n + 1.2$

Exercise 9.2.4

Given the arithmetic sequence, find a formula for the general term and use it to determine the 100^{th} term.

1. $3, 9, 15, 21, 27, \ldots$
2. $3, 8, 13, 18, 23, \ldots$
3. $-3, -7, -11, -15, -19, \ldots$
4. $-6, -14, -22, -30, -38, \ldots$
5. $-5, -10, -15, -20, -25, \ldots$
6. $2, 4, 6, 8, 10, \ldots$
7. $\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \frac{13}{2}, \frac{17}{2}, \ldots$
8. $-\frac{1}{3}, \frac{2}{3}, \frac{5}{3}, \frac{8}{3}, \frac{11}{3}, \ldots$
9. $\frac{1}{3}, 0, -\frac{1}{3}, -\frac{2}{3}, -1, \ldots$
10. $\frac{1}{4}, -\frac{1}{2}, -\frac{5}{4}, -2, -\frac{11}{4}, \ldots$
11. $0.8, 2, 3.2, 4.4, 5.6, \ldots$
12. $4.4, 7.5, 10.6, 13.7, 16.8, \ldots$
13. Find the 50^{th} positive odd integer.
14. Find the 50^{th} positive even integer.
15. Find the 40^{th} term in the sequence that consists of every other positive odd integer: $1, 5, 9, 13, \ldots$
16. Find the 40^{th} term in the sequence that consists of every other positive even integer: $2, 6, 10, 14, \ldots$
17. What number is the term 355 in the arithmetic sequence $-15, -5, 5, 15, 25, \ldots$?
18. What number is the term -172 in the arithmetic sequence $4, -4, -12, -20, -28, \ldots$?
19. Given the arithmetic sequence defined by the recurrence relation $a_n = a_{n-1} + 5$ where $a_1 = 2$ and $n > 1$, find an equation that gives the general term in terms of a_1 and the common difference d.
20. Given the arithmetic sequence defined by the recurrence relation $a_n = a_{n-1} - 9$ where $a_1 = 4$ and $n > 1$, find an equation that gives the general term in terms of a_1 and the common difference d.

Answer

1. $a_n = 6n - 3; a_{100} = 597$

3. $a_n = 1 - 4n; a_{100} = -399$

5. $a_n = -5n; a_{100} = -500$

7. $a_n = 2n - \frac{3}{2}; a_{100} = \frac{397}{2}$

9. $a_n = \frac{2}{3} - \frac{1}{3}n; a_{100} = -\frac{98}{3}$

11. $a_n = 1.2n - 0.4; a_{100} = 119.6$

13. 99

15. 157

17. 38

19. $a_n = 5n - 3$

Exercise 9.2.5

Given the terms of an arithmetic sequence, find a formula for the general term.

1. $a_1 = 6$ and $a_7 = 42$
2. $a_1 = -\frac{1}{2}$ and $a_{12} = -6$
3. $a_1 = -19$ and $a_{26} = 56$
4. $a_1 = -9$ and $a_{31} = 141$
5. $a_1 = \frac{1}{6}$ and $a_{10} = \frac{37}{6}$

6. $a_1 = \frac{5}{4}$ and $a_{11} = \frac{65}{4}$
7. $a_3 = 6$ and $a_{26} = -40$
8. $a_3 = 16$ and $a_{15} = 76$
9. $a_4 = -8$ and $a_{23} = 30$
10. $a_5 = -7$ and $a_{37} = -135$
11. $a_4 = -\frac{23}{10}$ and $a_{21} = -\frac{25}{2}$
12. $a_3 = \frac{1}{8}$ and $a_{12} = -\frac{11}{2}$
13. $a_5 = 13.2$ and $a_{26} = 61.5$
14. $a_4 = -1.2$ and $a_{13} = 12.3$

Answer

1. $a_n = 6n$

3. $a_n = 3n - 22$

5. $a_n = \frac{2}{3}n - \frac{1}{2}$

7. $a_n = 12 - 2n$

9. $a_n = 2n - 16$

11. $a_n = \frac{1}{10} - \frac{3}{5}n$

13. $a_n = 2.3n + 1.7$

Exercise 9.2.6

Find all arithmetic means between the given terms.

1. $a_1 = -3$ and $a_6 = 17$
2. $a_1 = 5$ and $a_5 = -7$
3. $a_2 = 4$ and $a_8 = 7$
4. $a_5 = \frac{1}{2}$ and $a_9 = -\frac{7}{2}$
5. $a_5 = 15$ and $a_7 = 21$
6. $a_6 = 4$ and $a_{11} = -1$

Answer

1. $1, 5, 9, 13$

3. $\frac{9}{2}, 5, \frac{11}{2}, 6, \frac{13}{2}$

5. 18

Exercise 9.2.7

Calculate the indicated sum given the formula for the general term.

1. $a_n = 3n + 5; S_{100}$
2. $a_n = 5n - 11; S_{100}$
3. $a_n = \frac{1}{2} - n, S_{70}$
4. $a_n = 1 - \frac{3}{2}n; S_{120}$
5. $a_n = \frac{1}{2}n - \frac{3}{4}; S_{20}$
6. $a_n = n - \frac{3}{5}; S_{150}$
7. $a_n = 45 - 5n; S_{65}$
8. $a_n = 2n - 48; S_{95}$
9. $a_n = 4.4 - 1.6n; S_{75}$
10. $a_n = 6.5n - 3.3; S_{67}$

Answer

1. $15,650$
3. $-2,450$
5. 90
7. $-7,800$
9. $-4,230$

Exercise 9.2.8

Evaluate.

1. $\sum_{n=1}^{160}(3n)$
2. $\sum_{n=1}^{121}(-2n)$
3. $\sum_{n=1}^{250}(4n-3)$
4. $\sum_{n=1}^{120}(2n+12)$
5. $\sum_{n=1}^{70}(19-8n)$
6. $\sum_{n=1}^{220}(5-n)$
7. $\sum_{n=1}^{60}\left(\frac{5}{2}-\frac{1}{2}n\right)$
8. $\sum_{n=1}^{51}\left(\frac{3}{8}n+\frac{1}{4}\right)$
9. $\sum_{n=1}^{120}(1.5n-2.6)$
10. $\sum_{n=1}^{175}(-0.2n-1.6)$
11. Find the sum of the first 200 positive integers.
12. Find the sum of the first 400 positive integers.

Answer

1. $38,640$
3. $124,750$
5. $-18,550$
7. -765
9. $10,578$
11. $20,100$

Exercise 9.2.9

The general term for the sequence of positive odd integers is given by $a_n=2n-1$ and the general term for the sequence of positive even integers is given by $a_n=2n$. Find the following.

1. The sum of the first 50 positive odd integers.
2. The sum of the first 200 positive odd integers.
3. The sum of the first 50 positive even integers.
4. The sum of the first 200 positive even integers.
5. The sum of the first k positive odd integers.
6. The sum of the first k positive even integers.
7. The first row of seating in a small theater consists of 8 seats. Each row thereafter consists of 3 more seats than the previous row. If there are 12 rows, how many total seats are in the theater?
8. The first row of seating in an outdoor amphitheater contains 42 seats, the second row contains 44 seats, the third row contains 46 seats, and so on. If there are 22 rows, what is the total seating capacity of the theater?
9. If a triangular stack of bricks has 37 bricks on the bottom row, 34 bricks on the second row and so on with one brick on top. How many bricks are in the stack?
10. Each successive row of a triangular stack of bricks has one less brick until there is only one brick on top. How many rows does the stack have if there are 210 total bricks?

11. A 10-year salary contract offers $\$65,000$ for the first year with a $\$3,200$ increase each additional year. Determine the total salary obligation over the 10 year period.
12. A clock tower strikes its bell the number of times indicated by the hour. At one o'clock it strikes once, at two o'clock it strikes twice and so on. How many times does the clock tower strike its bell in a day?

Answer

1. $2,500$
3. $2,550$
5. k^2
7. 294 seats
9. 247 bricks
11. $\$794,000$

Exercise $9.2.10$

1. Is the Fibonacci sequence an arithmetic sequence? Explain.
2. Use the formula for the nth partial sum of an arithmetic sequence $S_n = \frac{n(a_1+a_n)}{2}$ and the formula for the general term $a_n = a_1 + (n-1)d$ to derive a new formula for the nth partial sum $S_n = \frac{n}{2}[2a_1 + (n-1)d]$. Under what circumstances would this formula be useful? Explain using an example of your own making.
3. Discuss methods for calculating sums where the index does not start at 1. For example, $\sum_{n=1}^{35}(3n+4) = 1,659$.
4. A famous story involves Carl Friedrich Gauss misbehaving at school. As punishment, his teacher assigned him the task of adding the first 100 integers. The legend is that young Gauss answered correctly within seconds. What is the answer and how do you think he was able to find the sum so quickly?

Answer

1. Answer may vary
3. Answer may vary

Footnotes

[12] A sequence of numbers where each successive number is the sum of the previous number and some constant d.

[13] Used when referring to an arithmetic sequence.

[14] The constant d that is obtained from subtracting any two successive terms of an arithmetic sequence; $a_n - a_{n-1} = d$.

[15] The terms between given terms of an arithmetic sequence.

[16] The sum of the terms of an arithmetic sequence.

[17] The sum of the first n terms of an arithmetic sequence given by the formula: $S_n = \frac{n(a_1+a_n)}{2}$.

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9.3: Geometric Sequences and Series

Learning Objectives

- Identify the common ratio of a geometric sequence.
- Find a formula for the general term of a geometric sequence.
- Calculate the nth partial sum of a geometric sequence.
- Calculate the sum of an infinite geometric series when it exists.

Geometric Sequences

A **geometric sequence**[18], or **geometric progression**[19], is a sequence of numbers where each successive number is the product of the previous number and some constant r.

$$a_n = ra_{n-1} \quad \textit{Geometric Sequence} \tag{9.3.1}$$

And because $\frac{a_n}{a_{n-1}} = r$, the constant factor r is called the **common ratio**[20]. For example, the following is a geometric sequence,

$$9, 27, 81, 243, 729 \ldots \tag{9.3.2}$$

Here $a_1 = 9$ and the ratio between any two successive terms is 3. We can construct the general term $a_n = 3a_{n-1}$ where,

$$\begin{aligned} a_1 &= 9 \\ a_2 &= 3a_1 = 3(9) = 27 \\ a_3 &= 3a_2 = 3(27) = 81 \\ a_4 &= 3a_3 = 3(81) = 243 \\ a_5 &= 3a_4 = 3(243) = 729 \\ &\vdots \end{aligned} \tag{9.3.3}$$

In general, given the first term a_1 and the common ratio r of a geometric sequence we can write the following:

$$\begin{aligned} a_2 &= ra_1 \\ a_3 &= ra_2 = r\left(a_1 r\right) = a_1 r^2 \\ a_4 &= ra_3 = r\left(a_1 r^2\right) = a_1 r^3 \\ a_5 &= ra_3 = r\left(a_1 r^3\right) = a_1 r^4 \\ &\vdots \end{aligned}$$

From this we see that any geometric sequence can be written in terms of its first element, its common ratio, and the index as follows:

$$a_n = a_1 r^{n-1} \quad \textit{Geometric Sequence}$$

In fact, any general term that is exponential in n is a geometric sequence.

Example 9.3.1:

Find an equation for the general term of the given geometric sequence and use it to calculate its 10^{th} term: $3, 6, 12, 24, 48 \ldots$

Solution

Begin by finding the common ratio,

$$r = \frac{6}{3} = 2$$

Note that the ratio between any two successive terms is 2. The sequence is indeed a geometric progression where $a_1 = 3$ and $r = 2$.

$$\begin{aligned} a_n &= a_1 r^{n-1} \\ &= 3(2)^{n-1} \end{aligned}$$

Therefore, we can write the general term $a_n = 3(2)^{n-1}$ and the 10^{th} term can be calculated as follows:

$$\begin{aligned} a_{10} &= 3(2)^{10-1} \\ &= 3(2)^9 \\ &= 1,536 \end{aligned}$$

Answer:

$a_n = 3(2)^{n-1}; a_{10} = 1,536$

The terms between given terms of a geometric sequence are called **geometric means**[21].

Example 9.3.2:

Find all terms between $a_1 = -5$ and $a_4 = -135$ of a geometric sequence. In other words, find all geometric means between the 1^{st} and 4^{th} terms.

Solution

Begin by finding the common ratio r. In this case, we are given the first and fourth terms:

$$\begin{aligned} a_n &= a_1 r^{n-1} \quad \textit{Use } n=4 \\ a_4 &= a_1 r^{4-1} \\ a_4 &= a_1 r^3 \end{aligned}$$

Substitute $a_1 = -5$ and $a_4 = -135$ into the above equation and then solve for r.

$$\begin{aligned} -135 &= -5r^3 \\ 27 &= r^3 \\ 3 &= r \end{aligned}$$

Next use the first term $a_1 = -5$ and the common ratio $r = 3$ to find an equation for the nth term of the sequence.

$$\begin{aligned} a_n &= a_1 r^{n-1} \\ a_n &= -5(3)^{n-1} \end{aligned}$$

Now we can use $a_n = -5(3)^{n-1}$ where n is a positive integer to determine the missing terms.

$$\left.\begin{aligned} a_1 &= -5(3)^{1-1} = -5 \cdot 3^0 = -5 \\ a_2 &= -5(3)^{2-1} = -5 \cdot 3^1 = -15 \\ a_3 &= -5(3)^{3-1} = -5 \cdot 3^2 = -45 \\ a_4 &= -5(3)^{4-1} = -5 \cdot 3^3 = -135 \end{aligned}\right\} \textit{geometric means}$$

Answer:

$-15, -45$

The first term of a geometric sequence may not be given.

Example 9.3.3:

Find the general term of a geometric sequence where $a_2 = -2$ and $a_5 = \frac{2}{125}$.

Solution

To determine a formula for the general term we need a_1 and r. A nonlinear system with these as variables can be formed using the given information and $a_n = a_1 r^{n-1}$ ::

$$\left\{\begin{aligned} a_2 &= a_1 r^{2-1} \\ a_5 &= a_1 r^{5-1} \end{aligned}\right. \Longrightarrow \left\{\begin{aligned} -2 &= a_1 r & &\textit{Use } a_2 = -2. \\ \tfrac{2}{125} &= a_1 r^4 & &\textit{Use } a_5 = \tfrac{2}{125}. \end{aligned}\right.$$

Solve for a_1 in the first equation,

$$\begin{aligned} -2 &= a_1 r \quad \Rightarrow \quad \tfrac{-2}{r} = a_1 \\ \tfrac{2}{125} &= a_1 r^4 \end{aligned}$$

Substitute $a_1 = \frac{-2}{r}$ into the second equation and solve for r.

$$\begin{aligned} \tfrac{2}{125} &= a_1 r^4 \\ \tfrac{2}{125} &= \left(\tfrac{-2}{r}\right) r^4 \\ \tfrac{2}{125} &= -2r^3 \\ -\tfrac{1}{125} &= r^3 \\ -\tfrac{1}{5} &= r \end{aligned}$$

Back substitute to find a_1:

$$\begin{aligned} a_1 &= \frac{-2}{r} \\ &= \frac{-2}{\left(-\frac{1}{5}\right)} \\ &= 10 \end{aligned}$$

Therefore, $a_1 = 10$ and $r = -\frac{1}{5}$.

Answer:

$a_n = 10\left(-\frac{1}{5}\right)^{n-1}$

Exercise 9.3.1

Find an equation for the general term of the given geometric sequence and use it to calculate its 6^{th} term: $2, \frac{4}{3}, \frac{8}{9}, \ldots$

Answer

$a_n = 2\left(\frac{2}{3}\right)^{n-1}; a_6 = \frac{64}{243}$

www.youtube.com/v/IGPEl9vloLY

Geometric Series

A **geometric series**[22] is the sum of the terms of a geometric sequence. For example, the sum of the first 5 terms of the geometric sequence defined by $a_n = 3^{n+1}$ follows:

$$\begin{aligned} S_5 &= \sum_{n=1}^{5} 3^{n+1} \\ &= 3^{1+1} + 3^{2+1} + 3^{3+1} + 3^{4+1} + 3^{5+1} \\ &= 3^2 + 3^3 + 3^4 + 3^5 + 3^6 \\ &= 9 + 27 + 81 + 3^5 + 3^6 \\ &= 1,089 \end{aligned}$$

Adding 5 positive integers is manageable. However, the task of adding a large number of terms is not. Therefore, we next develop a formula that can be used to calculate the sum of the first n terms of any geometric sequence. In general,

$$S_n = a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-1}$$

Multiplying both sides by r we can write,

$$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^n$$

Subtracting these two equations we then obtain,

$$\begin{aligned} S_n - rS_n &= a_1 - a_1 r^n \\ S_n(1-r) &= a_1\left(1 - r^n\right) \end{aligned}$$

Assuming $r \neq 1$ dividing both sides by $(1-r)$ leads us to the formula for the n**th partial sum of a geometric sequence**[23]:

$$S_n = \frac{a_1(1-r^n)}{1-r} (r \neq 1)$$

In other words, the nth partial sum of any geometric sequence can be calculated using the first term and the common ratio. For example, to calculate the sum of the first 15 terms of the geometric sequence defined by $a_n = 3^{n+1}$, use the formula with $a_1 = 9$ and $r = 3$.

$$\begin{aligned} S_{15} &= \frac{a_1\left(1 - r^{15}\right)}{1-r} \\ &= \frac{9 \cdot \left(1 - 3^{15}\right)}{1-3} \\ &= \frac{9(-14,348,906)}{-2} \\ &= 64,570,077 \end{aligned}$$

Example 9.3.4:

Find the sum of the first 10 terms of the given sequence: $4, -8, 16, -32, 64, \ldots$

Solution

Determine whether or not there is a common ratio between the given terms.

$$r = \frac{-8}{4} = -2$$

Note that the ratio between any two successive terms is -2; hence, the given sequence is a geometric sequence. Use $r = -2$ and the fact that $a_1 = 4$ to calculate the sum of the first 10 terms,

$$\begin{aligned} S_n &= \frac{a_1\left(1-r^n\right)}{1-r} \\ S_{10} &= \frac{4[1-(-2)^{10}]}{1-(-2)}] \\ &= \frac{4(1-1,024)}{1+2} \\ &= \frac{4(-1,023)}{3} \\ &= -1,364 \end{aligned}$$

Answer:

$S_{10} = -1,364$

Example 9.3.5:

Evaluate: $\sum_{n=1}^{6} 2(-5)^n$.

Solution

In this case, we are asked to find the sum of the first 6 terms of a geometric sequence with general term $a_n = 2(-5)^n$. Use this to determine the 1^{st} term and the common ratio r:

$$a_1 = 2(-5)^1 = -10$$

To show that there is a common ratio we can use successive terms in general as follows:

$$\begin{aligned} r &= \frac{a_n}{a_{n-1}} \\ &= \frac{2(-5)^n}{2(-5)^{n-1}} \\ &= (-5)^{n-(n-1)} \\ &= (-5)^1 \\ &= -5 \end{aligned}$$

Use $a_1 = -10$ and $r = -5$ to calculate the 6^{th} partial sum.

$$\begin{aligned} S_n &= \frac{a_1\left(1-r^n\right)}{1-r} \\ S_6 &= \frac{-10[1-(-5)^6]}{1-(-5)} \\ &= \frac{-10(1-15,625)}{1+5} \\ &= \frac{-10(-15,624)}{6} \\ &= 26,040 \end{aligned}$$

Answer:

$26,040$

Exercise 9.3.2

Find the sum of the first 9 terms of the given sequence: $-2, 1, -1/2, \ldots$

Answer

$S_9 = -\frac{171}{128}$

www.youtube.com/v/v-t3P95rWe8

If the common ratio r of an infinite geometric sequence is a fraction where $|r| < 1$ (that is $-1 < r < 1$), then the factor $(1 - r^n)$ found in the formula for the nth partial sum tends toward 1 as n increases. For example, if $r = \frac{1}{10}$ and $n = 2, 4, 6$ we have,

$$1 - \left(\frac{1}{10}\right)^2 = 1 - 0.01 = 0.99$$
$$1 - \left(\frac{1}{10}\right)^4 = 1 - 0.0001 = 0.9999$$
$$1 - \left(\frac{1}{10}\right)^6 = 1 - 0.00001 = 0.999999$$

Here we can see that this factor gets closer and closer to 1 for increasingly larger values of n. This illustrates the idea of a limit, an important concept used extensively in higher-level mathematics, which is expressed using the following notation:

$$\lim_{n \to \infty} (1 - r^n) = 1 \text{ where } |r| < 1$$

This is read, "the limit of $(1 - r^n)$ as n approaches infinity equals 1." While this gives a preview of what is to come in your continuing study of mathematics, at this point we are concerned with developing a formula for special infinite geometric series. Consider the nth partial sum of any geometric sequence,

$$S_n = \frac{a_1(1-r^n)}{1-r} = \frac{a_1}{1-r}(1 - r^n)$$

If $|r| < 1$ then the limit of the partial sums as n approaches infinity exists and we can write,

$$S_n = \frac{a_1}{1-r}(1 - r^n) \quad n \overset{\Longrightarrow}{\rightarrow} \infty \quad S_\infty = \frac{a_1}{1-r} \cdot 1$$

Therefore, a **convergent geometric series**[24] is an infinite geometric series where $|r| < 1$; its sum can be calculated using the formula:

$$S_\infty = \frac{a_1}{1-r}$$

Example 9.3.6:

Find the sum of the infinite geometric series: $\frac{3}{2} + \frac{1}{2} + \frac{1}{6} + \frac{1}{18} + \frac{1}{54} + \ldots$

Solution

Determine the common ratio, Since the common ratio $r = \frac{1}{3}$ is a fraction between -1 and 1, this is a convergent geometric series. Use the first term $a_1 = \frac{3}{2}$ and the common ratio to calculate its sum

$$\begin{aligned} S_\infty &= \frac{a_1}{1-r} \\ &= \frac{\frac{3}{2}}{1-\left(\frac{1}{3}\right)} \\ &= \frac{\frac{3}{3}}{\frac{2}{3}} \\ &= \frac{3}{2} \cdot \frac{3}{2} \\ &= \frac{9}{4} \end{aligned}$$

Answer:

$S_\infty = \frac{9}{4}$

Note

In the case of an infinite geometric series where $|r| \geq 1$, the series diverges and we say that there is no sum. For example, if $a_n = (5)^{n-1}$ then $r = 5$ and we have

$$S_{\infty} = \sum_{n=1}^{\infty} (5)^{n-1} = 1 + 5 + 25 + \cdots$$

We can see that this sum grows without bound and has no sum.

Exercise 9.3.3

Find the sum of the infinite geometric series: $\sum_{n=1}^{\infty} -2\left(\frac{5}{9}\right)^{n-1}$

Answer

$-\frac{9}{2}$

www.youtube.com/v/KxsPVUyle_A

A repeating decimal can be written as an infinite geometric series whose common ratio is a power of $1/10$. Therefore, the formula for a convergent geometric series can be used to convert a repeating decimal into a fraction.

Example 9.3.7:

Write as a fraction: $1.181818\ldots$

Solution

Begin by identifying the repeating digits to the right of the decimal and rewrite it as a geometric progression.

$$\begin{aligned} 0.181818\ldots &= 0.18 + 0.0018 + 0.000018 + \ldots \\ &= \frac{18}{100} + \frac{18}{10,000} + \frac{18}{1,000,000} + \ldots \end{aligned}$$

In this form we can determine the common ratio,

$$\begin{aligned} r &= \frac{\frac{18}{10,000}}{\frac{18}{100}} \\ &= \frac{18}{10,000} \times \frac{100}{18} \\ &= \frac{1}{100} \end{aligned}$$

Note that the ratio between any two successive terms is $\frac{1}{100}$. Use this and the fact that $a_1 = \frac{18}{100}$ to calculate the infinite sum:

$$\begin{aligned} S_{\infty} &= \frac{a_1}{1-r} \\ &= \frac{\frac{18}{100}}{1-\left(\frac{1}{100}\right)} \\ &= \frac{\frac{18}{100}}{\frac{90}{100}} \\ &= \frac{18}{100} \cdot \frac{100}{99} \\ &= \frac{2}{11} \end{aligned}$$

Therefore, $0.181818\ldots = \frac{2}{11}$ and we have,

$$1.181818\ldots = 1 + \frac{2}{11} = 1\frac{2}{11}$$

Answer:

$1\frac{2}{11}$

Example 9.3.8:

A certain ball bounces back to two-thirds of the height it fell from. If this ball is initially dropped from 27 feet, approximate the total distance the ball travels.

Solution

We can calculate the height of each successive bounce:

$$\begin{aligned}27\cdot\tfrac{2}{3}&=18\text{ feet} && \color{#1ca3d8}{\textit{Height of the first bounce}}\\18\cdot\tfrac{2}{3}&=12\text{ feet} && \color{#1ca3d8}{\textit{Height of the second bounce}}\\12\cdot\tfrac{2}{3}&=8\text{ feet} && \color{#1ca3d8}{\textit{Height of the third bounce}}\end{aligned}$$

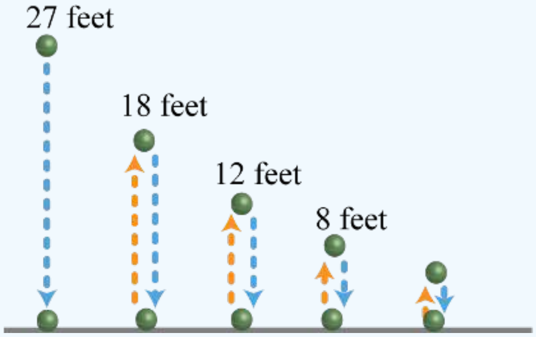

Figure $9.3.1$

The total distance that the ball travels is the sum of the distances the ball is falling and the distances the ball is rising. The distances the ball falls forms a geometric series,

$$27+18+12+\ldots \quad \color{#1ca3d8}{\textit{Distance the ball is falling}}$$

where $a_1=27$ and $r=\frac{2}{3}$. Because r is a fraction between -1 and 1, this sum can be calculated as follows:

$$\begin{aligned}S_\infty&=\frac{a_1}{1-r}\\&=\frac{27}{1-\frac{2}{3}}\\&=\frac{27}{\frac{1}{3}}\\&=81\end{aligned}$$

Therefore, the ball is falling a total distance of 81 feet. The distances the ball rises forms a geometric series,

$$18+12+8+\cdots \quad \color{#1ca3d8}{\textit{Distance the ball is rising}}$$

where $a_1=18$ and $r=\frac{2}{3}$. Calculate this sum in a similar manner:

$$\begin{aligned}S_\infty&=\frac{a_1}{1-r}\\&=\frac{18}{1-\frac{2}{3}}\\&=\frac{18}{\frac{1}{3}}\\&=54\end{aligned}$$

Therefore, the ball is rising a total distance of 54 feet. Approximate the total distance traveled by adding the total rising and falling distances:

$$81+54=135\text{ feet}$$

Answer:

135 feet

Key Takeaways

- A geometric sequence is a sequence where the ratio r between successive terms is constant.
- The general term of a geometric sequence can be written in terms of its first term a_1, common ratio r, and index n as follows: $a_n = a_1 r^{n-1}$.
- A geometric series is the sum of the terms of a geometric sequence.
- The nth partial sum of a geometric sequence can be calculated using the first term a_1 and common ratio r as follows: $S_n = \frac{a_1(1-r^n)}{1-r}$.
- The infinite sum of a geometric sequence can be calculated if the common ratio is a fraction between -1 and 1 (that is $|r| < 1$) as follows: $S_\infty = \frac{a_1}{1-r}$. If $|r| \geq 1$, then no sum exists.

Exercise 9.3.4

Write the first 5 terms of the geometric sequence given its first term and common ratio. Find a formula for its general term.

1. $a_1 = 1; r = 5$
2. $a_1 = 1; r = 3$
3. $a_1 = 2; r = 3$
4. $a_1 = 5; r = 4$
5. $a_1 = 2; r = -3$
6. $a_1 = 6; r = -2$
7. $a_1 = 3; r = \frac{2}{3}$
8. $a_1 = 6; r = \frac{1}{2}$
9. $a_1 = 1.2; r = 0.6$
10. $a_1 = -0.6; r = -3$

Answer

1. $1, 5, 25, 125, 625; a_n = 5^{n-1}$

3. $2, 6, 18, 54, 162; a_n = 2(3)^{n-1}$

5. $2, -6, 18, -54, 162; a_n = 2(-3)^{n-1}$

7. $3, 2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27}; a_n = 3\left(\frac{2}{3}\right)^{n-1}$

9. $1.2, 0.72, 0.432, 0.2592, 0.15552; a_n = 1.2(0.6)^{n-1}$

Exercise 9.3.5

Given the geometric sequence, find a formula for the general term and use it to determine the 5^{th} term in the sequence.

1. $7, 28, 112, \ldots$
2. $-2, -10, -50, \ldots$
3. $2, \frac{1}{2}, \frac{1}{8}, \ldots$
4. $1, \frac{2}{5}, \frac{4}{25}, \ldots$
5. $8, 4, 2, \ldots$
6. $6, 2, \frac{2}{3}, \ldots$
7. $-1, \frac{2}{3}, -\frac{4}{9}, \ldots$
8. $2, -\frac{3}{2}, \frac{9}{8}, \ldots$
9. $\frac{1}{3}, -2, 12, \ldots$
10. $\frac{2}{5}, -2, 10, \ldots$
11. $-3.6, -4.32, -5.184, \ldots$
12. $0.8, -2.08, 5.408, \ldots$
13. Find the general term and use it to determine the 20^{th} term in the sequence: $1, \frac{x}{2}, \frac{x^2}{4}, \ldots$
14. Find the general term and use it to determine the 20^{th} term in the sequence: $2, -6x, 18x^2 \ldots$
15. The number of cells in a culture of a certain bacteria doubles every 4 hours. If 200 cells are initially present, write a sequence that shows the population of cells after every nth 4-hour period for one day. Write a formula that gives the number of cells after any 4-hour period.

16. A certain ball bounces back at one-half of the height it fell from. If this ball is initially dropped from 12 feet, find a formula that gives the height of the ball on the nth bounce and use it to find the height of the ball on the 6^{th} bounce.
17. Given a geometric sequence defined by the recurrence relation $a_n = 4a_{n-1}$ where $a_1 = 2$ and $n > 1$, find an equation that gives the general term in terms of a_1 and the common ratio r.
18. Given the geometric sequence defined by the recurrence relation $a_n = 6a_{n-1}$ where $a_1 = \frac{1}{2}$ and $n > 1$, find an equation that gives the general term in terms of a_1 and the common ratio r.

Answer

1. $a_n = 7(4)^{n-1}, a_5 = 1,792$
3. $a_n = 2\left(\frac{1}{4}\right)^{n-1}, a_5 = \frac{1}{128}$
5. $a_n = 8\left(\frac{1}{2}\right)^{n-1}, a_5 = \frac{1}{2}$
7. $a_n = -\left(-\frac{2}{3}\right)^{n-1}, a_5 = -\frac{16}{81}$
9. $a_n = \frac{1}{3}(-6)^{n-1}, a_5 = 432$
11. $a_n = -3.6(1.2)^{n-1}, a_5 = -7.46496$
13. $a_n = \left(\frac{x}{2}\right)^{n-1}; a_{20} = \frac{x^{19}}{2^{19}}$
15. 400 cells; 800 cells; $1,600$ cells; $3,200$ cells; $6,400$ cells; $12,800$ cells; $p_n = 400(2)^{n-1}$ cells
17. $a_n = 2(4)^{n-1}$

Exercise 9.3.6

Given the terms of a geometric sequence, find a formula for the general term.

1. $a_1 = -3$ and $a_6 = -96$
2. $a_1 = 5$ and $a_4 = -40$
3. $a_1 = -2$ and $a_8 = -\frac{1}{64}$
4. $a_1 = \frac{3}{4}$ and $a_4 = -\frac{1}{36}$
5. $a_2 = 18$ and $a_5 = 486$
6. $a_2 = 10$ and $a_7 = 320$
7. $a_4 = -2$ and $a_9 = 64$
8. $a_3 = -\frac{4}{3}$ and $a_6 = \frac{32}{81}$
9. $a_5 = 153.6$ and $a_8 = 9,830.4$
10. $a_4 = -2.4 \times 10^{-3}$ and $a_9 = -7.68 \times 10^{-7}$

Answer

1. $a_n = -3(2)^{n-1}$
3. $a_n = -2\left(\frac{1}{2}\right)^{n-1}$
5. $a_n = 6(3)^{n-1}$
7. $a_n = \frac{1}{4}(-2)^{n-1}$
9. $a_n = 0.6(4)^{n-1}$

Exercise 9.3.7

Find all geometric means between the given terms.

1. $a_1 = 2$ and $a_4 = 250$
2. $a_1 = \frac{1}{3}$ and $a_6 = -\frac{1}{96}$
3. $a_2 = -20$ and $a_5 = -20,000$
4. $a_3 = 49$ and $a_6 = -16,807$

Answer

1. $10, 50$
3. $-200; -2,000$

Exercise $9.3.8$

Calculate the indicated sum.

1. $a_n = 2^{n+1}; S_{12}$
2. $a_n = (-2)^{n+1}; S_{12}$
3. $a_n = \left(\frac{1}{2}\right)^n; S_7$
4. $a_n = \left(\frac{2}{3}\right)^{n-1}; S_6$
5. $a_n = 5(-3)^{n-1}; S_5$
6. $a_n = -7(-4)^n; S_5$
7. $a_n = 2\left(-\frac{1}{4}\right)^n; S_5$
8. $a_n = \frac{1}{3}(2)^{n+1}; S_{10}$
9. $\sum_{n=1}^{5} 5^n$
10. $\sum_{n=1}^{6} (-4)^n$
11. $\sum_{k=1}^{10} 2^{k+1}$
12. $\sum_{k=1}^{14} 2^{k-1}$
13. $\sum_{k=1}^{10} -2(3)^k$
14. $\sum_{k=1}^{8} 5(-2)^k$
15. $\sum_{n=1}^{5} 2\left(\frac{1}{2}\right)^{n+2}$
16. $\sum_{n=1}^{4} -3\left(\frac{2}{3}\right)^n$
17. $a_n = \left(\frac{1}{5}\right)^n; S_\infty$
18. $a_n = \left(\frac{2}{3}\right)^{n-1}; S_\infty$
19. $a_n = 2\left(-\frac{3}{4}\right)^{n-1}; S_\infty$
20. $a_n = 3\left(-\frac{1}{6}\right)^n; S_\infty$
21. $a_n = -2\left(\frac{1}{2}\right)^{n+1}; S_\infty$
22. $a_n = -\frac{1}{3}\left(-\frac{1}{2}\right)^n; S_\infty$
23. $\sum_{n=1}^{\infty} 2\left(\frac{1}{3}\right)^{n-1}$
24. $\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$
25. $\sum_{n=1}^{\infty} 3(2)^{n-2}$
26. $\sum_{n=1}^{\infty} -\frac{1}{4}(3)^{n-2}$
27. $\sum_{n=1}^{\infty} \frac{1}{2}\left(-\frac{1}{6}\right)^n$
28. $\sum_{n=1}^{\infty} \frac{1}{3}\left(-\frac{2}{5}\right)^n$

Answer

1. $16,380$
3. $\frac{127}{128}$
5. 305
7. $-\frac{205}{512}$
9. $3,905$
11. $4,092$
13. $-177,144$
15. $\frac{31}{64}$
17. $\frac{1}{4}$
19. $\frac{8}{7}$
21. -1
23. 3

25. No sum

27. $-\frac{1}{14}$

Exercise $9.3.9$

Write as a mixed number.

1. $1.222\ldots$
2. $5.777\ldots$
3. $2.252525\ldots$
4. $3.272727\ldots$
5. $1.999\ldots$
6. $1.090909\ldots$
7. Suppose you agreed to work for pennies a day for 30 days. You will earn 1 penny on the first day, 2 pennies the second day, 4 pennies the third day, and so on. How many total pennies will you have earned at the end of the 30 day period? What is the dollar amount?
8. An initial roulette wager of \$$100$ is placed (on red) and lost. To make up the difference, the player doubles the bet and places a \$$200$ wager and loses. Again, to make up the difference, the player doubles the wager to \$$400$ and loses. If the player continues doubling his bet in this manner and loses 7 times in a row, how much will he have lost in total?
9. A certain ball bounces back to one-half of the height it fell from. If this ball is initially dropped from 12 feet, approximate the total distance the ball travels.
10. A golf ball bounces back off of a cement sidewalk three-quarters of the height it fell from. If the ball is initially dropped from 8 meters, approximate the total distance the ball travels.
11. A structured settlement yields an amount in dollars each year, represented by n, according to the formula $p_n = 6,000(0.80)^{n-1}$. What is the total amount gained from the settlement after 10 years?
12. Beginning with a square, where each side measures 1 unit, inscribe another square by connecting the midpoints of each side. Continue inscribing squares in this manner indefinitely, as pictured:

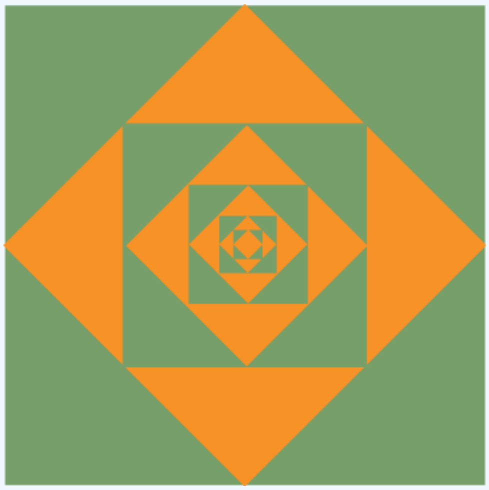

Figure $9.3.2$

Find the sum of the area of all squares in the figure. (Hint: Begin by finding the sequence formed using the areas of each square.)

Answer

1. $1\frac{2}{9}$

3. $2\frac{25}{99}$

5. 2

7. $1,073,741,823$ pennies; $\$10,737,418.23$

9. 36 feet

11. $\$26,778.77$

Exercise $9.3.10$

Categorize the sequence as arithmetic, geometric, or neither. Give the common difference or ratio, if it exists.

1. $-12, 24, -48, \ldots$
2. $-7, -5, -3, \ldots$
3. $-3, -11, -19, \ldots$
4. $4, 9, 16, \ldots$
5. $2, \frac{3}{2}, \frac{4}{3}, \ldots$
6. $\frac{4}{3}, \frac{8}{9}, \frac{16}{27}, \ldots$
7. $\frac{1}{6}, -\frac{1}{6}, -\frac{1}{2}, \ldots$
8. $\frac{1}{3}, \frac{1}{4}, \frac{3}{16}, \ldots$
9. $\frac{1}{2}, \frac{1}{4}, \frac{1}{6} \ldots$
10. $-\frac{1}{10}, -\frac{1}{5}, -\frac{3}{10}, \ldots$
11. $1.26, 0.252, 0.0504, \ldots$
12. $0.02, 0.08, 0.18, \ldots$
13. $1, -1, 1, -1, \ldots$
14. $0, 0, 0, \ldots$

Answer

1. Geometric; $r=-2$

3. Arithmetic; $d=-8$

5. Neither

7. Arithmetic; $d=-\frac{1}{3}$

9. Neither

11. Geometric; $r=0.2$

13. Geometric; $r=-1$

Exercise $9.3.11$

Categorize the sequence as arithmetic or geometric, and then calculate the indicated sum.

1. $a_n=3(5)^{n-1}; S_8$
2. $a_n=5-6n; S_{22}$
3. $a_n=2n; S_{14}$
4. $a_n=2^n; S_{10}$
5. $a_n=-2\left(\frac{1}{7}\right)^{n-1}; S_\infty$
6. $a_n=-2+\frac{1}{7}n; S_8$

Answer

1. Geometric; $292,968$

3. Arithmetic; 210

5. Geometric; $-\frac{7}{3}$

Exercise 9.3.12

Calculate the indicated sum.

1. $\sum_{n=1}^{50}(3n-5)$
2. $\sum_{n=1}^{25}(4-8n)$
3. $\sum_{n=1}^{12}(-2)^{n-1}$
4. $\sum_{n=1}^{\infty} 5\left(-\frac{1}{2}\right)^{n-1}$
5. $\sum_{n=1}^{40} 5$
6. $\sum_{n=1}^{\infty} 0.6^{n}$

Answer

1. $3,575$

3. $-1,365$

5. 200

Exercise 9.3.13

1. Use the techniques found in this section to explain why $0.999\ldots=1$.
2. Construct a geometric sequence where $r=1$. Explore the nth partial sum of such a sequence. What conclusions can we make?

Answer

1. Answer may vary

Footnotes

[18] A sequence of numbers where each successive number is the product of the previous number and some constant r.

[19] Used when referring to a geometric sequence.

[20] The constant r that is obtained from dividing any two successive terms of a geometric sequence; $\frac{a_n}{a_{n-1}}=r$.

[21] The terms between given terms of a geometric sequence.

[22] The sum of the terms of a geometric sequence.

[23] The sum of the first n terms of a geometric sequence, given by the formula: $S_n=\frac{a_1(1-r^n)}{1-r}, r\neq 1$.

[24] An infinite geometric series where $|r|<1$ whose sum is given by the formula:$S_\infty=\frac{a_1}{1-r}$.

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9.4: Binomial Theorem

Learning Objectives

- Evaluate expressions involving factorials.
- Calculate binomial coefficients.
- Expand powers of binomials using the binomial theorem.

Factorials and the Binomial Coefficient

We begin by defining the *factorial*[25] of a natural number n, denoted $n!$, as the product of all natural numbers less than or equal to n.

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$

For example,

$$\begin{aligned}
&7! = 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 5,040 && \textit{Seven factorial}\\
&5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120 && \textit{Five factorial}\\
&3! = 3\cdot 2\cdot 1 = 6 && \textit{Three factorial}\\
&1! = 1 = 1 && \textit{One factorial}
\end{aligned}$$

We define *zero factorial*[26] to be equal to 1,

$$0! = 1 \quad \textit{Zero factorial}$$

The factorial of a negative number is not defined.

> On most modern calculators you will find a factorial function. Some calculators do not provide a button dedicated to it. However, it usually can be found in the menu system if one is provided.

The factorial can also be expressed using the following recurrence relation,

$$n! = n(n-1)!$$

For example, the factorial of 8 can be expressed as the product of 8 and $7!$:

$$\begin{aligned}
8! &= 8\cdot 7!\\
&= 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1\\
&= 40,320
\end{aligned}$$

When working with ratios involving factorials, it is often the case that many of the factors cancel.

Example 9.4.1

Evaluate: $\frac{12!}{6!}$.

Solution

$$\begin{aligned}
\frac{12!}{6!} &= \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}\\
&= \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot \cancel{6!}}{\cancel{6!}}\\
&= 12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\\
&= 665,280
\end{aligned}$$

Answer

$665,280$

The *binomial coefficient*[27], denoted ${}_nC_k = \left(\begin{array}{c} n \\ k \end{array}\right)$, is read “$n$ choose k” and is given by the following formula:

$${}_nC_k = \left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!}$$

This formula is very important in a branch of mathematics called combinatorics. It gives the number of ways k elements can be chosen from a set of n elements where order does not matter. In this section, we are concerned with the ability to calculate this quantity.

Example 9.4.2

Calculate $\left(\begin{array}{c} 7 \\ 3 \end{array} \right)$.

Solution

Use the formula for the binomial coefficient,

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!}$$

where $n = 7$ and $k = 3$. After substituting, look for factors to cancel.

$$\begin{aligned} \left(\begin{array}{c} 7 \\ 3 \end{array} \right) &= \frac{7!}{3!(7-3)!} \\ &= \frac{7!}{3!4!} \\ &= \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4!}}{3! \cancel{4!}} \\ &= \frac{210}{6} \\ &= 35 \end{aligned}$$

Answer:

35

Check the menu system of your calculator for a function that calculates this quantity. Look for the notation ${}_nC_k$ in the probability subsection.

Exercise 9.4.1

Calculate $\left(\begin{array}{c} 8 \\ 5 \end{array} \right)$.

Answer

56

www.youtube.com/v/Rpb8KD1HQGc

Consider the following binomial raised to the 3^{rd} power in its expanded form:

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

Compare it to the following calculations,

$$\begin{aligned} \left(\begin{array}{c} 3 \\ 0 \end{array} \right) &= \frac{3!}{0!(3-0)!} = \frac{3!}{1 \cdot 3!} = 1 \\ \left(\begin{array}{c} 3 \\ 1 \end{array} \right) &= d\frac{3!}{1!(3-1)!} = \frac{3 \cdot 2!}{1 \cdot 2!} = 3 \end{aligned}$$

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \frac{3!}{2!(3-2)!} = \frac{3 \cdot 2!}{2!} = 3$$
$$\begin{pmatrix} 3 \\ 3 \end{pmatrix} = \frac{3!}{3!(3-3)!} = \frac{3!}{3!0!} = 1$$

Notice that there appears to be a connection between these calculations and the coefficients of the expanded binomial. This observation is generalized in the next section.

Binomial Theorem

Consider expanding $(x+2)^5$:

$$(x+2)^5 = (x+2)(x+2)(x+2)(x+2)(x+2)$$

One quickly realizes that this is a very tedious calculation involving multiple applications of the distributive property. The **binomial theorem**[28] provides a method of expanding binomials raised to powers without directly multiplying each factor:

$$(x+y)^n = \begin{pmatrix} n \\ 0 \end{pmatrix} x^n y^0 + \begin{pmatrix} n \\ 1 \end{pmatrix} x^{n-1} y^1 + \begin{pmatrix} n \\ 2 \end{pmatrix} x^{n-2} y^2 + \ldots + \begin{pmatrix} n \\ n-1 \end{pmatrix} x^1 y^{n-1}$$

More compactly we can write,

$$(x+y)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} x^{n-k} y^k \quad \textit{Binomial theorem}$$

Example 9.4.3

Expand using the binomial theorem: $(x+2)^5$.

Solution

Use the binomial theorem where $n=5$ and $y=2$.

$$(x+2)^5 = \begin{pmatrix} 5 \\ 0 \end{pmatrix} x^5 2^0 + \begin{pmatrix} 5 \\ 1 \end{pmatrix} x^4 2^1 + \begin{pmatrix} 5 \\ 2 \end{pmatrix} x^3 2^2 + \begin{pmatrix} 5 \\ 3 \end{pmatrix} x^2 2^3 + \begin{pmatrix} 5 \\ 4 \end{pmatrix} x^1 2^4$$

Sometimes it is helpful to identify the pattern that results from applying the binomial theorem. Notice that powers of the variable x start at 5 and decrease to zero. The powers of the constant term start at 0 and increase to 5. The binomial coefficients can be calculated off to the side and are left to the reader as an exercise.

$$\begin{aligned} (x+2)^5 &= \begin{pmatrix} 5 \\ 0 \end{pmatrix} x^5 2^0 + \begin{pmatrix} 5 \\ 1 \end{pmatrix} x^4 2^1 + \begin{pmatrix} 5 \\ 2 \end{pmatrix} x^3 2^2 + \begin{pmatrix} 5 \\ 3 \end{pmatrix} x^2 2^3 + \begin{pmatrix} 5 \\ 4 \end{pmatrix} x^1 2^4 \\ &= 1x^5 \times 1 + 5x^4 \times 2 + 10x^3 \times 4 + 10x^2 \times 8 + 5x^1 \times 16 + 1 \times 1 \\ &= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32 \end{aligned}$$

Answer

$x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$

The binomial may have negative terms, in which case we will obtain an alternating series.

Example 9.4.4

Expand using the binomial theorem: $(u-2v)^4$.

Solution

Use the binomial theorem where $n=4, x=u$, and $y=-2v$ and then simplify each term.

$$\begin{aligned} (u-2v)^4 &= \begin{pmatrix} 4 \\ 0 \end{pmatrix} u^4(-2v)^0 + \begin{pmatrix} 4 \\ 1 \end{pmatrix} u^3(-2v)^1 + \begin{pmatrix} 4 \\ 2 \end{pmatrix} u^2(-2v)^2 + \begin{pmatrix} 4 \\ 3 \end{pmatrix} u^1(-2v)^3 + \begin{pmatrix} 4 \\ 4 \end{pmatrix} u^0(-2v)^4 \\ &= 1 \times u^4 \times 1 + 4u^3(-2v) + 6u^2\left(4v^2\right) + 4u\left(-8v^3\right) + 16v^4 \\ &= u^4 - 8u^3 v + 24u^2 v^2 - 32uv^3 + 16v^4 \end{aligned}$$

Answer

$u^4 - 8u^3v + 24u^2v^2 - 32uv^3 + 16v^4$

Exercise 9.4.2

Expand using the binomial theorem: $(a^2 - 3)^4$

Answer

$a^8 - 12a^6 + 54a^4 - 108a^2 + 81$

www.youtube.com/v/wICbqmoa4T4

Next we study the coefficients of the expansions of $(x+y)^n$ starting with $n = 0$:

$$\begin{aligned}
(x+y)^0 &= 1 \\
(x+y)^1 &= x+y \\
(x+y)^2 &= x+y \\
(x+y)^3 &= x^2+2xy+y^2 \\
(x+y)^3 &= x^3+3x^2y+3xy^2+y^3 \\
(x+y)^4 &= x^4+4x^3y+6x^2y^2+4xy^3+y^4
\end{aligned}$$

Write the coefficients in a triangular array and note that each number below is the sum of the two numbers above it, always leaving a 1 on either end.

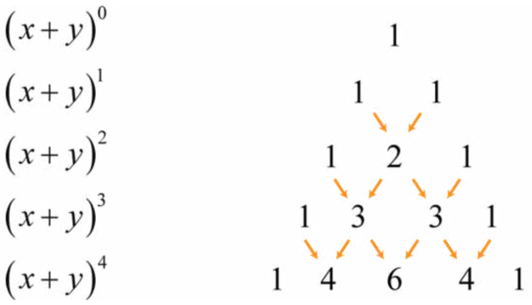

Figure $9.4.1$

This is **Pascal’s triangle**[29]; it provides a quick method for calculating the binomial coefficients. Use this in conjunction with the binomial theorem to streamline the process of expanding binomials raised to powers. For example, to expand $(x-1)^6$ we would need two more rows of Pascal’s triangle,

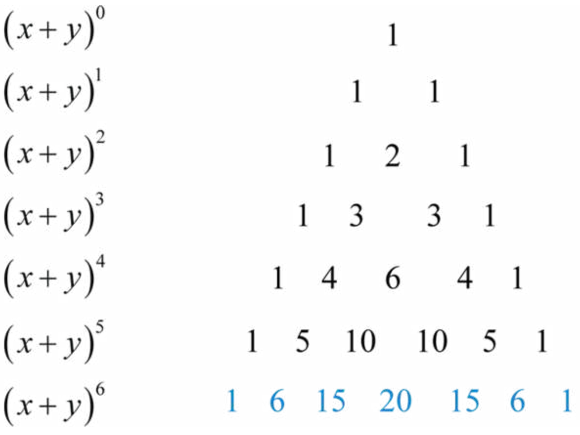

Figure $9.4.2$

The binomial coefficients that we need are in blue. Use these numbers and the binomial theorem to quickly expand $(x-1)^6$ as follows:

$$\begin{aligned}(x-1)^6 &= 1x^6(-1)^0+6x^5(-1)^1+15x^4(-1)^2+20x^3(-1)^3+15x^2(-1)^4+6x(-1)^5\\ &= x^6-6x^5+15x^4-20x^3+15x^2-6x+1\end{aligned}$$

Example 9.4.5

Expand using the binomial theorem and Pascal's triangle: $(2x-5)^4$.

Solution

From Pascal's triangle we can see that when $n=4$ the binomial coefficients are $1, 4, 6, 4$, and 1.Use these numbers and the binomial theorem as follows:

$$\begin{aligned}(2x-5)^4 &= 1(2x)^4(-5)^0+4(2x)^3(-5)^1+6(2x)^2(-5)^2+4(2x)^1(-5)^3+(2x)^0(-5)^4\\ &= 16x^4\cdot 1+4\cdot 8x^3(-5)+6\cdot 4x^2\cdot 25+4\cdot 2x(-125)+1\cdot 625\\ &= 16x^4-160x^3+600x^2-1,000x+625\end{aligned}$$

Answer:

$16x^4-160x^3+600x^2-1,000x+625$

Key Takeaways

- To calculate the factorial of a natural number, multiply that number by all natural numbers less than it: $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$. Remember that we have defined $0! = 1$.
- The binomial coefficients are the integers calculated using the formula:

$$\left(\begin{array}{l} n \\ k \end{array}\right)=\frac{n!}{k!(n-k)!}.$$

- The binomial theorem provides a method for expanding binomials raised to powers without directly multiplying each factor:

$$(x+y)^n=\sum_{k=0}^{n}\left(\begin{array}{l} n \\ k \end{array}\right) x^{n-k}y^k$$

- Use Pascal's triangle to quickly determine the binomial coefficients.

Exercise 9.4.3

Evaluate.

1. $6!$
2. $4!$
3. $10!$
4. $9!$
5. $\frac{6!}{3!}$
6. $\frac{8!}{4!}$
7. $\frac{13!}{9!}$
8. $\frac{15!}{10!}$
9. $\frac{12!}{3!7!}$
10. $\frac{10!}{2!5!}$
11. $\frac{n!}{(n-2)!}$
12. $\frac{(n+1)!}{(n-1)!}$
13. (a) $4!+3!$ (b) $(4+3)!$
14. (a) $4!-3!$ (b) $(4-3)!$

Answer

1. 720

3. $3,628,800$

5. 120

7. $17,160$

9. $15,840$

11. n^2-n

13. a. 30 b. $5,040$

Exercise 9.4.4

Rewrite using factorial notation.

1. $1\times2\times3\times4\times5\times6\times7$
2. $1\times2\times3\times4\times5$
3. $15\times14\times13$
4. $10\times9\times8\times7$
5. 13
6. 8×7
7. $n(n-1)(n-2)$
8. $1\times2\times3\times\cdots\times n\times(n+1)$

Answer

1. $7!$

3. $\frac{15!}{12!}$

5. $\frac{13!}{12!}$

7. $\frac{n!}{(n-3)!}$

Exercise 9.4.5

Calculate the indicated binomial coefficient.

1. $\begin{pmatrix}6\\4\end{pmatrix}$
2. $\begin{pmatrix}8\\4\end{pmatrix}$
3. $\begin{pmatrix}7\\2\end{pmatrix}$
4. $\begin{pmatrix}9\\5\end{pmatrix}$
5. $\begin{pmatrix}9\\0\end{pmatrix}$
6. $\begin{pmatrix}13\\12\end{pmatrix}$
7. $\begin{pmatrix}n\\0\end{pmatrix}$
8. $\begin{pmatrix}n\\n\end{pmatrix}$
9. $\begin{pmatrix}n\\1\end{pmatrix}$
10. $\begin{pmatrix}n\\n-1\end{pmatrix}$
11. ${}_{10}C_8$
12. ${}_{5}C_1$
13. ${}_{12}C_{12}$

14. ${}_{10}C_5$
15. ${}_{n}C_{n-2}$
16. ${}_{n}C_{n-3}$

Answer

1. 15

3. 21

5. 1

7. 1

9. n

11. 45

13. 1

15. $\frac{n^2-n}{2}$

Exercise 9.4.6

Expand using the binomial theorem.

1. $(4x-3)^3$
2. $(2x-5)^3$
3. $\left(\frac{x}{2}+y\right)^3$
4. $\left(x+\frac{1}{y}\right)^3$
5. $(x+3)^4$
6. $(x+5)^4$
7. $(x-4)^4$
8. $(x-2)^4$
9. $\left(x+\frac{2}{y}\right)^4$
10. $\left(\frac{x}{3}-y\right)^4$
11. $(x+1)^5$
12. $(x-3)^5$
13. $(x-2)^6$
14. $(x+1)^6$
15. $(x-1)^7$
16. $(x+1)^7$
17. $(5x-1)^4$
18. $(3x-2)^4$
19. $(4u+v)^4$
20. $(3u-v)^4$
21. $(u-5v)^5$
22. $(2u+3v)^5$
23. $\left(a-b^2\right)^5$
24. $\left(a^2+b^2\right)^4$
25. $\left(a^2+b^4\right)^6$
26. $\left(a^5+b^2\right)^5$
27. $(x+\sqrt{2})^3$
28. $(x-\sqrt{2})^4$
29. $(\sqrt{x}-\sqrt{y})^4, x, y \geq 0$
30. $(\sqrt{x}+2\sqrt{y})^5, x, y \geq 0$
31. $(x+y)^7$
32. $(x+y)^8$

33. $(x+y)^9$
34. $(x-y)^7$
35. $(x-y)^8$
36. $(x-y)^9$

Answer

1. $64x^3-144x^2+108x-27$
3. $\frac{x^3}{8}+\frac{3x^2y}{4}+\frac{3xy^2}{2}+y^3$
5. $x^4+12x^3+54x^2+108x+81$
7. $x^4-16x^3+96x^2-256x+256$
9. $x^4+\frac{8x^3}{y}+\frac{24x^2}{y^2}+\frac{32x}{y^3}+\frac{16}{y^4}$
11. $x^5+5x^4+10x^3+10x^2+5x+1$
13. $x^6-12x^5+60x^4-160x^3+240x^2-192x+64$
15. $x^7-7x^6+21x^5-35x^4+35x^3-21x^2+7x-1$
17. $625x^4-500x^3+150x^2-20x+1$
19. $256u^4+256u^3v+96u^2v^2+16uv^3+v^4$
21. $u^5-25u^4v+250u^3v^2-1,250u^2v^3+3,125uv^4-3,125v^5$
23. $a^5-5a^4b^2+10a^3b^4-10a^2b^6+5ab^8-b^{10}$
25. $a^{12}+6a^{10}b^4+15a^8b^8+20a^6b^{12}+15a^4b^{16}+6a^2b^{20}+b^{24}$
27. $x^3+3\sqrt{2}x^2+6x+2\sqrt{2}$
29. $x^2-4x\sqrt{xy}+6xy-4y\sqrt{xy}+y^2$
31. $x^7+7x^6y+21x^5y^2+35x^4y^3+35x^3y^4+21x^2y^5+7xy^6+y^7$
33. $x^9+9x^8y+36x^7y^2+84x^6y^3+126x^5y^4+126x^4y^5+84x^3y^6+36x^2y^7+9xy^8+y^9$
35. $x^8-8x^7y+28x^6y^2-56x^5y^3+70x^4y^4-56x^3y^5+28x^2y^6-8xy^7+y^8$

Exercise 9.4.7

1. Determine the factorials of the integers $5, 10, 15, 20$, and 25. Which grows faster, the common exponential function $a_n = 10^n$ or the factorial function $a_n = n!$? Explain.
2. Research and discuss the history of the binomial theorem.

Answer

1. Answer may vary

Footnotes

[25] The product of all natural numbers less than or equal to a given natural number, denoted $n!$.

[26] The factorial of zero is defined to be equal to $1; 0! = 1$.

[27] An integer that is calculated using the formula: $\left(\begin{array}{l} n \\ k \end{array}\right)=\frac{n!}{k!(n-k)!}$

[28] Describes the algebraic expansion of binomials raised to powers: $(x+y)^n=\sum_{k=0}^{n}\left(\begin{array}{l} n \\ k \end{array}\right) x^{n-k}y^k$.

[29] A triangular array of numbers that correspond to the binomial coefficients.

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9.E: Sequences, Series, and the Binomial Theorem (Exercises)

Exercise $9.E.1$

Find the first 5 terms of the sequence as well as the 30^{th} term.

1. $a_n = 5n - 3$
2. $a_n = -4n + 3$
3. $a_n = -10n$
4. $a_n = 3n$
5. $a_n = (-1)^n (n-2)^2$
6. $a_n = \frac{(-1)^n}{2n-1}$
7. $a_n = \frac{2n+1}{n}$
8. $a_n = (-1)^{n+1}(n-1)$

Answer

1. $2, 7, 12, 17, 22; a_{30} = 147$

3. $-10, -20, -30, -40, -50; a_{30} = -300$

5. $-1, 0, -1, 4, -9; a_{30} = 784$

7. $3, \frac{5}{2}, \frac{7}{3}, \frac{9}{4}, \frac{11}{5}; a_{30} = \frac{61}{30}$

Exercise $9.E.2$

Find the first 5 terms of the sequence.

1. $a_n = \frac{nx^n}{2n+1}$
2. $a_n = \frac{(-1)^{n-1}x^{n+2}}{n}$
3. $a_n = 2^n x^{2n}$
4. $a_n = (-3x)^{n-1}$
5. $a_n = a_{n-1} + 5$ where $a_1 = 0$
6. $a_n = 4a_{n-1} + 1$ where $a_1 = -2$
7. $a_n = a_{n-2} - 3a_{n-1}$ where $a_1 = 0$ and $a_2 = -3$
8. $a_n = 5a_{n-2} - a_{n-1}$ where $a_1 = -1$ and $a_2 = 0$

Answer

1. $\frac{x}{3}, \frac{2x^2}{5}, \frac{3x^3}{7}, \frac{4x^4}{9}, \frac{5x^5}{11}$

3. $2x^2, 4x^4, 8x^6, 16x^8, 32x^{10}$

5. $0, 5, 10, 15, 20$

7. $0, -3, 9, -30, 99$

Exercise $9.E.3$

Find the indicated partial sum.

1. $1, 4, 7, 10, 13, \ldots; S_5$
2. $3, 1, -1, -3, -5, \ldots; S_5$
3. $-1, 3, -5, 7, -9, \ldots; S_4$
4. $a_n = (-1)^n n^2; S_4$
5. $a_n = -3(n-2)^2; S_4$
6. $a_n = \left(-\frac{1}{5}\right)^{n-2}; S_4$

Answer

1. 35

3. -5
5. -18

Exercise $9.E.4$

Evaluate.

1. $\sum_{k=1}^{6}(1-2k)$
2. $\sum_{k=1}^{4}(-1)^{k}3k^{2}$
3. $\sum_{n=1}^{3}\frac{n+1}{n}$
4. $\sum_{n=1}^{7}5(-1)^{n-1}$
5. $\sum_{k=4}^{8}(1-k)^{2}$
6. $\sum_{k=-2}^{2}\left(\frac{2}{3}\right)^{k}$

Answer

1. -36
3. $\frac{29}{6}$
5. 135

Exercise $9.E.5$

Write the first 5 terms of the arithmetic sequence given its first term and common difference. Find a formula for its general term.

1. $a_1=6; d=5$
2. $a_1=5; d=7$
3. $a_1=5; d=-3$
4. $a_1=-\frac{3}{2}; d=-\frac{1}{2}$
5. $a_1=-\frac{3}{4}; d=-\frac{3}{4}$
6. $a_1=-3.6; d=1.2$
7. $a_1=7; d=0$
8. $a_1=1; d=1$

Answer

1. $6, 11, 16, 21, 26; a_n=5n+1$
3. $5, 2, -1, -4, -7; a_n=8-3n$
5. $-\frac{3}{4}, -\frac{3}{2}, -\frac{9}{4}, -3, -\frac{15}{4}; a_n=-\frac{3}{4}n$
7. $7, 7, 7, 7, 7; a_n=7$

Exercise $9.E.6$

Given the terms of an arithmetic sequence, find a formula for the general term.

1. $10, 20, 30, 40, 50, \ldots$
2. $-7, -5, -3, -1, 1, \ldots$
3. $-2, -5, -8, -11, -14, \ldots$
4. $-\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}, 1, \ldots$
5. $a_4=11$ and $a_9=26$
6. $a_5=-5$ and $a_{10}=-15$
7. $a_6=6$ and $a_{24}=15$
8. $a_3=-1.4$ and $a_7=1$

Answer

1. $a_n=10n$

3. $a_n = 1 - 3n$
5. $a_n = 3n - 1$
7. $a_n = \frac{1}{2}n + 3$

Exercise $9.E.7$

Calculate the indicated sum given the formula for the general term of an arithmetic sequence.

1. $a_n = 4n - 3; S_{60}$
2. $a_n = -2n + 9; S_{35}$
3. $a_n = \frac{1}{5}n - \frac{1}{2}; S_{15}$
4. $a_n = -n + \frac{1}{4}; S_{20}$
5. $a_n = 1.8n - 4.2; S_{45}$
6. $a_n = -6.5n + 3; S_{35}$

Answer

1. $7,140$
3. $\frac{33}{2}$
5. $1,674$

Exercise $9.E.8$

Evaluate.

1. $\sum_{n=1}^{22}(7n - 5)$
2. $\sum_{n=1}^{100}(1 - 4n)$
3. $\sum_{n=1}^{35}\left(\frac{2}{3}n\right)$
4. $\sum_{n=1}^{30}\left(-\frac{1}{4}n + 1\right)$
5. $\sum_{n=1}^{40}(2.3n - 1.1)$
6. $\sum_{n=1}^{300} n$
7. Find the sum of the first 175 positive odd integers.
8. Find the sum of the first 175 positive even integers.
9. Find all arithmetic means between $a_1 = \frac{2}{3}$ and $a_5 = -\frac{2}{3}$
10. Find all arithmetic means between $a_3 = -7$ and $a_7 = 13$.
11. A 5-year salary contract offers $\$58,200$ for the first year with a $\$4,200$ increase each additional year. Determine the total salary obligation over the 5-year period.
12. The first row of seating in a theater consists of 10 seats. Each successive row consists of four more seats than the previous row. If there are 14 rows, how many total seats are there in the theater?

Answer

1. $1,661$
3. 420
5. $1,842$
7. $30,625$
9. $\frac{1}{3}, 0, -\frac{1}{3}$
11. $\$333,000$

Exercise $9.E.9$

Write the first 5 terms of the geometric sequence given its first term and common ratio. Find a formula for its general term.

1. $a_1 = 5; r = 2$
2. $a_1 = 3; r = -2$

3. $a_1 = 1; r = -\frac{3}{2}$
4. $a_1 = -4; r = \frac{1}{3}$
5. $a_1 = 1.2; r = 0.2$
6. $a_1 = -5.4; r = -0.1$

Answer

1. $5, 10, 20, 40, 80; a_n = 5(2)^{n-1}$

3. $1, -\frac{3}{2}, \frac{9}{4}, -\frac{27}{8}, \frac{81}{16}; a_n = \left(-\frac{3}{2}\right)^{n-1}$

5. $1.2, 0.24, 0.048, 0.0096, 0.00192; a_n = 1.2(0.2)^{n-1}$

Exercise $9.E.10$

Given the terms of a geometric sequence, find a formula for the general term.

1. $4, 40, 400, \ldots$
2. $-6, -30, -150, \ldots$
3. $6, \frac{9}{2}, \frac{27}{8}, \ldots$
4. $1, \frac{3}{5}, \frac{9}{25}, \ldots$
5. $a_4 = -4$ and $a_9 = 128$
6. $a_2 = -1$ and $a_5 = -64$
7. $a_2 = -\frac{5}{2}$ and $a_5 = -\frac{625}{16}$
8. $a_3 = 50$ and $a_6 = -6,250$
9. Find all geometric means between $a_1 = -1$ and $a_4 = 64$.
10. Find all geometric means between $a_3 = 6$ and $a_6 = 162$.

Answer

1. $a_n = 4(10)^{n-1}$

3. $a_n = 6\left(\frac{3}{4}\right)^{n-1}$

5. $a_1 = \frac{1}{2}(-2)^{n-1}$

7. $a_n = -\left(\frac{5}{2}\right)^{n-1}$

9. $4, 16$

Exercise $9.E.11$

Calculate the indicated sum given the formula for the general term of a geometric sequence.

1. $a_n = 3(4)^{n-1}; S_6$
2. $a_n = -5(3)^{n-1}; S_{10}$
3. $a_n = \frac{3}{2}(-2)^n; S_{14}$
4. $a_n = \frac{1}{5}(-3)^{n+1}; S_{12}$
5. $a_n = 8\left(\frac{1}{2}\right)^{n+2}; S_8$
6. $a_n = \frac{1}{8}(-2)^{n+2}; S_{10}$

Answer

1. $4,095$

3. $16,383$

5. $\frac{255}{128}$

Exercise $9.E.12$

Evaluate.

1. $\sum_{n=1}^{10} 3(-4)^n$

2. $\sum_{n=1}^{9} -\frac{3}{5}(-2)^{n-1}$
3. $\sum_{n=1}^{\infty} -3\left(\frac{2}{3}\right)^{n}$
4. $\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{4}{5}\right)^{n+1}$
5. $\sum_{n=1}^{\infty} \frac{1}{2}\left(-\frac{3}{2}\right)^{n}$
6. $\sum_{n=1}^{\infty} \frac{3}{2}\left(-\frac{1}{2}\right)^{n}$
7. After the first year of operation, the value of a company van was reported to be $\$40,000$. Because of depreciation, after the second year of operation the van was reported to have a value of $\$32,000$ and then $\$25,600$ after the third year of operation. Write a formula that gives the value of the van after the nth year of operation. Use it to determine the value of the van after 10 years of operation.
8. The number of cells in a culture of bacteria doubles every 6 hours. If 250 cells are initially present, write a sequence that shows the number of cells present after every 6-hour period for one day. Write a formula that gives the number of cells after the nth 6-hour period.
9. A ball bounces back to one-half of the height that it fell from. If dropped from 32 feet, approximate the total distance the ball travels.
10. A structured settlement yields an amount in dollars each year n according to the formula $p_n = 12,500(0.75)^{n-1}$. What is the total value of a 10-year settlement?

Answer

1. $2,516,580$

3. -6

5. No sum

7. $v_n = 40,000(0.8)^{n-1}; v_{10} = \$5,368.71$

9. 96 feet

Exercise $9.E.13$

Classify the sequence as arithmetic, geometric, or neither.

1. $4, 9, 14, \ldots$
2. $6, 18, 54, \ldots$
3. $-1, -\frac{1}{2}, 0, \ldots$
4. $10, 30, 60, \ldots$
5. $0, 1, 8, \ldots$
6. $-1, \frac{2}{3}, -\frac{4}{9}, \ldots$

Answer

1. Arithmetic; $d = 5$

3. Arithmetic; $d = \frac{1}{2}$

5. Neither

Exercise $9.E.14$

Evaluate.

1. $\sum_{n=1}^{4} n^2$
2. $\sum_{n=1}^{4} n^3$
3. $\sum_{n=1}^{32} (-4n+5)$
4. $\sum_{n=1}^{\infty} -2\left(\frac{1}{5}\right)^{n-1}$
5. $\sum_{n=1}^{8} \frac{1}{3}(-3)^{n}$
6. $\sum_{n=1}^{46} \left(\frac{1}{4}n - \frac{1}{2}\right)$
7. $\sum_{n=1}^{22} (3-n)$
8. $\sum_{n=1}^{31} 2n$
9. $\sum_{n=1}^{28} 3$

10. $\sum_{n=1}^{30} 3(-1)^{n-1}$
11. $\sum_{n=1}^{31} 3(-1)^{n-1}$

Answer

1. 30
3. $-1,952$
5. $1,640$
7. -187
9. 84
11. 3

Exercise $9.E.15$

Evaluate.

1. $8!$
2. $11!$
3. $\frac{10!}{2!6!}$
4. $\frac{9!3!}{8!}$
5. $\frac{(n+3)!}{n!}$
6. $\frac{(n-2)!}{(n+1)!}$

Answer

2. $39,916,800$
4. 54
6. $\frac{1}{n(n+1)(n-1)}$

Exercise $9.E.16$

Calculate the indicated binomial coefficient.

1. $\binom{7}{4}$
2. $\binom{8}{3}$
3. $\binom{10}{5}$
4. $\binom{11}{10}$
5. $\binom{12}{0}$
6. $\binom{n+1}{n-1}$
7. $\binom{n}{n-2}$

Answer

2. 56
4. 11
6. $\frac{n(n+1)}{2}$

Exercise $9.E.17$

Expand using the binomial theorem.

1. $(x+7)^3$
2. $(x-9)^3$
3. $(2y-3)^4$
4. $(y+4)^4$
5. $(x+2y)^5$
6. $(3x-y)^5$
7. $(u-v)^6$
8. $(u+v)^6$
9. $\left(5x^2+2y^2\right)^4$
10. $\left(x^3-2y^2\right)^4$

Answer

1. $x^3+21x^2+147x+343$

3. $16y^4-96y^3+216y^2-216y+81$

5. $x^5+10x^4y+40x^3y^2+80x^2y^3+80xy^4+32y^5$

7. $u^6-6u^5v+15u^4v^2-20u^3v^3+15u^2v^4-6uv^5+v^6$

9. $625x^8+1,000x^6y^2+600x^4y^4+160x^2y^6+16y^8$

Sample Exam

Exercise $9.E.18$

Find the first 5 terms of the sequence.

1. $a_n=6n-15$
2. $a_n=5(-4)^{n-2}$
3. $a_n=\frac{n-1}{2n-1}$
4. $a_n=(-1)^{n-1}x^{2n}$

Answer

1. $-9,-3,3,9,15$

3. $0,\frac{1}{3},\frac{2}{5},\frac{3}{7},\frac{4}{9}$

Exercise $9.E.19$

Find the indicated partial sum

1. $a_n=(n-1)n^2; S_4$
2. $\sum_{k=1}^{5}(-1)^k 2^{k-2}$

Answer

1. 70

Exercise $9.E.20$

Classify the sequence as arithmetic, geometric, or neither.

1. $-1,-\frac{3}{2},-2,\ldots$
2. $1,-6,36,\ldots$
3. $\frac{3}{8},-\frac{3}{4},\frac{3}{2},\ldots$
4. $\frac{1}{2},\frac{1}{4},\frac{2}{9},\ldots$

Answer

1. Arithmetic

3. Geometric

Exercise $9.E.21$

Given the terms of an arithmetic sequence, find a formula for the general term.

1. $10, 5, 0, -5, -10, \ldots$
2. $a_4 = -\frac{1}{2}$ and $a_9 = 2$

Answer

1. $a_n = 15 - 5n$

Exercise $9.E.22$

Given the terms of a geometric sequence, find a formula for the general term.

1. $-\frac{1}{8}, -\frac{1}{2}, -2, -8, -32, \ldots$
2. $a_3 = 1$ and $a_8 = -32$

Answer

1. $a_n = -\frac{1}{8}(4)^{n-1}$

Exercise $9.E.23$

Calculate the indicated sum.

1. $a_n = 5 - n; S_{44}$
2. $a_n = (-2)^{n+2}; S_{12}$
3. $\sum_{n=1}^{\infty} 4\left(-\frac{1}{2}\right)^{n-1}$
4. $\sum_{n=1}^{100}\left(2n - \frac{3}{2}\right)$

Answer

1. -770

3. $\frac{8}{3}$

Exercise $9.E.24$

Evaluate.

1. $\frac{14!}{10!6!}$
2. $\binom{9}{7}$
3. Determine the sum of the first 48 positive odd integers.
4. The first row of seating in a theater consists of 14 seats. Each successive row consists of two more seats than the previous row. If there are 22 rows, how many total seats are there in the theater?
5. A ball bounces back to one-third of the height that it fell from. If dropped from 27 feet, approximate the total distance the ball travels.

Answer

1. $\frac{1{,}001}{30}$

3. $2,304$

5. 54 feet

Exercise $9.E.25$

Expand using the binomial theorem.

1. $(x - 5y)^4$

2. $(3a+b^2)^5$

Answer

2. $243a^5+405a^4b^2+270a^3b^4{+}90a^2b^6+15ab^8+b^{10}$

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Glossary

absolute value | The distance from the graph of a number *a* to zero on a number line, denoted $|a|$.

absolute value function | The function defined by $f(x) = |x|$.

AC method | Method used for factoring trinomials by replacing the middle term with two terms that allow us to factor the resulting four-term polynomial by grouping.

addition property of equations | If *A*, *B*, *C*, and *D* are algebraic expressions, where *A* = *B* and *C* = *D*, then *A* + *C* = *B* + *D*.

addition property of equations | If *A*, *B*, *C*, and *D* are algebraic expressions, where *A* = *B* and *C* = *D*, then *A* + *C* = *B* + *D*.

algebraic expressions | Combinations of variables and numbers along with mathematical operations used to generalize specific arithmetic operations.

argument of the absolute value | The number or expression inside the absolute value.

argument of the absolute value | The number or expression inside the absolute value.

argument of the function | The value or algebraic expression used as input when using function notation.

arithmetic means | The terms between given terms of an arithmetic sequence.

arithmetic progression | Used when referring to an arithmetic sequence.

arithmetic series | The sum of the terms of an arithmetic sequence.

arithmetic series | The sum of the terms of an arithmetic sequence.

augmented matrix | The coefficient matrix with the column of constants included.

augmented matrix | The coefficient matrix with the column of constants included.

average cost | The total cost divided by the number of units produced, which can be represented by $\overline{C}(x) = \frac{C(x)}{x}$, where $C(x)$ is a cost function.

average cost | The total cost divided by the number of units produced, which can be represented by $\overline{C}(x) = \frac{C(x)}{x}$, where $C(x)$ is a cost function.

axis of symmetry | A term used when referencing the line of symmetry.

axis of symmetry | A term used when referencing the line of symmetry.

Back substitute | Once a value is found for a variable, substitute it back into one of the original equations, or its equivalent, to determine the corresponding value of the other variable.

Back substitute | Once a value is found for a variable, substitute it back into one of the original equations, or its equivalent, to determine the corresponding value of the other variable.

binomial | Polynomial with two terms.

binomial coefficient | An integer that is calculated using the formula: $=\binom{n}{k} \quad \frac{n!}{k!(n\ \ k)!}$.

binomial coefficient | An integer that is calculated using the formula: $=\binom{n}{k} \quad \frac{n!}{k!(n\ \ k)!}$.

binomial theorem | Describes the algebraic expansion of binomials raised to powers: $(x+y)^n$

$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.

binomial theorem | Describes the algebraic expansion of binomials raised to powers: $(x+y)^n$

$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.

breakeven point | The point at which profit is neither negative nor positive; profit is equal to zero.

breakeven point | The point at which profit is neither negative nor positive; profit is equal to zero.

Cartesian coordinate system | Term used in honor of René Descartes when referring to the rectangular coordinate system.

Cartesian coordinate system | Term used in honor of René Descartes when referring to the rectangular coordinate system.

change of base formula | msubloga x=logb xlogb a; we can write any base-*a* logarithm in terms of base-*b* logarithms using this formula.

change of base formula | msubloga x=logb xlogb a; we can write any base-*a* logarithm in terms of base-*b* logarithms using this formula.

circle in general form | The equation of a circle written in the form $x^2 + y^2 + cx + dy + e = 0$.

circle in general form | The equation of a circle written in the form $x^2 + y^2 + cx + dy + e = 0$.

circle in standard form | The equation of a circle written in the form $(x-h)^2 + (y-k)^2 = r^2$ where (h,k) is the center and *r* is the radius.

circle in standard form | The equation of a circle written in the form $(x-h)^2 + (y-k)^2 = r^2$ where (h,k) is the center and *r* is the radius.

co-vertices | Points on the ellipse that mark the endpoints of the minor axis.

co-vertices | Points on the ellipse that mark the endpoints of the minor axis.

codomain | Used when referencing the range.

codomain | Used when referencing the range.

coefficient matrix | The matrix of coefficients of a linear system in standard form written as they appear lined up without the variables or operations.

coefficient matrix | The matrix of coefficients of a linear system in standard form written as they appear lined up without the variables or operations.

combining like terms | Adding or subtracting like terms within an algebraic expression to obtain a single term with the same variable part.

combining like terms | Adding or subtracting like terms within an algebraic expression to obtain a single term with the same variable part.

common denominator | A denominator that is shared by more than one fraction.

common denominator | A denominator that is shared by more than one fraction.

common difference | The constant *d* that is obtained from subtracting any two successive terms of an arithmetic sequence; $a_n \quad a_{n\ 1} \quad d$.

common difference | The constant *d* that is obtained from subtracting any two successive terms of an arithmetic sequence; $a_n \quad a_{n\ 1} \quad d$.

common factor | A factor that is shared by more than one real number.

common factor | A factor that is shared by more than one real number.

common logarithm | The logarithm base 10, denoted $\log x$.

common logarithm | The logarithm base 10, denoted $\log x$.

common ratio | The constant *r* that is obtained from dividing any two successive terms of a geometric sequence; $\frac{a_n}{a_{n-1}} = r$.

common ratio | The constant *r* that is obtained from dividing any two successive terms of a geometric sequence; $\frac{a_n}{a_{n-1}} = r$.

completely factored | A polynomial that is prime or written as a product of prime polynomials.

completely factored | A polynomial that is prime or written as a product of prime polynomials.

completing the square | The process of rewriting a quadratic equation to be in the form $(x-p)^2 = q$.

completing the square | The process of rewriting a quadratic equation to be in the form $(x-p)^2 = q$.

complex conjugate | Two complex numbers whose real parts are the same and imaginary parts are opposite. If given $a + bi$, then its complex conjugate is $a - bi$.

complex conjugate | Two complex numbers whose real parts are the same and imaginary parts are opposite. If given $a + bi$, then its complex conjugate is $a - bi$.

complex rational expression | A rational expression that contains one or more rational expressions in the numerator or denominator or both.

complex rational expression | A rational expression that contains one or more rational expressions in the numerator or denominator or both.

composition operator | The open dot used to indicate the function composition $(f \circ g)(x) = f(g(x))$.

composition operator | The open dot used to indicate the function composition $(f \circ g)(x) = f(g(x))$.

compound inequalities | Two or more inequalities in one statement joined by the word “and” or by the word “or.”

compound inequalities | Two or more inequalities in one statement joined by the word “and” or by the word “or.”

compound interest formula | A formula that gives the amount accumulated by earning interest on principal and interest over time: $A(t) = P(1 + \frac{r}{n})^{nt}$.

compound interest formula | A formula that gives the amount accumulated by earning interest on principal and interest over time: $A(t) = P(1 + \frac{r}{n})^{nt}$.

conic section | A curve obtained from the intersection of a right circular cone and a plane.

conic section | A curve obtained from the intersection of a right circular cone and a plane.

conjugate axis | A line segment through the center of a hyperbola that is perpendicular to the transverse axis.

conjugate axis | A line segment through the center of a hyperbola that is perpendicular to the transverse axis.

conjugate binomials | The binomials $(a+b)$ and $(a-b)$.

conjugate binomials | The binomials $(a+b)$ and $(a-b)$.

conjugates | The factors $(a+b)$ and $(a-b)$ are conjugates.

conjugates | The factors $(a+b)$ and $(a-b)$ are conjugates.

constant function | Any function of the form $f(x)=c$ where *c* is a real number.

constant function | Any function of the form $f(x)=c$ where *c* is a real number.

constant of proportionality | Used when referring to the constant of variation.

constant of proportionality | Used when referring to the constant of variation.

constant of variation | The nonzero multiple *k*, when quantities vary directly or inversely.

constant of variation | The nonzero multiple *k*, when quantities vary directly or inversely.

constant polynomial | A polynomial with degree 0.

constant term | A term written without a variable factor.

constant term | A term written without a variable factor.

continuously compounding interest formula | A formula that gives the amount accumulated by earning continuously compounded interest: $A(t)=Pe^{rt}$.

continuously compounding interest formula | A formula that gives the amount accumulated by earning continuously compounded interest: $A(t)=Pe^{rt}$.

contradiction | An equation that is never true and has no solution.

contradiction | An equation that is never true and has no solution.

convergent geometric series | An infinite geometric series where $|r|<1$ whose sum is given by the formula: $S_\infty=\frac{a_1}{1-r}$.

convergent geometric series | An infinite geometric series where $|r|<1$ whose sum is given by the formula: $S_\infty=\frac{a_1}{1-r}$.

cost function | A function that models the cost of producing a number of units.

cost function | A function that models the cost of producing a number of units.

Cramer's rule | The solution to an independent system of linear equations expressed in terms of determinants.

Cramer's rule | The solution to an independent system of linear equations expressed in terms of determinants.

critical numbers | The values in the domain of a function that separate regions that produce positive or negative results.

critical numbers | The values in the domain of a function that separate regions that produce positive or negative results.

cross multiplication | If $\frac{a}{b} \quad \frac{c}{d}$ then $ad \quad bc$.

cross multiplication | If $\frac{a}{b} \quad \frac{c}{d}$ then $ad \quad bc$.

cube | The result when the exponent of any real number is 3.

cube | The result when the exponent of any real number is 3.

cube root function | The function defined by $f(x)=\sqrt[3]{x}$.

cube root function | The function defined by $f(x)=\sqrt[3]{x}$.

cubing function | The cubic function defined by $f(x)=x^3$.

cubing function | The cubic function defined by $f(x)=x^3$.

degree of a polynomial | The largest degree of all of its terms.

degree of a polynomial | The largest degree of all of its terms.

degree of a term | The exponent of the variable. If there is more than one variable in the term, the degree of the term is the sum their exponents.

degree of a term | The exponent of the variable. If there is more than one variable in the term, the degree of the term is the sum their exponents.

dependent system | A linear system with two variables that consists of equivalent equations. It has infinitely many ordered pair solutions, denoted by $(x, mx+b)$.

dependent system | A linear system with two variables that consists of equivalent equations. It has infinitely many ordered pair solutions, denoted by $(x, mx+b)$.

dependent variable | The variable whose value is determined by the value of the independent variable. Usually we think of the *y*-value of an ordered pair (*x*, *y*) as the dependent variable.

dependent variable | The variable whose value is determined by the value of the independent variable. Usually we think of the *y*-value of an ordered pair (*x*, *y*) as the dependent variable.

determinant | A real number associated with a square matrix.

determinant | A real number associated with a square matrix.

diameter | The length of a line segment passing through the center of a circle whose endpoints are on the circle.

diameter | The length of a line segment passing through the center of a circle whose endpoints are on the circle.

difference | The result of subtracting.

difference | The result of subtracting.

difference of cubes | $\begin{array}{l} a^3-b^3 \\ =(a-b)(a^2+ab+b^2) \end{array}$, where *a* and *b* represent algebraic expressions.

difference of cubes | $\begin{array}{l} a^3 \quad b^3 \\ =(a \quad b)(a^2+ab+b^2) \end{array}$, where *a* and *b* represent algebraic expressions.

difference of squares | The special product obtained by multiplying conjugate binomials $(a+b)(a \quad b)=a^2 \quad b^2$.

difference of squares | The special product obtained by multiplying conjugate binomials $(a+b)(a \quad b) \quad a^2 \quad b^2$.

difference of squares | $a^2 \quad b^2 \quad (a+b)(a \quad b)$, where *a* and *b* represent algebraic expressions.

difference of squares | $a^2 \quad b^2 \quad (a+b)(a \quad b)$, where *a* and *b* represent algebraic expressions.

difference quotient | The mathematical quantity $\frac{f(x+h)-f(x)}{h}$, where $h\neq 0$, which represents the slope of a secant line through a function *f*.

difference quotient | The mathematical quantity $\frac{f(x+h)-f(x)}{h}$, where $h\neq 0$, which represents the slope of a secant line through a function *f*.

dilation | A non-rigid transformation, produced by multiplying functions by a nonzero real number, which appears to stretch the graph either vertically or horizontally.

dilation | A non-rigid transformation, produced by multiplying functions by a nonzero real number, which appears to stretch the graph either vertically or horizontally.

discriminant | The expression inside the radical of the quadratic formula, b^2-4ac.

discriminant | The expression inside the radical of the quadratic formula, b^2-4ac.

distance formula | Given two points (x_1, y_1) and (x_2, y_2), calculate the distance *d* between them using the formula $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

distance formula | Given two points (x_1, y_1) and (x_2, y_2), calculate the distance *d* between them using the formula $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

distance formula | Given two points (x_1, y_1) and (x_2, y_2), the distance *d* between them is given by $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

distance formula | Given two points (x_1, y_1) and (x_2, y_2), the distance *d* between them is given by $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

distributive property | Given any real numbers *a*, *b*, and *c*, $a(b+c)=ab+ac$ or $(b+c)a=ba+ca$.

distributive property | Given any real numbers *a*, *b*, and *c*, $a(b+c)=ab+ac$ or $(b+c)a=ba+ca$.

division | Divide functions as indicated by the notation: $(f/g)(x)=\frac{f(x)}{g(x)}$, where $g(x)\neq 0$.

division | Divide functions as indicated by the notation: $(f/g)(x)=\frac{f(x)}{g(x)}$, where $g(x)\neq 0$.

double root | A root that is repeated twice.

double root | A root that is repeated twice.

double-negative property | The opposite of a negative number is positive: −(−*a*) = *a*.

double-negative property | The opposite of a negative number is positive: −(−*a*) = *a*.

doubling time | The period of time it takes a quantity to double.

doubling time | The period of time it takes a quantity to double.

element | An object within a set.

element | An object within a set.

elementary row operations | Operations that can be performed to obtain equivalent linear systems.

elementary row operations | Operations that can be performed to obtain equivalent linear systems.

ellipse | The set of points in a plane whose distances from two fixed points have a sum that is equal to a positive constant.

ellipse | The set of points in a plane whose distances from two fixed points have a sum that is equal to a positive constant.

ellipse in general form | The equation of an ellipse written in the form px2+qy2+cx+dy+e=0 where $p, q > 0$.

ellipse in general form | The equation of an ellipse written in the form px2+qy2+cx+dy+e=0 where $p, q > 0$.

ellipse in standard form | The equation of an ellipse written in the form mfrac2a2+(y−k)2b2=1. The center is (h, k) and the larger of *a* and *b* is the major radius and the smaller is the minor radius.

ellipse in standard form | The equation of an ellipse written in the form mfrac2a2+(y−k)2b2=1. The center is (h, k) and the larger of *a* and *b* is the major radius and the smaller is the minor radius.

empty set | A subset with no elements, denoted Ø or { }.

empty set | A subset with no elements, denoted Ø or { }.

equivalent equations | Equations with the same solution set.

equivalent equations | Equations with the same solution set.

equivalent fractions | Two equal fractions expressed using different numerators and denominators.

equivalent fractions | Two equal fractions expressed using different numerators and denominators.

equivalent inequality | Inequalities that share the same solution set.

equivalent inequality | Inequalities that share the same solution set.

equivalent system | A system consisting of equivalent equations that share the same solution set.

equivalent system | A system consisting of equivalent equations that share the same solution set.

evaluating | The process of performing the operations of an algebraic expression for given values of the variables.

evaluating | The process of performing the operations of an algebraic expression for given values of the variables.

even integers | Integers that are divisible by 2.

even integers | Integers that are divisible by 2.

exponent | The positive integer *n* in the exponential notation a^n that indicates the number of times the base is used as a factor.

exponent | The positive integer *n* in the exponential notation a^n that indicates the number of times the base is used as a factor.

exponential form | An equivalent expression written using a rational exponent.

exponential form | An equivalent expression written using a rational exponent.

exponential function | Any function with a definition of the form $f(x) = b^x$ where $b > 0$ and $b \neq 1$.

exponential function | Any function with a definition of the form $f(x) = b^x$ where $b > 0$ and $b \neq 1$.

exponential growth/decay formula | A formula that models exponential growth or decay: $P(t) =$ P0ekt.

exponential growth/decay formula | A formula that models exponential growth or decay: $P(t) =$P0ekt.

exponential notation | The compact notation a^n used when a factor *a* is repeated *n* times.

exponential notation | The compact notation a^n used when a factor *a* is repeated *n* times.

extracting the root | Applying the square root property as a means of solving a quadratic equation.

extracting the root | Applying the square root property as a means of solving a quadratic equation.

extraneous solutions | A solution that does not solve the original equation.

extraneous solutions | A solution that does not solve the original equation.

extraneous solutions | A properly found solution that does not solve the original equation.

extraneous solutions | A properly found solution that does not solve the original equation.

extrapolation | Using a linear function to estimate values that extend beyond the given data points.

extrapolation | Using a linear function to estimate values that extend beyond the given data points.

factorial | The product of all natural numbers less than or equal to a given natural number, denoted *n*!.

factorial | The product of all natural numbers less than or equal to a given natural number, denoted *n*!.

Factoring by grouping | A technique for factoring polynomials with four terms.

Factoring by grouping | A technique for factoring polynomials with four terms.

Factoring out the greatest common factor (GCF) | The process of rewriting a polynomial as a product using the GCF of all of its terms.

Factoring out the greatest common factor (GCF) | The process of rewriting a polynomial as a product using the GCF of all of its terms.

factors | Any of the numbers that form a product.

factors | Any of the numbers that form a product.

factors | Any of the numbers or expressions that form a product.

factors | Any of the numbers or expressions that form a product.

finite sequence | A sequence whose domain is $\{1, 2, 3, \ldots, k\}$ where *k* is a natural number.

finite sequence | A sequence whose domain is $\{1, 2, 3, \ldots, k\}$ where *k* is a natural number.

floor function | A term used when referring to the greatest integer function.

floor function | A term used when referring to the greatest integer function.

formulas | A reusable mathematical model using algebraic expressions to describe a common application.

formulas | A reusable mathematical model using algebraic expressions to describe a common application.

fraction | A rational number written as a quotient of two integers: $\frac{a}{b}$, where $b \neq 0$.

fraction | A rational number written as a quotient of two integers: $\frac{a}{b}$, where $b \neq 0$.

function | A relation where each element in the domain corresponds to exactly one element in the range.

function | A relation where each element in the domain corresponds to exactly one element in the range.

function notation | The notation $f(x) = y$, which reads "*f* of *x* is equal to *y*." Given a function, *y* and $f(x)$ can be used interchangeably.

function notation | The notation $f(x) = y$, which reads "*f* of *x* is equal to *y*." Given a function, *y* and $f(x)$ can be used interchangeably.

fundamental rectangle | The rectangle formed using the endpoints of a hyperbolas, transverse and conjugate axes.

fundamental rectangle | The rectangle formed using the endpoints of a hyperbolas, transverse and conjugate axes.

fundamental theorem of algebra | Guarantees that there will be as many (or fewer) roots to a polynomial function with one variable as its degree.

fundamental theorem of algebra | Guarantees that there will be as many (or fewer) roots to a polynomial function with one variable as its degree.

fundamental theorem of algebra | If multiple roots and complex roots are counted, then every polynomial with one variable will have as many roots as its degree.

fundamental theorem of algebra | If multiple roots and complex roots are counted, then every polynomial with one variable will have as many roots as its degree.

Gaussian elimination | Steps used to obtain an equivalent linear system in upper triangular form so that it can be solved using back substitution.

Gaussian elimination | Steps used to obtain an equivalent linear system in upper triangular form so that it can be solved using back substitution.

general term of a sequence | An equation that defines the *n*th term of a sequence commonly denoted using subscripts a_n.

general term of a sequence | An equation that defines the *n*th term of a sequence commonly denoted using subscripts a_n.

geometric means | The terms between given terms of a geometric sequence.

geometric means | The terms between given terms of a geometric sequence.

geometric progression | Used when referring to a geometric sequence.

geometric progression | Used when referring to a geometric sequence.

geometric series | The sum of the terms of a geometric sequence.

geometric series | The sum of the terms of a geometric sequence.

graph | A visual representation of a relation on a rectangular coordinate plane.

graph | A visual representation of a relation on a rectangular coordinate plane.

graph of the solution set | Solutions to an algebraic expression expressed on a number line.

graph of the solution set | Solutions to an algebraic expression expressed on a number line.

graphing method | A means of solving a system by graphing the equations on the same set of axes and determining where they intersect.

graphing method | A means of solving a system by graphing the equations on the same set of axes and determining where they intersect.

greatest common factor (GCF). | The largest shared factor of any number of integers.

greatest common factor (GCF). | The largest shared factor of any number of integers.

greatest common monomial factor (GCF) | The product of the common variable factors and the GCF of the coefficients.

greatest common monomial factor (GCF) | The product of the common variable factors and the GCF of the coefficients.

greatest integer function | The function that assigns any real number *x* to the greatest integer less than or equal to *x* denoted $f(x) = [x]$.

greatest integer function | The function that assigns any real number *x* to the greatest integer less than or equal to *x* denoted $f(x) = [x]$.

grouping symbols | Parentheses, brackets, braces, and the fraction bar are the common symbols used to group expressions and mathematical operations within a computation.

grouping symbols | Parentheses, brackets, braces, and the fraction bar are the common symbols used to group expressions and mathematical operations within a computation.

half-life | The period of time it takes a quantity to decay to one-half of the initial amount.

half-life | The period of time it takes a quantity to decay to one-half of the initial amount.

horizontal asymptote | A horizontal line to which a graph becomes infinitely close where the *x*-values tend toward ±∞.

horizontal asymptote | A horizontal line to which a graph becomes infinitely close where the *x*-values tend toward ±∞.

horizontal line test | If a horizontal line intersects the graph of a function more than once, then it is not one-to-one.

horizontal line test | If a horizontal line intersects the graph of a function more than once, then it is not one-to-one.

horizontal translation | A rigid transformation that shifts a graph left or right.

horizontal translation | A rigid transformation that shifts a graph left or right.

hyperbola | The set of points in a plane whose distances from two fixed points, called foci, has an absolute difference that is equal to a positive constant.

hyperbola | The set of points in a plane whose distances from two fixed points, called foci, has an absolute difference that is equal to a positive constant.

hyperbola in general form | The equation of a hyperbola written in the form $\begin{array}{r}px^2 - qy^2 + cx \\ +dy + e \;=\; 0\end{array}$ or $\begin{array}{r}qy^2 - px^2 - cx \\ +dy + e \;=\; 0\end{array}$ where $p, q > 0$.

hyperbola in general form | The equation of a hyperbola written in the form $\begin{array}{r}px^2 \quad qy^2 + cx \\ +dy + e \;=\; 0\end{array}$ or $\begin{array}{r}qy^2 \quad px^2 \quad cx \\ +dy + e \;=\; 0\end{array}$ where $p, q > 0$.

hyperbola opening left and right in standard form | The equation of a hyperbola written in the form $\frac{(x-h)^2}{a^2} -$ mfrac2b2=1. The center is (h, k), *a* defines the transverse axis, and *b* defines the conjugate axis.

hyperbola opening left and right in standard form | The equation of a hyperbola written in the form $\frac{(x-h)^2}{a^2}$ mfrac2b2=1. The center is (h, k), *a* defines the transverse axis, and *b* defines the conjugate axis.

hyperbola opening upward and downward in standard form | The equation of a hyperbola written in the form $\frac{(y-k)^2}{b^2} -$ mfrac2a2=1. The center is (h, k), *b* defines the transverse axis, and *a* defines the conjugate axis.

hyperbola opening upward and downward in standard form | The equation of a hyperbola written in the form $\frac{(y-k)^2}{b^2} -$ mfrac2a2=1. The center is (h, k), *b* defines the transverse axis, and *a* defines the conjugate axis.

identity function | The linear function defined by $f(x) = x$.

identity function | The linear function defined by $f(x) = x$.

imaginary number | A square root of any negative real number.

imaginary number | A square root of any negative real number.

imaginary part | The real number *b* of a complex number $a + bi$.

imaginary part | The real number *b* of a complex number $a + bi$.

imaginary unit | Defined as $i = \sqrt{-1}$ where $i^2 = -1$.

imaginary unit | Defined as $i = \sqrt{-1}$ where $i^2 = -1$.

inclusive inequalities | Use the symbol ≤ to express quantities that are “less than or equal to” and ≥ for quantities that are “greater than or equal to” each other.

inclusive inequalities | Use the symbol ≤ to express quantities that are “less than or equal to” and ≥ for quantities that are “greater than or equal to” each other.

inconsistent system | A system with no simultaneous solution.

inconsistent system | A system with no simultaneous solution.

independent system | A linear system with two variables that has exactly one ordered pair solution.

independent system | A linear system with two variables that has exactly one ordered pair solution.

indeterminate | A quotient such as $\frac{0}{0}$ is a quantity that is uncertain or ambiguous.

indeterminate | A quotient such as $\frac{0}{0}$ is a quantity that is uncertain or ambiguous.

index | The positive integer *n* in the notation $\sqrt[n]{\ }$ that is used to indicate an *n*th root.

index | The positive integer *n* in the notation $\sqrt[n]{\ }$ that is used to indicate an *n*th root.

index | The positive integer *n* in the notation $\sqrt[n]{\ }$ that is used to indicate an *n*th root.

index | The positive integer *n* in the notation $\sqrt[n]{\ }$ that is used to indicate an *n*th root.

index of summation | The variable used in sigma notation to indicate the lower and upper bounds of the summation.

index of summation | The variable used in sigma notation to indicate the lower and upper bounds of the summation.

infinite sequence | A sequence whose domain is the set of natural numbers $\{1, 2, 3, \ldots\}$.

infinite sequence | A sequence whose domain is the set of natural numbers $\{1, 2, 3, \ldots\}$.

infinity | The symbol ∞ indicates the interval is unbounded to the right.

infinity | The symbol ∞ indicates the interval is unbounded to the right.

integers | The set of positive and negative whole numbers combined with zero: {…, −3, −2, −1, 0, 1, 2, 3, …}.

integers | The set of positive and negative whole numbers combined with zero: {…, −3, −2, −1, 0, 1, 2, 3, …}.

interpolation | Using a linear function to estimate a value between given data points.

interpolation | Using a linear function to estimate a value between given data points.

intersection | The set formed by the shared values of the individual solution sets that is indicated by the logical use of the word “and,” denoted with the symbol ∩.

intersection | The set formed by the shared values of the individual solution sets that is indicated by the logical use of the word “and,” denoted with the symbol ∩.

inverse properties of the logarithm | Given $b > 0$ we have msublogb bx=x and msupblogb x=x when $x > 0$.

inverse properties of the logarithm | Given $b > 0$ we have msublogb bx=x and msupblogb x=x when $x > 0$.

inversely proportional | Used when referring to inverse variation.

inversely proportional | Used when referring to inverse variation.

Irrational numbers | Numbers that cannot be written as a ratio of two integers.

Irrational numbers | Numbers that cannot be written as a ratio of two integers.

joint variation | Describes a quantity *y* that varies directly as the product of two other quantities *x* and *z*: y =kxz.

joint variation | Describes a quantity *y* that varies directly as the product of two other quantities *x* and *z*: y =kxz.

leading coefficient | The coefficient of the term with the largest degree.

least common denominator | The least common multiple of a set of denominators.

least common denominator | The least common multiple of a set of denominators.

linear equation with one variable | An equation that can be written in the standard form $ax + b = 0$, where *a* and *b* are real numbers and $a \neq 0$.

linear equation with one variable | An equation that can be written in the standard form $ax + b = 0$, where *a* and *b* are real numbers and $a \neq 0$.

linear function | Any function that can be written in the form $f(x) \quad mx + b$

linear function | Any function that can be written in the form $f(x) \quad mx + b$

linear inequality | Linear expressions related with the symbols ≤, <, ≥, and >.

linear inequality | Linear expressions related with the symbols $\leq$, $<$, $\geq$, and $>$.

linear inequality with two variables | An inequality relating linear expressions with two variables. The solution set is a region defining half of the plane.

linear inequality with two variables | An inequality relating linear expressions with two variables. The solution set is a region defining half of the plane.

linear systems | A set of two or more linear equations with the same variables.

linear systems | A set of two or more linear equations with the same variables.

logarithm base *b* | The exponent to which the base *b* is raised in order to obtain a specific value. In other words, $y =$ msublogb x is equivalent to $b^y = x$.

logarithm base *b* | The exponent to which the base *b* is raised in order to obtain a specific value. In other words, $y =$ msublogb x is equivalent to $b^y = x$.

logarithmic equation | An equation that involves a logarithm with a variable argument.

logarithmic equation | An equation that involves a logarithm with a variable argument.

mathematical modeling | Using data to find mathematical equations that describe, or model, real-world applications.

mathematical modeling | Using data to find mathematical equations that describe, or model, real-world applications.

midpoint | Given two points (x_1, y_1) and (x_2, y_2), the midpoint is an ordered pair given by (mfracx22, y1+y22).

midpoint | Given two points (x_1, y_1) and (x_2, y_2), the midpoint is an ordered pair given by (mfracx22, y1+y22).

minor | The determinant of the matrix that results after eliminating a row and column of a square matrix.

minor | The determinant of the matrix that results after eliminating a row and column of a square matrix.

monomial | Polynomial with one term.

natural logarithm | The logarithm base *e*, denoted $\ln x$.

natural logarithm | The logarithm base *e*, denoted $\ln x$.

negative exponents | $x^{-n} =$ mfrac1xn , given any integer *n*, where *x* is nonzero.

negative exponents | $x^{-n} =$ mfrac1xn , given any integer *n*, where *x* is nonzero.

negative infinity | The symbol $-\infty$ indicates the interval is unbounded to the left.

negative infinity | The symbol $-\infty$ indicates the interval is unbounded to the left.

non-rigid transformation | A set of operations that change the size and/or shape of a graph in a coordinate plane.

non-rigid transformation | A set of operations that change the size and/or shape of a graph in a coordinate plane.

nonlinear system | A system of equations where at least one equation is not linear.

nonlinear system | A system of equations where at least one equation is not linear.

***n*th partial sum of a geometric sequence** | The sum of the first *n* terms of a geometric sequence, given by the formula: $S_n = \frac{a_1(1-r^n)}{1-r}$, $r \neq 1$.

***n*th partial sum of a geometric sequence** | The sum of the first *n* terms of a geometric sequence, given by the formula: $S_n = \frac{a_1(1-r^n)}{1-r}$, $r \neq 1$.

***n*th partial sum of an arithmetic sequence** | The sum of the first *n* terms of an arithmetic sequence given by the formula: $S_n = \frac{n(a_1+a_n)}{2}$.

***n*th partial sum of an arithmetic sequence** | The sum of the first *n* terms of an arithmetic sequence given by the formula: $S_n = \frac{n(a_1+a_n)}{2}$.

***n*th root** | A number that when raised to the *n*th power $(n \geq 2)$ yields the original number.

***n*th root** | A number that when raised to the *n*th power $(n \geq 2)$ yields the original number.

odd integers | Nonzero integers that are not divisible by 2.

odd integers | Nonzero integers that are not divisible by 2.

one-to-one property of exponential functions | Given $b > 0$ and $b \neq 1$ we have $b^x = b^y$ if and only if $x = y$.

one-to-one property of exponential functions | Given $b > 0$ and $b \neq 1$ we have $b^x = b^y$ if and only if $x = y$.

one-to-one property of logarithms | Given $b > 0$ and $b \neq 1$ where $x, y > 0$ we have msublogb x=logb y if and only if $x = y$.

one-to-one property of logarithms | Given $b > 0$ and $b \neq 1$ where $x, y > 0$ we have msublogb x=logb y if and only if $x = y$.

opposite | Real numbers whose graphs are on opposite sides of the origin with the same distance to the origin.

opposite | Real numbers whose graphs are on opposite sides of the origin with the same distance to the origin.

opposite binomial property | If given a binomial $a - b$, then the opposite is $-(a-b) = b - a$.

opposite binomial property | If given a binomial $a - b$, then the opposite is $-(a-b) = b - a$.

opposite reciprocals | Two real numbers whose product is −1. Given a real number $\frac{a}{b}$, the opposite reciprocal is $-\frac{b}{a}$.

opposite reciprocals | Two real numbers whose product is −1. Given a real number $\frac{a}{b}$, the opposite reciprocal is $-\frac{b}{a}$.

ordered triple | Triples (*x*, *y*, *z*) that identify position relative to the origin in three-dimensional space.

ordered triple | Triples (*x*, *y*, *z*) that identify position relative to the origin in three-dimensional space.

origin | The point on the number line that represents zero.

origin | The point on the number line that represents zero.

origin | The point where the *x*- and *y*-axes cross, denoted by (0, 0).

origin | The point where the *x*- and *y*-axes cross, denoted by (0, 0).

parabola | The curved graph formed by the squaring function.

parabola | The curved graph formed by the squaring function.

parabola | The U-shaped graph of any quadratic function defined by $f(x) =$ax2+bx+c, where *a*, *b*, and *c* are real numbers and $a \neq 0$.

parabola | The U-shaped graph of any quadratic function defined by $f(x) =$ax2+bx+c, where *a*, *b*, and *c* are real numbers and $a \neq 0$.

parabola | The set of points in a plane equidistant from a given line, called the directrix, and a point not on the line, called the focus.

parabola | The set of points in a plane equidistant from a given line, called the directrix, and a point not on the line, called the focus.

parabola in standard form | The equation of a parabola written in the form $y = a(x-h)^2 + k$ or $x = a(y-k)^2 + h$.

parabola in standard form | The equation of a parabola written in the form $y = a(x-h)^2 + k$ or $x = a(y-k)^2 + h$.

parallel lines | Lines in the same plane that do not intersect; their slopes are the same.

parallel lines | Lines in the same plane that do not intersect; their slopes are the same.

partial sum | The sum of the first *n* terms in a sequence denoted S_n.

partial sum | The sum of the first *n* terms in a sequence denoted S_n.

Pascal's triangle | A triangular array of numbers that correspond to the binomial coefficients.

Pascal's triangle | A triangular array of numbers that correspond to the binomial coefficients.

perfect square trinomials | The trinomials obtained by squaring the binomials $(a+b)^2 = a^2$+2ab+b2 and $(a-b)^2 = a^2 - 2ab + b^2$.

perfect square trinomials | The trinomials obtained by squaring the binomials $(a+b)^2 = a^2$+2ab+b2 and $(a-b)^2 = a^2 - 2ab + b^2$.

piecewise definition | A definition that changes depending on the value of the variable.

piecewise definition | A definition that changes depending on the value of the variable.

placeholders | Terms with zero coefficients used to fill in all missing exponents within a polynomial.

placeholders | Terms with zero coefficients used to fill in all missing exponents within a polynomial.

plane | Any flat two-dimensional surface.

plane | Any flat two-dimensional surface.

plotting points | A way of determining a graph using a finite number of representative ordered pair solutions.

plotting points | A way of determining a graph using a finite number of representative ordered pair solutions.

point-slope form | Any nonvertical line can be written in the form $y - y_1 = m(x - x_1)$, where *m* is the slope and (x_1, y_1) is any point on the line.

point-slope form | Any nonvertical line can be written in the form $y - y_1 = m(x - x_1)$, where *m* is the slope and (x_1, y_1) is any point on the line.

polynomial | An algebraic expression consisting of terms with real number coefficients and variables with whole number exponents.

polynomial | An algebraic expression consisting of terms with real number coefficients and variables with whole number exponents.

polynomial inequality | A mathematical statement that relates a polynomial expression as either less than or greater than another.

polynomial inequality | A mathematical statement that relates a polynomial expression as either less than or greater than another.

polynomial long division | The process of dividing two polynomials using the division algorithm.

polynomial long division | The process of dividing two polynomials using the division algorithm.

polynomials with one variable | A polynomial where each term has the form anxn, where a_n is any real number and *n* is any whole number.

polynomials with one variable | A polynomial where each term has the form anxn, where a_n is any real number and *n* is any whole number.

power property of equality | Given any positive integer *n* and real numbers *a* and *b* where $a = b$, then $a^n = b^n$.

power property of equality | Given any positive integer *n* and real numbers *a* and *b* where $a = b$, then $a^n = b^n$.

power property of logarithms | msublogb xn=nlogb x; the logarithm of a quantity raised to a power is equal to that power times the logarithm of the quantity.

power property of logarithms | msublogb xn=nlogb x; the logarithm of a quantity raised to a power is equal to that power times the logarithm of the quantity.

power rule for exponents | $(x^m)^n =$ msupxmn; a power raised to a power can be simplified by multiplying the exponents.

power rule for exponents | $(x^m)^n =$ msupxmn; a power raised to a power can be simplified by multiplying the exponents.

prime factorization | The unique factorization of a natural number written as a product of primes.

prime factorization | The unique factorization of a natural number written as a product of primes.

prime number | Integer greater than 1 that is divisible only by 1 and itself.

prime number | Integer greater than 1 that is divisible only by 1 and itself.

prime polynomial | A polynomial with integer coefficients that cannot be factored as a product of polynomials with integer coefficients other than 1 and itself.

prime polynomial | A polynomial with integer coefficients that cannot be factored as a product of polynomials with integer coefficients other than 1 and itself.

principal (nonnegative) *n*th root | The positive *n*th root when *n* is even.

principal (nonnegative) *n*th root | The positive *n*th root when *n* is even.

principal (nonnegative) square root | The non-negative square root.

principal (nonnegative) square root | The non-negative square root.

principal (nonnegative) square root | The positive square root of a positive real number, denoted with the symbol $\sqrt{\ }$.

principal (nonnegative) square root | The positive square root of a positive real number, denoted with the symbol $\sqrt{\ }$.

product of complex conjugates | The real number that results from multiplying complex conjugates: $(a + bi)(a - bi) = a^2 + b^2$.

product of complex conjugates | The real number that results from multiplying complex conjugates: $(a + bi)(a - bi) = a^2 + b^2$.

product property of logarithms | msublogb (xy)=logb x+logb y; the logarithm of a product is equal to the sum of the logarithm of the factors.

product property of logarithms | msublogb (xy)=logb x+logb y; the logarithm of a product is equal to the sum of the logarithm of the factors.

product rule for exponents | $x^m \cdot x^n = x^{m+n}$; the product of two expressions with the same base can be simplified by adding the exponents.

product rule for exponents | $x^m \cdot x^n = x^{m+n}$; the product of two expressions with the same base can be simplified by adding the exponents.

profit function | A function that models the profit as revenue less cost.

profit function | A function that models the profit as revenue less cost.

properties of equality | Properties that allow us to obtain equivalent equations by adding, subtracting, multiplying, and dividing both sides of an equation by nonzero real numbers.

properties of equality | Properties that allow us to obtain equivalent equations by adding, subtracting, multiplying, and dividing both sides of an equation by nonzero real numbers.

properties of inequalities | Properties used to obtain equivalent inequalities and used as a means to solve them.

properties of inequalities | Properties used to obtain equivalent inequalities and used as a means to solve them.

proportion | A statement of equality of two ratios.

proportion | A statement of equality of two ratios.

Pythagorean theorem | The hypotenuse of any right triangle is equal to the square root of the sum of the squares of the lengths of the triangle's legs.

Pythagorean theorem | The hypotenuse of any right triangle is equal to the square root of the sum of the squares of the lengths of the triangle's legs.

quadrants | The four regions of a rectangular coordinate plane partly bounded by the *x*- and *y*-axes and numbered using the Roman numerals I, II, III, and IV.

quadrants | The four regions of a rectangular coordinate plane partly bounded by the *x*- and *y*-axes and numbered using the Roman numerals I, II, III, and IV.

quadratic form | An equation of the form au2+bu+c=0 where *a*, *b* and *c* are real numbers and *u* represents an algebraic expression.

quadratic form | An equation of the form au2+bu+c=0 where *a*, *b* and *c* are real numbers and *u* represents an algebraic expression.

quadratic formula | The formula $x = \frac{-b\pm\sqrt{b^2\ \ 4ac}}{2a}$, which gives the solutions to any quadratic equation in the standard form ax2+bx+c=0, where *a*, *b*, and *c* are real numbers and $a \neq 0$.

quadratic formula | The formula $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, which gives the solutions to any quadratic equation in the standard form ax2+bx+c=0, where *a*, *b*, and *c* are real numbers and $a \neq 0$.

quadratic inequality | A mathematical statement that relates a quadratic expression as either less than or greater than another.

quadratic inequality | A mathematical statement that relates a quadratic expression as either less than or greater than another.

quotient | The result of division.

quotient | The result of dividing.

quotient | The result of division.

quotient | The result of dividing.

quotient property of logarithms | msublogb (xy)=logb x−logb y; the logarithm of a quotient is equal to the difference of the logarithm of the numerator and the logarithm of the denominator.

quotient property of logarithms | msublogb (xy)=logb x−logb y; the logarithm of a quotient is equal to the difference of the logarithm of the numerator and the logarithm of the denominator.

quotient rule for radicals | Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$, mrootABn=AnBn where $B \neq 0$.

quotient rule for radicals | Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$, mrootABn=AnBn where $B \neq 0$.

quotients with negative exponents | $\frac{x^{-n}}{y^{-m}} =$ mfracymxn, given any integers *m* and *n*, where $x \neq 0$ and $y \neq 0$.

quotients with negative exponents | $\frac{x^{-n}}{y^{-m}} =$ mfracymxn, given any integers *m* and *n*, where $x \neq 0$ and $y \neq 0$.

radical | Used when referring to an expression of the form $\sqrt[n]{A}$.

radical | Used when referring to an expression of the form $\sqrt[n]{A}$.

radical equation | Any equation that contains one or more radicals with a variable in the radicand.

radical equation | Any equation that contains one or more radicals with a variable in the radicand.

radical expression | An algebraic expression that contains radicals.

radical expression | An algebraic expression that contains radicals.

radical is simplified | A radical where the radicand does not consist of any factors that can be written as perfect powers of the index.

radical is simplified | A radical where the radicand does not consist of any factors that can be written as perfect powers of the index.

radicand | The number within a radical.

radicand | The number within a radical.

radicand | The expression *A* within a radical sign, $\sqrt[n]{A}$.

radicand | The expression *A* within a radical sign, $\sqrt[n]{A}$.

rational equation | An equation containing at least one rational expression.

rational equation | An equation containing at least one rational expression.

rational inequality | A mathematical statement that relates a rational expression as either less than or greater than another.

rational inequality | A mathematical statement that relates a rational expression as either less than or greater than another.

rationalizing the denominator | The process of determining an equivalent radical expression with a rational denominator.

rationalizing the denominator | The process of determining an equivalent radical expression with a rational denominator.

real numbers | The set of all rational and irrational numbers.

real numbers | The set of all rational and irrational numbers.

reciprocal function | The function defined by $f(x) = \frac{1}{x}$.

reciprocal function | The function defined by $f(x) = \frac{1}{x}$.

reciprocals | Two real numbers whose product is 1.

reciprocals | Two real numbers whose product is 1.

recurrence relation | A formula that uses previous terms of a sequence to describe subsequent terms.

recurrence relation | A formula that uses previous terms of a sequence to describe subsequent terms.

reducing | The process of finding equivalent fractions by dividing the numerator and the denominator by common factors.

reducing | The process of finding equivalent fractions by dividing the numerator and the denominator by common factors.

reflection | A transformation that produces a mirror image of the graph about an axis.

reflection | A transformation that produces a mirror image of the graph about an axis.

relation | Any set of ordered pairs.

relation | Any set of ordered pairs.

relatively prime | Expressions that share no common factors other than 1.

relatively prime | Expressions that share no common factors other than 1.

Restrictions | The set of real numbers for which a rational function is not defined.

Restrictions | The set of real numbers for which a rational function is not defined.

revenue function | A function that models income based on a number of units sold.

revenue function | A function that models income based on a number of units sold.

root | A value in the domain of a function that results in zero.

root | A value in the domain of a function that results in zero.

row echelon form | A matrix in triangular form where the leading nonzero element of each row is 1.

row echelon form | A matrix in triangular form where the leading nonzero element of each row is 1.

run | The horizontal change between any two points on a line.

run | The horizontal change between any two points on a line.

scientific notation | Real numbers expressed the form a×10n, where *n* is an integer and $1 \leq a < 10$.

scientific notation | Real numbers expressed the form a×10n, where *n* is an integer and $1 \leq a < 10$.

secant line | Line that intersects two points on the graph of a function.

secant line | Line that intersects two points on the graph of a function.

set notation | Notation used to describe a set using mathematical symbols.

set notation | Notation used to describe a set using mathematical symbols.

sign chart | A model of a function using a number line and signs (+ or −) to indicate regions in the domain where the function is positive or negative.

sign chart | A model of a function using a number line and signs (+ or −) to indicate regions in the domain where the function is positive or negative.

similar radicals | Term used when referring to like radicals.

similar radicals | Term used when referring to like radicals.

similar terms | Used when referring to like terms.

similar terms | Used when referring to like terms.

Simple interest | Modeled by the formula I =prt, where *p* represents the principal amount invested at an annual interest rate *r* for *t* years.

Simple interest | Modeled by the formula I =prt, where *p* represents the principal amount invested at an annual interest rate *r* for *t* years.

simplified radical | A radical where the radicand does not consist of any factors that can be written as perfect powers of the index.

simplified radical | A radical where the radicand does not consist of any factors that can be written as perfect powers of the index.

simplifying the expression | The process of combining like terms until the expression contains no more similar terms.

simplifying the expression | The process of combining like terms until the expression contains no more similar terms.

simultaneous solution | Used when referring to a solution of a system of equations.

simultaneous solution | Used when referring to a solution of a system of equations.

slope formula | The slope of the line through the points (x_1, y_1) and (x_2, y_2) is given by the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.

slope formula | The slope of the line through the points (x_1, y_1) and (x_2, y_2) is given by the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.

slope-intercept form | Any nonvertical line can be written in the form $y = mx + b$, where *m* is the slope and (0, *b*) is the *y*-intercept.

slope-intercept form | Any nonvertical line can be written in the form $y = mx + b$, where *m* is the slope and (0, *b*) is the *y*-intercept.

solution | Any value that can replace the variable in an equation to produce a true statement.

solution | Any value that can replace the variable in an equation to produce a true statement.

solution to a linear inequality | A real number that produces a true statement when its value is substituted for the variable.

solution to a linear inequality | A real number that produces a true statement when its value is substituted for the variable.

Solutions | Values that can be used in place of the variable to satisfy the given condition.

Solutions | Values that can be used in place of the variable to satisfy the given condition.

solve by factoring | The process of solving an equation that is equal to zero by factoring it and then setting each variable factor equal to zero.

solve by factoring | The process of solving an equation that is equal to zero by factoring it and then setting each variable factor equal to zero.

split function | A term used when referring to a piecewise function.

split function | A term used when referring to a piecewise function.

square matrix | A matrix with the same number of rows and columns.

square matrix | A matrix with the same number of rows and columns.

square root function | The function defined by $f(x) = \sqrt{x}$.

square root function | The function defined by $f(x) = \sqrt{x}$.

square root function | The function defined by $f\,(x) = \sqrt{x}$.

square root function | The function defined by $f\,(x) = \sqrt{x}$.

square root property | For any real number *k*, if $x^2 = k$, then $x = \pm\sqrt{k}$.

square root property | For any real number *k*, if $x^2 = k$, then $x = \pm\sqrt{k}$.

squaring function | The quadratic function defined by $f(x) = x^2$.

squaring function | The quadratic function defined by $f(x) = x^2$.

squaring property of equality | Given real numbers *a* and *b*, where $a = b$, then $a^2 = b^2$.

squaring property of equality | Given real numbers *a* and *b*, where $a = b$, then $a^2 = b^2$.

standard form | Any quadratic equation in the form ax2+bx+c=0, where *a*, *b*, and *c* are real numbers and $a \neq 0$.

standard form | Any quadratic equation in the form ax2+bx+c=0, where *a*, *b*, and *c* are real numbers and $a \neq 0$.

substitute | The act of replacing a variable with an equivalent quantity.

substitute | The act of replacing a variable with an equivalent quantity.

substitution method | A means of solving a linear system by solving for one of the variables and substituting the result into the other equation.

substitution method | A means of solving a linear system by solving for one of the variables and substituting the result into the other equation.

subtraction | Subtract functions as indicated by the notation: $(f \quad g)\,(x) = f(x) \quad g\,(x)$.

subtraction | Subtract functions as indicated by the notation: $(f \quad g)\,(x) \quad f(x) \quad g\,(x)$.

sum of squares | $a^2 + b^2$, where *a* and *b* represent algebraic expressions. This does not have a general factored equivalent.

sum of squares | $a^2 + b^2$, where *a* and *b* represent algebraic expressions. This does not have a general factored equivalent.

summation | Used when referring to sigma notation.

summation | Used when referring to sigma notation.

symmetric property | Allows you to solve for the variable on either side of the equal sign, because $x=5$ is equivalent to $5=x$.

symmetric property | Allows you to solve for the variable on either side of the equal sign, because $x=5$ is equivalent to $5=x$.

system of inequalities | A set of two or more inequalities with the same variables.

system of inequalities | A set of two or more inequalities with the same variables.

TBA | Given any real number *a*, $a+0=0+a=a$.

TBA | Given any real number *a*, $a+(-a)=(-a)+a=0$.

TBA | Given real numbers *a*, *b* and *c*, $(a+b)+c=a+(b+c)$.

TBA | Given real numbers *a* and *b*, $a+b=b+a$.

TBA | Given any real number *a*, $a\cdot 0=0\cdot a=0$.

TBA | Given any real number *a*, $a\cdot 1=1\cdot a=a$.

TBA | Given any real numbers *a*, *b* and *c*, $(a\cdot b)\cdot c=a\cdot(b\cdot c)$.

TBA | Given any real numbers *a* and *b*, $a\cdot b=b\cdot a$.

TBA | Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$, $\sqrt[n]{A\cdot B}=\sqrt[n]{A}\cdot\sqrt[n]{B}$.

TBA | Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$, mrootABn=AnBn.

TBA | $(xy)^n$ =xnyn; if a product is raised to a power, then apply that power to each factor in the product.

TBA | $\left(\frac{x}{y}\right)^n=$ mfracxnyn ; if a quotient is raised to a power, then apply that power to the numerator and the denominator.

TBA | mfracxmxn=xm−n; the quotient of two expressions with the same base can be simplified by subtracting the exponents.

TBA | A polynomial with degree 1.

TBA | A polynomial with degree 2.

TBA | A polynomial with degree 3.

TBA | Given any real number *a*, $a+0=0+a=a$.

TBA | Given any real number *a*, $a+(-a)=(-a)+a=0$.

TBA | Given real numbers *a*, *b* and *c*, $(a+b)+c=a+(b+c)$.

TBA | Given real numbers *a* and *b*, $a+b=b+a$.

TBA | Given any real number *a*, $a\cdot 0=0\cdot a=0$.

TBA | Given any real number *a*, $a\cdot 1=1\cdot a=a$.

TBA | Given any real numbers *a*, *b* and *c*, $(a\quad b)\quad c=a\quad(b\quad c)$.

TBA | Given any real numbers *a* and *b*, $a\quad b=b\quad a$.

TBA | Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$, $\sqrt[n]{A\quad B}=\sqrt[n]{A}\quad\sqrt[n]{B}$.

TBA | Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$, mrootABn=AnBn.

TBA | $(xy)^n$ xnyn; if a product is raised to a power, then apply that power to each factor in the product.

TBA | $\left(\frac{x}{y}\right)^n$ mfracxnyn ; if a quotient is raised to a power, then apply that power to the numerator and the denominator.

TBA | mfracxmxn=xm−n; the quotient of two expressions with the same base can be simplified by subtracting the exponents.

TBA | Used when referring to direct variation.

TBA | Used when referring to joint variation.

TBA | Used when referring to direct variation.

TBA | Used when referring to joint variation.

test points | A point not on the boundary of the linear inequality used as a means to determine in which half-plane the solutions lie.

test points | A point not on the boundary of the linear inequality used as a means to determine in which half-plane the solutions lie.

trial and error (or guess and check) method | Describes the method of factoring a trinomial by systematically checking factors to see if their product is the original trinomial.

trial and error (or guess and check) method | Describes the method of factoring a trinomial by systematically checking factors to see if their product is the original trinomial.

trinomial | Polynomial with three terms.

***u*-substitution** | A technique in algebra using substitution to transform equations into familiar forms.

***u*-substitution** | A technique in algebra using substitution to transform equations into familiar forms.

undefined | A quotient such as $\frac{5}{0}$ is left without meaning and is not assigned an interpretation.

undefined | A quotient such as $\frac{5}{0}$ is left without meaning and is not assigned an interpretation.

uniform motion | The distance *D* after traveling at an average rate *r* for some time *t* can be calculated using the formula $D=rt$.

uniform motion | The distance *D* after traveling at an average rate *r* for some time *t* can be calculated using the formula $D=rt$.

Uniform motion (or distance) | Described by the formula $D=rt$, where the distance *D* is given as the product of the average rate *r* and the time *t* traveled at that rate.

Uniform motion (or distance) | Described by the formula $D=rt$, where the distance *D* is given as the product of the average rate *r* and the time *t* traveled at that rate.

union | The set formed by joining the individual solution sets indicated by the logical use of the word "or" and denoted with the symbol ∪.

union | The set formed by joining the individual solution sets indicated by the logical use of the word "or" and denoted with the symbol ∪.

unit circle | The circle centered at the origin with radius 1; its equation is $x^2+y^2=1$.

unit circle | The circle centered at the origin with radius 1; its equation is $x^2+y^2=1$.

upper triangular form | A linear system consisting of equations with three variables in standard form arranged so that the variable *x* does not appear after the first equation and the variable *y* does not appear after the second equation.

upper triangular form | A linear system consisting of equations with three variables in standard form arranged so that the variable *x* does not appear after the first equation and the variable *y* does not appear after the second equation.

variables | Letters used to represent numbers.

variables | Letters used to represent numbers.

vertex form | A quadratic function written in the form $f(x)=a(x-h)^2+k$.

vertex form | A quadratic function written in the form $f(x)=a(x-h)^2+k$.

vertex form | The equation of a parabola written in standard form is often called vertex form. In this form the vertex is apparent: (h,k) .

vertex form | The equation of a parabola written in standard form is often called vertex form. In this form the vertex is apparent: (h,k) .

vertical asymptote | A vertical line to which a graph becomes infinitely close.

vertical asymptote | A vertical line to which a graph becomes infinitely close.

vertical line test | If any vertical line intersects the graph more than once, then the graph does not represent a function.

vertical line test | If any vertical line intersects the graph more than once, then the graph does not represent a function.

vertical translation | A rigid transformation that shifts a graph up or down.

vertical translation | A rigid transformation that shifts a graph up or down.

vertices. | Points on the separate branches of a hyperbola where the distance is a minimum.

vertices. | Points on the separate branches of a hyperbola where the distance is a minimum.

whole numbers | The set of natural numbers combined with zero: {0, 1, 2, 3, 4, 5, …}.

whole numbers | The set of natural numbers combined with zero: {0, 1, 2, 3, 4, 5, …}.

work rate | The rate at which a task can be performed.

work rate | The rate at which a task can be performed.

work-rate formula | $\frac{1}{t_1}\cdot t+\frac{1}{t_2}\cdot t=1$, where mfrac1t1 and mfrac1t2 are the individual work rates and *t* is the time it takes to complete the task working together.

work-rate formula | $\frac{1}{t_1}\cdot t+\frac{1}{t_2}\cdot t=1$, where mfrac1t1 and mfrac1t2 are the individual work rates and *t* is the time it takes to complete the task working together.

***x*-intercept** | The point (or points) where a graph intersects the *x*-axis, expressed as an ordered pair (*x*, 0).

***x*-intercept** | The point (or points) where a graph intersects the *x*-axis, expressed as an ordered pair (*x*, 0).

***y*-intercept** | The point (or points) where a graph intersects the *y*-axis, expressed as an ordered pair (0, *y*).

***y*-intercept** | The point (or points) where a graph intersects the *y*-axis, expressed as an ordered pair (0, *y*).

zero as an exponent | $x^0\quad 1$; any nonzero base raised to the 0 power is defined to be 1.

zero as an exponent | $x^0\quad 1$; any nonzero base raised to the 0 power is defined to be 1.

zero factorial | The factorial of zero is defined to be equal to 1; $0!=\ 1$.

zero factorial | The factorial of zero is defined to be equal to 1; $0!=\ 1$.

zero-product property | A product is equal to zero if and only if at least one of the factors is zero.

zero-product property | A product is equal to zero if and only if at least one of the factors is zero.

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